Computing Complete Solution Sets for Approximate Four-bar Path Synthesis

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Abstract

The approximate path synthesis of four-bar linkages has been framed and solved with many different optimization techniques. Here we present a polynomial objective that is invariant to the number of approximate design positions selected, and a solution technique capable of finding all minima. The invariance property caps compute time despite increasing the size of input task specification data. This is performed by collecting a variable amount of task data into an invariable number of polynomial coefficients, called moments, before numerical optimization begins. The minima are found by applying the method of random monodromy loops to the zero gradient polynomial system of the aforementioned objective. This procedure finds all critical points, including the local and global minimum, and provides an in-process estimate of the percentage of critical points found. We applied our methodology to four-bar path synthesis problems of various computational scales by altering dimensional pre-specifications. The most general case was estimated to have $1,820,238 \pm 3810$ critical points, while pre-specification of one or two ground pivots yielded 26,052 and 503 roots, respectively, as validated by a trace test. The results are applied to a variety of examples.

Keywords: Kinematic synthesis, Optimization, Homotopy continuation

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1. Introduction

The approximate path synthesis of linkages is often formulated and solved as an optimization problem. Algorithms constructed from the current selection of optimization techniques can be used to find (1) one design option, (2) several stochastically generated design options, or (3) a pareto front of design options. Apart from the choice of optimization technique, the choice in how to construct an objective(s) yields widely varying results. In this work, we aim to compute nearly complete solutions to approximate kinematic synthesis problems. That is, we form nonconvex polynomial objectives and then apply polynomial homotopy continuation to first-order conditions to compute critical points. Our work is primarily advantageous over past research in that it removes guesswork over the superiority of local minima, can find minima with small regions of attraction, and, since it aims to find all minima, we can cross-compare them over auxiliary considerations neglected by the objective.

In a sense, our approach is basic in that we outright compute all stationary points from first-order conditions. However, such computations have been prohibitive in the past due to the scale of the computation. The enabling technology is the new algorithms and advances in polynomial homotopy continuation [1], notably, the method of random monodromy loops [2, 3, 4, 5, 6]. Although computational power has increased dramatically over the past decades, without the algorithmic advances of homotopy continuation, there would be no route forward to solving the problems approached in this paper.

The problems solved in this paper are related to path synthesis for the four-bar linkage (schematically shown in Fig. [1]). In such a problem, a planar path is prescribed for a point connected to the coupler link of a four-bar to trace. The goal is to compute the dimensions of a four-bar which can approximately reproduce this path. The scale of the computation required to completely solve such an optimization problem can be varied by installing simplifications in the form of pre-specified dimensions, which essentially reduce the dimension of the design space. In this vein, we form three different optimization problems: approximate path synthesis for a four-bar with (1) no dimensions pre-specified, (2) one ground pivot pre-specified, and (3) both ground pivots pre-specified. The first uses the most computational resources while the last requires
the least. For each of these problems, we conduct an ab initio computation to estimate the generic number of critical points each optimization problem has. This is performed by forming numerically general versions of the first-order conditions and solving them with the method of random monodromy loops. The size of these finite root sets are analyzed statistically to place confidence bounds on its accuracy [6] and, if possible, certified to be complete using a trace test [7, 8]. The resulting numerically generic finite root datasets can be applied as starting points with parameter homotopies [9] to solve for practical (not numerically generic) engineering design problems. Such parameter homotopies track fewer paths and thus use less computational resources. We demonstrate this functionality for a variety of examples in this paper.

1.1. Literature Review

If all dimensions of a four-bar are set to be design variables, then it can be shown that the four-bar can move a coupler point exactly through nine prescribed points generically. Wampler et al. [10] applied homotopy continuation to this problem and found the relevant polynomial system to generically have 8,652 finite roots which naturally has a 2-way symmetry from relabeling and a 3-way symmetry from Roberts’ cognates [11]. Hence, there are 1,442 distinct four-bar coupler curves that pass exactly through nine prescribed points generically thereby solving Alt’s problem [12]. If the two ground pivots of a four-bar are pre-specified and the rest of the dimensions are set to be design variables, then it can be shown that the four-bar can move a coupler point exactly through five prescribed points. Several authors [13, 14, 15] applied homotopy continuation to this problem and found the relevant polynomial system to generically have 36 nondegenerate, finite roots. In this work, we address analogous problems but applied to the approximate case, that is, N-point approximate synthesis.

