Smooth Points on Semi-algebraic Sets

Katherine Harris[a], Jonathan D. Hauenstein[b], Agnes Szanto[c]

[a]*Department of Mathematics and Computer Science, Beloit College, 700 College Avenue, Beloit, Wisconsin, 53511, USA.*
[b]*Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, 102G Crowley Hall, Notre Dame, Indiana, 46556, USA.*
[c]*Department of Mathematics, North Carolina State University, Campus Box 8205, Raleigh, North Carolina, 27695, USA.*

Abstract

Many algorithms for determining properties of real algebraic or semi-algebraic sets rely upon the ability to compute smooth points. In this paper, we present a procedure based on computing the critical points of some well-chosen function that guarantees the computation of smooth points in each bounded connected component of a (real) atomic semi-algebraic set. Our technique is intuitive in principal, performs well on previously difficult examples, and is straightforward to implement using existing numerical algebraic geometry software. The practical efficiency of our approach is demonstrated by solving a conjecture on the number of equilibria of the Kuramoto model for the $n = 4$ case. We also apply our method to design an algorithm to compute the real dimension of algebraic sets, the original motivation for this research. We compare the efficiency of our method to existing methods to compute the real dimension on a benchmark family.

Key words: computational real algebraic geometry, real smooth points, real dimension, polar varieties, numerical algebraic geometry, Kuramoto model

1. Introduction

When studying an atomic semi-algebraic set $S \subset \mathbb{R}^n$, i.e.,

$$S = \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_s(x) = 0,\ q_1(x) > 0, \ldots, q_m(x) > 0\} \tag{1}$$

for some polynomials $f_1, \ldots, f_s, q_1, \ldots, q_m \in \mathbb{R}[x_1, \ldots, x_n]$, one often first considers the complex variety $V = \{x \in \mathbb{C}^n : f_1(x) = \cdots = f_s(x) = 0\}$, known as its *algebraic relaxation*, and deduces properties of S from the properties of V. In particular, if S contains a smooth point and V is irreducible, then S is Zariski dense in V, and so all of the algebraic information of S can be determined from V. Thus, deciding the existence of smooth points in semi-algebraic sets and finding such points is a central problem in real algebraic geometry with many applications. An example of such application is computing all typical ranks of tensors Bernardi et al. (2018); Kruskal (1989); Friedland (2012).

Email addresses: harriske@beloit.edu (Katherine Harris), hauenstein@nd.edu (Jonathan D. Hauenstein), aszanto@ncsu.edu (Agnes Szanto)

One of the main results of this paper is to give a new technique to compute smooth points on bounded connected components of atomic semi-algebraic sets. When V is equidimensional, our method is simple and suggests a natural implementation using numerical homotopy methods. It extends other approaches that compute sample points on real semi-algebraic sets, such as computing the critical points of the distance function, in the sense that our method also guarantees the smoothness of the sample points. We illustrate this advantage on "Thom's lips," in which critical points of the distance function are often at the singularities (Wu and Reid, 2013, Ex. 2.3), while our method always computes smooth points.

The main idea is straightforward when V is irreducible. If a polynomial g vanishes on the singular points of V, but does not vanish on all of V, then the set of extreme points of g on $V \cap \mathbb{R}^n$ must contain a smooth point on every bounded connected component of $V \cap \mathbb{R}^n$, if such points exist. We extend this idea beyond the case when V is irreducible and $S = V \cap \mathbb{R}^n$ to the general case. We handle the case when V is reducible and its irreducible components have different dimensions using *infinitesimal deformations* of V and limits. We show that this limiting approach is well-suited for numerical homotopy continuation methods after we translate an infinitesimal real deformation (that may only work for arbitrary small values) into a complex deformation that works along a real arc parameterized by the interval $(0, 1]$. Finally, we present a novel technique to compute the required g polynomial using *deflations*, and compare its degree bounds to traditional symbolic approaches (see Proposition 9.2). In fact, Corollary 11.5 proves that our Real Smooth Point Algorithm performs well if the depth of the deflations (i.e., the number of iterations) is small.

To demonstrate the practical efficiency of our new approach, we present the solution of a conjecture for the first time: counting the equilibria of the Kuramoto model in the $n = 4$ case given in Xin et al. (2016) (see Kuramoto (1975) for the original model and Coss et al. (2018) for a detailed historical overview and additional references).

We also apply our method to compute the real dimension of a semi-algebraic set. The difficulty of this problem, compared to its complex counterpart, is that in many cases the real part of a semi-algebraic set lies within the singular set of the complex variety containing it so that its real dimension is smaller than the complex one. In terms of worst case complexity bounds of the existing algorithms in the literature, it is an open problem if the real dimension can be computed within the same asymptotic complexity bounds as the complex dimension. The original motivation for this research was to try to find an algorithm for the real dimension that has worst case complexity comparable to its complex counterpart. Even though this paper is presented using computational tools from *numerical algebraic geometry* (c.f., Sommese and Wampler (2005); Bates et al. (2013)), all procedures can be translated to symbolic methods for polynomials with rational coefficients. In fact, after performing a worst case complexity estimate for a symbolic version, we unfortunately found that it does not improve the existing complexity bounds in the worst case (see Lairez and Safey El Din (2021) and the references therein). This is one of the reasons we present our results in a numerical algebraic geometry setting and give evidence of the efficiency with implementation on a benchmark family. As mentioned above, in Proposition 9.2 and Corollary 11.5, we give bounds on the degrees of the polynomials appearing in our algorithms and the number of homotopy paths they follow, highlighting the advantages and disadvantages of our approach compared to other purely symbolic techniques.

1.1. Related Work

There are many approaches in the literature to compute at least one real point on every connected component of a semi-algebraic set. Methods using projections to obtain a cell decomposition based on sign conditions go back to Collins' Cylindrical Algebraic Decomposition (CAD) algorithm described in Collins (1975). Improved symbolic methods using critical points or generalized critical points of functions along with infinitesimals and randomization can be found in Rouillier et al. (2000); Aubry et al. (2002); Safey El Din (2007); Faugère et al. (2008). The current state of the art deterministic symbolic algorithm is given in (Basu et al., 2006a, Alg. 13.3) which computes sample points on each connected component of all realizable sign conditions of a polynomial system and gives a complexity analysis. The most recent application of this technique is in Safey El Din et al. (2018, 2019) where the authors compute smooth points on real algebraic sets in order to compute the real radical of polynomial systems and analyze complexity.

Another line of work has been developed in parallel focuses on computing critical points while utilizing the tool of *polar varieties*, introduced and developed in Bank et al. (1997); Safey El Din and Schost (2003); Bank et al. (2004, 2009, 2010, 2015); Safey El Din and Spaenlehauer (2016). Alternatively, a homotopy-based approach computing the critical points of the distance function from a generic point or a line is presented in Hauenstein (2013); Wu and Reid (2013). It is important to note, however, that all of these methods only guarantee the finding at least one real point on every connected component of a semi-algebraic set, rather than real *smooth* points.

The real dimension problem has similarly been widely studied and the current state of the art deterministic algorithm is given by (Basu et al., 2006a, Alg. 14.10) computing all realizable sign conditions of a polynomial system. This approach improves on previous work in Vorobjov (1999) to obtain a complexity result with a better dependence on the number of polynomials in the input by utilizing a block elimination technique first proposed in Grigor′ev and Vorobjov (1988). More recent work has been presented giving probabilistic algorithms utilizing polar varieties which improve on complexity bounds even further in Safey El Din and Tsigaridas (2013); Bannwarth and Safey El Din (2015). Very recently, Lairez and Safey El Din (2021) gave a symbolic algorithm with improved complexity to compute the dimension of real algebraic sets using "level sets" and use them to reduce the problem to one of lower dimension. In the present paper, we use a benchmark family that appeared in Lairez and Safey El Din (2021) to demonstrate the efficiency of our method.