By alleviating the exactness requirement on the coupler trace, approximate synthesis techniques allow for a greater number of prescribed task points. These formulations lead to nonlinear optimization problems with many local minima. Examples of nonlinear programming techniques that find only a single minima include [16, 17]. As a slight improvement, the initial guess of the nonlinear program could be varied systematically [18] or randomly [19] to hopefully discover more minima. Other authors [20] have considered working directly with the first-order necessary conditions, similar to this work.

Metaheuristic algorithms [21, 22, 23, 24, 25, 26, 27, 28] are less prone to settling on an inferior local minimum. Additionally, these algorithms need no derivative information, no initial guess (usually), and are capable of generating a Pareto front to accommodate multiple objectives. However, metaheuristic algorithms are stochastic in nature, require hyper-parameter tuning, and do not necessarily find a global minimum or a complete view of the optimization landscape.

Approximate synthesis techniques accommodate the approximate nature of most practical design problems. Exact synthesis methods are often criticized as few practical design problems require such exactness. However, their appeal comes in the form of their deterministic nature and, assuming complete solutions are obtained, their ability to generate multiple design options of diverse forms. This paper aims to bring that aspect of completeness to approximate synthesis. Rather than using the approximate points directly in the formulation, our approach formulates an objective based off the moments of path points. Setting its gradient equal to zero leads to a square polynomial system in the design variables. Since this polynomial system is highly nonlinear, it possesses many roots, indicating the locations of critical points and potential minima. Polynomial homotopy continuation [29] is applied to a numerically general version of this system in order characterize the size of its solution set and compute start points for later parameter homotopies. Statistical estimates [6] yield confidence bounds on the root count and, when possible, a trace test [7, 8] is applied to certify the root counts from the previous step. Our work up until this point is numeric but nonetheless generic and conclusive. Parameter homotopies are used to compute results for specific design problems.

In the proceeding, we formulate synthesis equations and describe our numerical methods in Section 2. Next, we approach three four-bar path synthesis problems with various simplifications installed. We consider approximate path synthesis when no dimensions are pre-specified (Section 3), when one ground pivot is pre-specified (Section 4), and when both ground pivots are pre-specified (Section 5). In each case, we present practical design scenarios to showcase the utility of our approach. Section 6 summarizes the contribution.
2. Mathematical Formulation

2.1. Approximate Synthesis Equations

Consider the four-bar linkage shown in Fig. 1. Let A and B be the ground pivots of the linkage, and let \( l_1, l_2, l_3 \) be the moving link lengths with angular displacements, measured counter-clockwise from the \( x \)-axis, as \( \phi_1, \phi_2, \phi_3 \), respectively. The coupler trace point, normalized by the coupler base, \( l_2 \), is denoted in the coupler frame with the vector variable \( Q \) and in the global frame with the vector variable \( X \). The use of isotropic coordinates with complex variable/parameter and its respective conjugate rather than scalars afford simpler mathematical descriptions among other advantages.

Denote \( \Phi_k = e^{i\phi_k} \) for \( k = 1, 2, 3 \) be the 2D rotation operators. Then, the vector loop equations for the left and right dyads are, respectively,

\[
A + l_1 \Phi_1 + l_2 Q \Phi_2 = X, \quad (1)
\]
\[
B + l_3 \Phi_3 + l_2(Q - 1) \Phi_2 = X. \quad (2)
\]

Since we are working in isotropic coordinates, the conjugate relationship of the vector loops must be upheld. We denote conjugates with \( \ast \) and note that the conjugate of a rotation operator is its reciprocal. Additionally, since the link lengths are real values, \( l_k^* = l_k \). Hence, the conjugate loop equations are

\[
A^* + l_1^* \frac{1}{\Phi_1} + l_2^* Q^* \frac{1}{\Phi_2} = X^*, \quad (3)
\]
\[
B^* + l_3^* \frac{1}{\Phi_3} + l_2^*(Q - 1) \frac{1}{\Phi_2} = X^*. \quad (4)
\]

Since the rotation operators are not design specifications, they can be eliminated from the loop equations. Eliminating \( \Phi_1 \) using Eqs. (1) and (3) results in

\[
l_2 Q^*(A - X) + (XX^* - A^*X - AX^* - l_{1a}) \Phi_2 + l_2 Q(A^* - X^*) \Phi_2^2 = 0, \quad (5)
\]

where \( l_{1a} = l_1^2 - l_2^2 QQ^* - AA^* \). Likewise, we can eliminate \( \Phi_3 \) between Eqs. (2) and (4) to obtain

\[
l_2(Q^* - 1)(B - X) + (XX^* - B^*X - BX^* - l_{3a}) \Phi_2 + l_2(Q - 1)(B^* - X^*) \Phi_2^2 = 0, \quad (6)
\]

where \( l_{3a} = l_3^2 - l_2^2(Q - 1)(Q^* - 1) - BB^* \).