One can also compute the real dimension by computing the real radical of a semi-algebraic set, first studied in Becker and Neuhaus (1993) with improvements and implementations in Neuhaus (1998); Zeng (1999); Spang (2008); Chen et al. (2013). The most recent implementation can be found in Safey El Din et al. (2018, 2019) as mentioned above. Their approach is shown to be efficient in the case when the algebraic set is smooth, but the iterative computation of singularities can increase the complexity significantly in the worst case. An alternative method using semidefinite programming techniques was proposed by Wang (2016); Ma et al. (2016).

2. Preliminaries

2.1. Basic Definitions

The following collects some basic notions used throughout the paper, including atomic semi-algebraic sets, semi-algebraic sets, and real algebraic sets.

A set $S \subset \mathbb{R}^n$ is an *atomic semi-algebraic set* if it is of the form of (1). A set $T \subset \mathbb{R}^n$ is a *semi-algebraic set* if it is a finite union of atomic semi-algebraic sets. A set $U \subset \mathbb{R}^n$ is a *real algebraic set* if it is defined by polynomial equations only.

Smoothness on atomic semi-algebraic sets is described next.

Definition 2.1. Let $S \subset \mathbb{R}^n$ be an atomic semi-algebraic set as in (1). A point $z \in S$ is *smooth* (or nonsingular) in S if z is smooth in the algebraic set

$$V(f_1, \ldots, f_s) = \{x \in \mathbb{C}^n \ : \ f_1(x) = \cdots = f_s(x) = 0\},$$

i.e., if there exists a unique irreducible component $V \subset V(f_1, \ldots, f_s)$ containing z such that

$$\dim T_z(V) = \dim V$$

where $T_z(V)$ is the tangent space of V at z. We denote by $\mathrm{Sing}(S)$ the set of singular (or non-smooth) points in S.

An algebraic set $V \subset \mathbb{C}^n$ is *equidimensional* of dimension d if every irreducible component of V has dimension d. The following defines the (local) real dimension of semi-algebraic sets from (Basu et al., 2006a, §5.3).

Definition 2.2. For a semi-algebraic set $S \subset \mathbb{R}^n$, its *real dimension* $\dim_{\mathbb{R}} S$ is the largest k such that there exists an injective semi-algebraic map from $(0,1)^k$ to S. Here, a map $\varphi : (0,1)^k \to S$ is semi-algebraic if the graph of φ in $\mathbb{R}^{n+k}$ is semi-algebraic. By convention, the dimension (real or complex) of the empty set is -1.

Definition 2.3. Consider a point $z \in S \subset \mathbb{R}^n$, where S is a semi-algebraic set. The *local real dimension* of S at z is the maximal real dimension of the closure of every connected component C_j of S such that $z \in \overline{C_j}$.

The main ingredient in our results is the following theorem that was proven in (Marshall, 2008, Theorem 12.6.1).

Theorem 2.4. *Let $V \subset \mathbb{C}^n$ be an irreducible algebraic set. Then*

$$\dim_{\mathbb{R}} (V \cap \mathbb{R}^n) = \dim_{\mathbb{C}} V$$

if and only if there exists $z \in V \cap \mathbb{R}^n$ that is smooth.

2.2. Semi-algebraic to Algebraic

In this subsection, we show that our problem on atomic semi-algebraic sets can be reformulated as a problem on real algebraic sets. This will allow us to use homotopy continuation methods for solving polynomial equations.

The following shows that smooth points on each connected component of an atomic semi-algebraic set S can be obtained as projections of smooth points of some real algebraic set.

Proposition 2.5. *Let S be an atomic semi-algebraic set as in* (1) *and*

$$W := \left\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m \ : \ f_1(x) = \cdots = f_s(x) = 0,\ z_1^2 q_1(x) - 1 = \cdots = z_m^2 q_m(x) - 1 = 0\right\}.$$

If $y \in W$ is smooth, then $\pi_x(y) \in S$ is also smooth. Conversely, if $x \in S$ is smooth, then (x,z) is smooth in W for all $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ such that $(x,z) \in W$.

Proof. Without loss of generality, we can assume that $f_1, \ldots, f_s$ generate a prime ideal. The Jacobian matrix of the polynomial system defining W has the block structure

$$J(x,z) = \begin{array}{|c|c|} \hline Jf(x) & 0 \\ \hline * & \mathrm{diag}(2z_i q_i(x)) \\ \hline \end{array}$$

Since for $(x,z) \in W$ we have $z_i g_i(x) \neq 0$, the Jacobian matrix $Jf(x)$ has full column rank if and only if $J(x,z)$ has full column rank, which proves the claim. $\square$

Therefore, for the rest of the paper, we assume that we are given a *real algebraic set* and the goal is to compute smooth points on each connected component.

Remark 2.6. Applying Proposition 2.5 to reduce to the case of algebraic sets may not be the only possibility. We give this procedure for simplicity of presentation and implementation, but it is possible that another symbolically equivalent reduction method would produce a more efficient numerical algorithm. While a rigorous analysis of this falls outside the scope of this paper, we suggest the reader explores this more if they are studying an example or application that requires more computational efficiency.

2.3. Boundedness

The next reduction is to replace an arbitrary real algebraic set with a compact one.

Proposition 2.7. *Let $f_1, \ldots, f_{s-1} \in \mathbb{R}[x_1, \ldots, x_{n-1}]$ and consider $p = (p_1, \ldots, p_{n-1}) \in \mathbb{R}^{n-1}$. Let $\delta \in \mathbb{R}_+$, introduce a new variable x_n, and consider*

$$f_s := (x_1 - p_1)^2 + \cdots + (x_{n-1} - p_{n-1})^2 + x_n^2 - \delta.$$

Then, $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ is bounded and

$$\pi_{n-1}\left(V(f_1, \ldots, f_s) \cap \mathbb{R}^n\right) = V(f_1, \ldots, f_{s-1}) \cap \left\{z \in \mathbb{R}^{n-1} \ : \ \|z - p\|^2 \leq \delta\right\}$$

where $\pi_{n-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$.

Remark 2.8. The definition of f_s above is based on a standard trick used in real algebraic geometry to make an arbitrary real algebraic set bounded (e.g., see Basu et al. (2006b)). In general, $V \cap \mathbb{R}^{n-1}$ is embedded into a sphere in $\mathbb{R}^n$ around the origin of radius $1/\zeta$ where ζ is infinitesimal. Since numerical homotopy continuation methods are incompatible with infinitesimal variables, in this paper we are only interested in computing points with bounded coordinates, so it is sufficient to embed its intersection with a closed ball around p of radius $\sqrt{\delta}$ for some fixed $\delta \in \mathbb{R}_+$. In particular, we will not use infinitesimal variables in our algorithms.

Later in the paper, when we assume that $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ is bounded, we assume that we have applied Proposition 2.7 if necessary.

2.4. Genericity Assumptions

The algorithms described in this paper make assumptions that certain points, matrices, or linear polynomials are generically chosen from a vector space (over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). In all of these cases, there exists a proper Zariski closed subset of the corresponding vector space such that all choices outside this set yield correct results. Therefore, a generic choice means it is outside of this proper Zariski closed subset. For algorithms which depend on generic choices, we follow the convention from the literature that they compute the correct solutions with *algebraic probability one* (Sommese and Wampler, 2005, Chap. 4). Effective probability bounds can be obtained from bounds on the degrees of the proper Zariski closed sets containing the "bad" choices. See (Krick et al., 2001, Prop. 4.5) and Elliott and Schost (2019) for such bounds for linear changes of variables for Noetherian position and transversality, respectively.

2.5. Witness Sets

In this subsection, we discuss some main ideas from numerical algebraic geometry following Bates et al. (2013). In particular, we consider positive-dimensional algebraic sets and utilize a data structure for them that allows computation using classical homotopy continuation methods for square non-singular systems. The key is the notion of *witness sets*, which will rely on the idea of slicing an algebraic set with a generic linear space.

Definition 2.9. If an algebraic set $V \subset \mathbb{C}^n$ is equidimensional with $\dim(V) = k$, a *witness set* for V is the triple (F, L, W) such that

- $F \subset \mathbb{C}[x]$ is a *witness system* for V, i.e., each irreducible component of V is an irreducible component of $V(F)$,
- $L \subset \mathbb{C}[x]$ is a *linear system* where $V(L)$ is a linear space of codimension k that intersects V transversely, and
- $W \subset \mathbb{C}^n$ is a *witness point set* which is equal to $V \cap V(L)$.