To eliminate \( \Phi_2 \), note that the operator appears in both Eqs. (5) and (6) quadratically. One can define a Sylvester’s matrix representation using these two polynomials as follows:

\[
\begin{vmatrix}
Q^*(A - X) & g(X, X^*) & l_2 Q(A^* - X^*) & 0 \\
0 & l_2 Q^*(A - X) & g(X, X^*) & Q(A^* - X^*) \\
(Q^* - 1)(B - X) & h(X, X^*) & l_2(Q - 1)(B^* - X^*) & 0 \\
0 & l_2(Q^* - 1)(B - X) & h(X, X^*) & (Q - 1)(B^* - X^*)
\end{vmatrix} = 0,
\]
where

\[ g(X, X^*) = XX^* - A^*X - AX^* - l_{1s} \quad \text{and} \quad h(X, X^*) = XX^* - B^*X - BX^* - l_{3s}. \]

The determinant of the above Sylvester’s matrix eliminates \( \Phi_2 \) and describes a polynomial representation of the coupler trace of a four-bar linkage. This determinant is real-valued, of degree six for the design positions \((X_j, X_j^*)\), and of circularity \([11]\). We capitalize on the further numerical advantages provided by substituting \( l_{2s} = l_2^2 \) in the expanded form and by recognizing this condition is upheld for all four-bar linkages and their respective Roberts’ cognates \([11]\).

We denote this determinant condition as \( \eta_j(d, p) \) where \( d = \{A, A^*, B, B^*, l_{1s}, l_{2s}, l_{3s}, Q, Q^*\} \) are the design variables and \( p \) are the design parameters.

Consider the path generation problem for positions \((X_j, X_j^*)\) for \( j = 1, \ldots, N \) where \( N \) is the number of positions. It is well known that Alt’s problem \([12]\) for \( N = 9 \) generic points is equivalent to solving the nine-dimensional square system \( \eta_j(d, p) = 0 \) for \( j = 1, \ldots, 9 \). When \( N > 9 \) and the design positions are generic, the exact path synthesis problem has no solutions, therefore, one can describe a formulation for an approximate path synthesis. Our method of approximation is to minimize the residuals of the coupler-trace equation previously described based on an \( L_2 \)-norm measure. This measure preserves the system’s polynomial nature and is real-valued. That is, one aims to solve the unconstrained optimization problem

\[ \min_d \sum_{j=1}^{N} \eta_j^2(d, p). \tag{7} \]

The respective first-order optimality conditions

\[ \sum_{j=1}^{N} \eta_j \frac{\partial \eta_j}{\partial d} = 0. \tag{8} \]

Preliminary work by the current authors has been completed for problems where the design parameters are the design positions \( p = (X_j, X_j^*), j = 1, \ldots, N \) in cases where \( N \) is as large as twenty specified positions \([31]\). Since these design positions appear nonlinearly in \( \eta_j \), we design an alternative parameterization.

Mathematically, moments provide a descriptive measurements, e.g., the first moment of a probability distribution is the mean which describes the center of mass. By expanding the objective function in \((7)\), one observes that it depends linearly upon 47 moments of the points \((X_j, X_j^*)\) for \( j = 1, \ldots, N \) of the form

\[ \frac{1}{N} \sum_{j=1}^{N} X^aX^b \tag{9} \]

where \( 0 \leq a, b \leq 6 \) except \((a, b) = (0, 0)\) and \((a, b) = (6, 6)\). The former is excluded since that moment would simply be 1 and the latter does not appear in the expansion. In particular, let \( g = \{g_1, g_2, \ldots, g_{47}\} \) with

\[ g_1 = \frac{1}{N} \sum_{j=1}^{N} X_j, \ldots, g_{47} = \frac{1}{N} \sum_{j=1}^{N} X_j^6X_j^5 \]

with \([\text{Appendix A}]\) providing a full ordering of monomial moments. Hence, one can write the objective function as \( \psi(d; g) \) which is linear in \( g \) so that the first-order optimality conditions yield \( \nabla_d \psi(d; g) = 0 \) which is also linear in the parameters \( g \).

A natural question is to consider the image of the map from design positions to these moments. This, for example, can be analyzed using \([32, \text{Lemma 3}]\) which yields the following.

**Proposition 1.** The image of the map from the design positions to the 47-dimensional moment space is full dimensional for sufficiently large \( N \). In fact, this holds when \( 2N > 47 \).

Since the image from design positions to the moment space is dense, the generic number of roots using the moment formulation and the design position formulation agree when \( N \) is sufficiently large. One advantage
of using the moment formulation is that all parameters appear linearly in the system which can improve both local conditioning of the system and solving via random monodromy loops.

By using a moment formulation, one can consider discrete design position points and continuous families of design positions using the same formulation. For example, for a family \( (X(s), X^*(s)) \) with \( s_0 \leq s \leq s_1 \), one can replace \( \psi(d; g_s) = 0 \) with
\[
\int_{s_0}^{s_1} X(s)^a X^*(s)^b \, ds
\]

\[
\int_{s_0}^{s_1} ds.
\]

2.2. Random Monodromy Loops

Mathematically, monodromy is a group action that permutes solution sets of a system as the parameters are deformed along loops. A random monodromy loop (RML) is a numerical continuation technique that starts with seeding set of solutions and aims to compute another set of solutions as the parameters are deformed along a random loop. As the solution paths travel through intermediary systems that are generic before returning to the original system, the ending solution set could consist of both previously known solutions and new solutions. By performing a collection of random monodromy loops with each loop starting with the set of all known solutions at the time, one aims to compute all solutions efficiently.

To start, we fixed a randomly selected set of moment parameters \( g_s \) that does not need to satisfy the conjugate relationship and obtained a seed solution of \( \psi(d; g_s) = 0 \). Such a seed solution can be obtained using a local method such as Newton’s method or a Newton homotopy. Then, we employed RML along a randomly selected triangular loop in the parameter space. Specifically, for each loop, we randomly selected moment parameters \( g_1, g_2 \) and, since the parameters appear linearly, we simply used three straight-line homotopies along the three edges of the triangle:

\[
H_1(d; t) = \psi(d; g_1) t + \psi(d; g_2)(1 - t), \quad t \in [0, 1],
\]
\[
H_2(d; t) = \psi(d; g_1) t + \psi(d; g_2)(1 - t), \quad t \in [0, 1],
\]
\[
H_3(d; t) = \psi(d; g_2) t + \psi(d; g_3)(1 - t), \quad t \in [0, 1].
\]

Thus, starting with a known set of solutions corresponding to parameters \( g_s \), tracking along such a loop yields another set of solutions for the same parameters. After each loop, the ending set of solutions is filtered to distinguish new solutions from previously found ones. The next loop then starts tracking from the updated known set of solutions and the process continues. Thus, ideally, starting from one seed solution, one knows 2 solutions after completing the first loop, 4 solutions after the second, 8 solutions after the third, and so on.

One additional way to reduce the number of paths to track is to utilize the fact that the system \( \psi \) and the corresponding solution paths are invariant under Roberts’ cognates. Therefore, one only needs to track one point from each cognate group as the others are obtained simply via Roberts’ cognates. As a result, only a fraction of solution paths need to be tracked providing great computational efficiency in larger systems.

Since the number of isolated solutions is finite, such a doubling process in the number of known solutions can not continue indefinitely. Thus, by comparing the number of new solutions obtained with the number of repeated solutions, one can obtain statistical estimates on the total number of solutions using a probabilistic “catch and release” model [9]. The Schnabel estimator relies on results from multiple loops to yield both an estimate on the total number of solutions and confidence intervals. For our experiments, we utilized data from the current and two previous random monodromy loops to obtain a Schnabel estimate with a moving window size of three. Explicit equations for this estimate and confidence intervals are provided in [6].

2.3. Trace Test

Although the number of new solutions found and the Schnabel estimates provide confidence on the total number of solutions, one may wish to verify that all solutions have indeed been found. This can be accomplished via a 2-homogeneous trace test [10]. The 2-homogenity arises from the design variables \( d \) and the moment parameters \( g \). With such a test, one needs to collect two solution sets. The first is as above with the moment parameters fixed. In the second, one selects a design variable to be a parameter and selects a moment parameter to be a variable, and repeats the solving process as above. By using these two solution sets, the 2-homogeneous trace test [10] can determine if the solution sets are complete or not.

For the computations in this paper, we employed the second derivative trace test from [8 § 2.3] to avoid
tracking additional paths. In particular, this local trace test approach simply relies upon computing local Jacobian and Hessian information of $\psi$ to perform the trace test.

The following three sections utilize the aforementioned techniques on three formulations: the general case with no pre-specified dimensions, pre-specification of one ground pivot, and pre-specification of both ground pivots, respectively. Each problem was first solved in an ab initio run using random monodromy loops with the size of the solution set tested using either a statistical probabilistic model on the RML iterations or a trace test computation, or both. Finally, in each of these three formulations, we present a real-world example application. All computations were run using Bertini\cite{Bertini2013} in parallel mode on a four node dual 192 core machine at the University of Notre Dame’s Center for Research Computing.

3. The General Case (No Pre-specification of Dimensions)

3.1. Ab Initio Computation

Following Section 2 the general case has design variables $d = \{A, A^*, B, B^*, l_{1s}, l_{2s}, l_{3s}, Q, Q^*\}$ and moment parameters $g = \{g_1, \ldots, g_{47}\}$ as listed in Appendix A. With Roberts’ cognates and relabeling, the solutions arise in groups of 6 with formulas listed in Appendix B. With this setup, the RML procedure was used to determine the generic root count. Figure 2 shows the ratio of repeated solutions for the iterations and the Schnabel estimates with the 95% confidence interval based on groups of solutions, i.e., one-sixth of the total number.

We note that to further improve the conditioning of the system when performing the random monodromy loops, we redefined the grouping of the system’s variables for the homotopy and allowed for retracking of different solutions in the group of 6 when a path failed between intermediary systems. This resulted in improving path success rate of approximately 95 – 97% per iteration. The increased success rate case at an increase in the computational cost with the total time taking approximately 174 hours (7.25 days).

Previous work with this approximate synthesis problem used a parameterization based on design positions \cite{31} and executed 26 iterations of RML. Thus, Fig. 2 also shows 26 iterations of RML for the moment-parameterized system. The final count of distinct solution groups after 26 RML iterations was 303,387 yielding $6 \times 303,387 = 1,820,322$ solutions in total. One can see in Fig. 2a that the initial RML iterations find almost exclusively new points while later iterations find almost no new points. For example, iterations 23-26 only produced 87 new solutions, hence there is strong confidence we have recovered $\sim 99.8\%$ of the solution set.
3.2. Solution Set Validation

From Fig. 2b, one observes that the 95% confidence interval shrinks quickly for the Schnabel estimate with a moving window of size 3 as the number of iterations increases. In particular, at iteration 26, the 95% confidence interval for the Schnabel estimate is $6 \times (303.373 \pm 635) = 1,820,238 \pm 3810$ solutions. This is within a rather tight bound when regarding the size of the system, and the RML computation count from the previous section lies within this bound. The total isolated solution count is estimated to be upwards of approximately two million solutions, cognate transformations included [31].

We did not employ a trace test on this system for two reasons. First, the set of known solutions is probably not complete and thus more iterations would be needed. Also, solving for the switched variable-parameter systems poses an equally arduous challenge. Therefore, for this problem, we rely upon the statistical estimates to provide that the solution set is nearly complete.

3.3. Applied Example

As an illustration of using this \textit{ab initio} computation to solve an applied problem, we consider approximately replicating the curve in [34, Fig. 6]. This curve is traced by a Stephenson III six-bar mechanism with a torsion spring link. The mechanism is a locomotive hopping machine with three main trajectory phases: stance phase without spring activation, stance phase with spring activation, and swing phase. To set up the parameter homotopy, we used the given parameters in [34] to replicate the curve. For simplicity, we assumed the torsion spring leg had constant length. We then extracted sample points from the curve that belonged to the stance phases with and without spring activation as well as a subset of the swing phase to mimic the lift-off trajectory of the mechanism foot’s from the ground. We re-parameterized the sample points to be equally spaced, rescaled them to the unit plane, generated an interpolating function, and used numerical integration to compute the continuous form of the 47 moment parameters needed for our parameter homotopy.

![Figure 3: Example solutions of applied problem for the general case formulation that exhibit branch and circuit defective behavior. Input singularities are noted by pink dots on the coupler curve.](image-url)
The parameter homotopy tracked 303,387 paths from the generic parameters of the ab initio solve to the physically meaningful system. Using only double precision path tracking in about 26 hours of computational time, 108,008 successfully tracked to nonsingular solutions. Of these, 71 corresponded to physically meaningful designs. Due to cognate transformations, listed in Appendix B, we computed the cognates of the 71 unique solutions and filtered for duplicate designs. As there were no duplicates, this resulted in a total of $6 \times 71 = 426$ potential physically meaningful solutions.

Of the physically meaningful coupler curves, less than ten traced the design positions extremely well. However, these solutions, including many others, were not free of branch defects and therefore impossible to physically implement. Those solution designs which did display defect-free coupler curves either exhibited non-ideal behavior in tracing the specified design positions or impractical physical implementation, again rendering their designs undesirable. One can see such designs in Figs. 3 where the synthesized coupler curve branches are shown in blue and the input singularities, which mark the branch boundaries, are designated by pink dots on the coupler curve.

This example suggests that the general case formulation is not computationally effective and one would benefit from working with a smaller problem, including by designating specifications on one or both ground pivots on the four-bar final design.

4. Pre-specification of One Ground Pivot

4.1. Ab Initio Computation

Consider the approximate synthesis problem obtained by specifying the ground pivot $B = B^* = 0$ with design variables $d = \{A, A^*, l_{1s}, l_{2s}, l_{3s}, Q, Q^*\}$. Due to the designation of the one ground pivot at the origin, some moments have a zero coefficient and thus disappear within the computation. Thus, with this pre-specification, the system admits 41 moment parameters which are provided in in Table A.2 in Appendix A. Note that Prop. 1 still holds when $2N > 41$ for this reduced list of moments.

Using the RML procedure as previously described, the system reached convergence in 23 iterations to a final total solution count of 26,052 solutions as shown in Fig. 4 in about three hours. Due to the specification of one ground pivot, solutions to this system come in cognate member groups of size 2 for 13,026 distinct solution groups with formulas listed in Appendix C. When using this cognate reduction to only track one path in each group as a check on our RML procedure and resulted in the same $2 \times 13,026$ solutions that took approximately 100 minutes to compute.

![Figure 4](image-url)

(a) Ratio of repeat solutions per RML iteration

(b) Schnabel estimates with 95% confidence intervals

Figure 4: Computational summary of the ab initio solving when one ground pivot is pre-specified.
4.2. Solution Set Validation

As observed in Fig. 4b, the Schanbel estimates with a moving window of size 3 quickly tighten with the estimate after the 23rd iteration being 26,052±193 solutions. With the tight Schnabel confidence interval and multiple runs that admitted the same number of solutions, this prompted using a trace test validation.

To utilize the 2-homogeneous trace test as summarized in Sec. 2.3, one needs to perform another solve where we switched the variable-parameter pair $t_{1s}$ and $g_1$. The resulting system produced 14,792 solutions and the 2-homogeneous trace test was then successfully applied to a total of $26,052 + 14,792 = 40,844$ solutions. Thus, this trace test confirmation shows that 26,052 is indeed the precise solution count.

4.3. Applied Example

This system was applied to a gripping mechanism inspired by the solution present in [35, Fig. 5]. The original positions were chosen from [35] and fitted to a polynomial interpolation in order to define additional points for a total of $N = 20$ positions provided in [Appendix D]. To align this solution design with our pivot specification that $B = B^* = 0$, the $B$ pivot of [35] was translated to the origin, and each real design position, including the additional interpolated points, was shifted by a difference of $-0.56 - 3.17\sqrt{-1}$ for $B$ and $-0.56 + 3.17\sqrt{-1}$ for $B^*$. After computing the corresponding moments, a parameter homotopy aiming to track 26,052 solutions was utilized using adaptive precision path tracking [36]. The total computation
time for this parameter homotopy was 73 hours and resulted in computing 22,570 nonsingular solutions. Of those nonsingular solutions, 75 obeyed the physically meaningful complex-conjugate condition. Recall that in addition to these real solutions, their respective cognates are also solutions. Since the $B$ and $B^*$ pivots are fixed at the origin, the valid cognate transformations must obey this pivot specification with formulas presented in Appendix C. After collecting all of the cognates and checking for repeats, this produced a total of 94 physically meaningful solutions.

Searching through the 94 solutions, we were able to recover not only the design similar to [35, Fig. 5], but other solutions that had ideal structures for the application. One such four-bar design and its respective synthesized path is presented in Figs. 5a and 5b. A model of the prototype of the gripping mechanism using legs with the selected four-bar design to pick up a small block is shown in Fig. 5c.

5. Pre-specification of Both Ground Pivots

5.1. Ab Initio Computation

The last problem under consideration is when both ground pivots are pre-specified. Consider fixing $A = A^* = 1$ and $B = B^* = 0$ so that the resulting variable list is $d = \{l_1 s, l_2 s, l_3 s, Q, Q^*\}$. Since we retain $B, B^*$ at the origin like the pre-specification of one ground pivot formulation in Sec. 4, this system also has 41 moment parameters listed in Appendix A. For this simplified system, the RML procedure computed all 503 solutions within 13 iterations and the path tracking success rate using double precision was 99%. Using only a single processor, the total computation time was approximately three minutes. Figure 6 shows the ratio of number of repeats per iteration.

Since this system is small, it can be solved directly using standard homotopy continuation techniques without the use of RML in Bertini, a method not feasible for the other systems. This direct solve also resulted in 503 solutions from 7,362 tracked paths and took approximately ten minutes on a single processor. These 503 solutions are all distinct as the designs come in cognate groups of size one due to fixed pivots.

5.2. Solution Set Validation

With such a small system that can be solved repeatedly and consistently reported 503 solutions, a statistical validation of the RML iterations was not necessary. Moreover, this solution count can be confirmed using the 2-homogeneous trace test summarized in Sec. 2.3. By switching the variable-parameter pair $l_{1s}$ and $g_{41}$, the resulting system produced 129 solutions which can be directly computed with Bertini. The 2-homogeneous trace test was then successfully applied to a total of $503 + 129 = 632$ solutions confirming that 503 is indeed the precise solution count.

Figure 6: Ratio of repeat solutions per RML iteration when both ground pivots are pre-specified.
5.3. Applied Example

We consider the application of deployment of a wing mechanism in an Unmanned Aerial Vehicle (UAV) as shown in Fig. 7. The wing of the UAV is made up of a planar 2R chain \( OCD \) with a proximal link \( OC \) and a distal link \( CD \). The proximal link is connected to the fuselage using a rotary joint at \( O \). Note that the 2R chain is by itself a two DoF system. For the deployment this wing, three design configurations must be met, namely, a stowed configuration, an intermediate configuration, and a deployed configuration. The objective of this design challenge is to size a four-bar linkage (shown in black) with given ground pivot locations \( A \) and \( B \), respectively, such that a chosen guide point \( X \) in the distal link of the 2R chain is guided approximately along the design positions, indicated in starred points, in a constrained manner.

We used the following specifications:

\[
O = 0, \quad A = 0.01 + 1.051\sqrt{-1}, \quad A^* = 0.01 - 1.051\sqrt{-1}, \\
B = 0.137 - 0.211\sqrt{-1}, \quad B^* = 0.137 + 0.211\sqrt{-1}, \quad |OC| = 2.563, \quad |CD| = 3.4, \quad |CX| = 0.34.
\]

The design positions to be met approximately are as listed in Appendix D. Note that the design positions specified are largely restricted to be within the reachable workspace of the guide-point \( X \) defined by the annular region as shown in Fig. 7. The desired curve is expected to intersect the workspace boundary in the stowed configuration and be tangential to it in the intermediate and deployed configurations.

Starting from 503 start points found during the \textit{ab initio} run, a parameter homotopy run is carried out to the target system which represents the design problem of deployable wing mechanism. The successful paths yielded a subset of 27 physical solutions, of which 7 are local minima and 20 are saddle points. One of the local minima is found to be particularly effective in terms of packaging the system in the stowed configuration, which is shown in Fig. 7 at the three significant configurations of interest. For this plot, some minor corrections to the link dimensions of the overall system was made to ensure that the tangency conditions between the four-bar coupler curve and the workspace boundary of the wing guide-point \( X \) are met exactly at the intermediate and deployed configurations.

6. Conclusions

Approximate kinematic synthesis is an appealing technique to find optimal designs of linkages. Previous optimization frameworks and solvers face the problem of settling on inferior local minima that may or may not depend on the initial solution guess and do not present a full set of the minima. Essentially, these methods fail to identify the landscape of the optimization problem. This paper presents a polynomial objective formulation to the approximate synthesis problem that can be solved via the polynomial homotopy continuation technique of random monodromy loops. Solving for the critical points of this formulation, known as the \textit{ab initio} solve, provides a starting solution set for which parameter homotopies can be applied.
to find real designs for real positions. To produce a parameterized system where the parameter appear linearly, we employed a moment formulation. The resulting system parameterized by moments was solved for three systems: with no pre-specifications on design dimensions (general case), a pre-specification on one ground pivot, and a pre-specification on both ground pivots. In each case, the resulting solutions sets were used to compute real design application examples. The resulting total generic solution set count for the general case, pre-specification of one ground pivot, and pre-specification of both ground pivots formulations are approximately $1,820,238 \pm 3810$, and exactly 26,052 and 503 points, respectively.

Acknowledgment

This material is based upon work supported by the National Science Foundation under Grant Nos. CMMI-2041789 and CMMI-2144732. We thank Dr. James Joo and Dr. Paul Fleitz from the Air Force Research Lab for raising the third example for our consideration. We would like to thank Nicholas Deluca, Gabrielle Myers, and Katie Pala for construction of the gripper prototype.

Appendix A. Moment Parameter Equations

Each moment in $g$ depends on the exponents $(a, b)$ as in (9). For the general case with no pre-specified dimensions, the 47 moments correspond with the exponents $(a, b)$ in Table A.1. When one or both pivot locations are pre-specified, the 41 moments correspond with the exponents $(a, b)$ in Table A.2.

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Table A.1: Exponents $(a, b)$ for the 47 moments in the general case

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Table A.2: Exponents $(a, b)$ for the 41 moments when one or both ground pivots are pre-specified

Appendix B. Cognate Transformations for the General Case Formulation

For the four-bar represented in Fig. 1 with design variables $v_1 = \{A, A^*, B, B^*, l_1, l_2, l_3, Q, Q^*\}$, the two other Roberts’ cognates can be computed with the following cognate transformations:

$$v_2 = \{B, B^*, A + Q(B - A), A^* + Q^*(B^* - A^*), l_2 \sqrt{(1 - Q)(1 - Q^*)}, l_3 \sqrt{(1 - Q)} \sqrt{(1 - Q^*)}, \frac{1}{1 - Q}, \frac{1}{1 - Q^*} \},$$

$$v_3 = \{A + Q(B - A), A^* + Q^*(B^* - A^*), A, A^*, l_3 \sqrt{QQ'}, l_1 \sqrt{QQ'}, l_2 \sqrt{QQ'}, \frac{Q - 1}{Q}, \frac{Q' - 1}{Q'} \}.$$
Additionally, symmetric representation arising from relabeling of $v_1$ is

$$v' = \{B, B^*, A, A^*, l_3, l_2, l_1, 1 - Q, 1 - Q^*\}.$$

The three Roberts’ cognates and their symmetric representations yield $3 \cdot 2 = 6$ members to a group.

**Appendix C. Cognate Transformations for the Pre-specification of One Ground Pivot**

For the four-bar represented in Fig. 1, design variables $v_1 = \{A, A^*, B, B^*, l_1, l_2, l_3, Q, Q^*\}$ such that $B = B^* = 0$, the Roberts’ cognate that preserves the location of the fixed pivot can be computed as $v'_2$ via Appendix B. Thus, there are 2 members to a group.

**Appendix D. Design Positions for the Applied Examples**

Tables D.3 and D.4 list the design positions for the applied examples in Section 4 and 5 with one ground pivot and both ground pivots pre-specified, respectively. In these tables, $x$ is the real part and $y$ is the imaginary part of the design positions. Note the points’ respective complex conjugates are similar except for an opposite sign on the imaginary component.

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Table D.3: Design positions of the applied example for the pre-specification of one ground pivot formulation.

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Table D.4: Design positions of the applied example for the pre-specification of both ground pivots formulation

**References**