We note that the number of points in the witness point set W in the above definition will determine the number of paths we need to track with homotopy continuation methods, directly impacting the complexity of the numerical algebraic geometry computations. Note also that this number is an invariant of V called its *degree*.

Although the witness point set provides some information, a witness system is needed to perform any additional computations, such as deciding whether a given point lies on the algebraic set V. Membership Test Algorithm 1 follows the approach of (Bates et al., 2013, Section 8.4) in order to do this and is correct with algebraic probability one.

Algorithm 1 MembershipTest

Input: $p \in \mathbb{C}^n$ and (F, L, W) a witness set for some equidimensional algebraic set $V \subset \mathbb{C}^n$.

Output: TRUE if $p \in V$ and FALSE if $p \notin V$.

1. Choose generic linear polynomials L' with $p \in V(L')$ and $\dim(V(L')) = \dim(V(L))$.
2. $H(x, t) := [F(x), tL(x) + (1 - t)L'(x)]$.
3. Track the finitely many homotopy paths of $H(x, t)$ starting from the witness point set W for $t = 1$, obtaining the witness point set $W' := V \cap V(L')$ at $t = 0$.
4. If $p \in W'$, return TRUE. Else, return FALSE.

2.6. *Isosingular Deflation*

As we mentioned in the Introduction, one of the ingredients of our algorithm for computing smooth points on a real algebraic set $V \cap \mathbb{R}^n$ is a polynomial g that vanishes on the singular points of V, but does not vanish identically on the irreducible components of V. We give an algorithm to compute such a g in Section 9 using *isosingular deflation*. This subsection summarizes the basic definitions and results that we use in Sections 8 and 9. Further details on isosingular deflation can be found in Hauenstein and Wampler (2013).

Definition 2.10. Let $f_1, \ldots, f_s \in \mathbb{C}[x], F_0 = \{f_1, \ldots, f_s\}$, and $z \in V(F_0) \subset \mathbb{C}^n$. The *isosingular deflation operator* $\mathcal{D}$ is defined via

$$(F_1, z) := \mathcal{D}(F_0, z)$$

where $F_1 \subset \mathbb{C}[x]$ consists of F_0 and all $(r+1) \times (r+1)$ minors of the Jacobian matrix JF_0 for F_0 where $r = \operatorname{rank} JF_0(z)$. Thus, $z \in V(F_1)$, meaning that we can iterate this operator to construct a sequence of systems $F_j \subset \mathbb{C}[x]$ with $(F_j, z) = \mathcal{D}(F_{j-1}, z) = \mathcal{D}^j(F_0, z)$ for $j \geq 1$.

We say that $F \subset \mathbb{C}[x]$ is the *isosingular deflation* of F_0 at z if there exists a minimal $j \geq 0$ such that $(F, z) = \mathcal{D}^j(F_0, z)$ and $\dim \operatorname{NullSpace}(JF(z)) = \dim_F(z)$, where $\dim_F(z)$ is the maximal dimension of the irreducible components of $V(F)$ containing z (called the *local dimension* of z with respect to F).

To compute the isosingular deflation of F_0 at z we refer to (Hauenstein and Wampler, 2013, Algorithm 6.3).

Using the deflation operator, we can now formally define the isosingular sets and singular points of our algebraic set in this context.

Definition 2.11. Let $f_1, \ldots, f_s \in \mathbb{C}[x], F_0 = \{f_1, \ldots, f_s\}$, and $z \in V(F_0) \subset \mathbb{C}^n$. Let $\mathcal{D}$ be the isosingular deflation operator defined in Definition 2.10. We define

- The *deflation sequence* of F_0 at z is $\{d_k(F_0, z)\}_{k=0}^{\infty}$ where

  $$d_k(F_0, z) = \operatorname{dnull}(F_k, z) := \dim \operatorname{NullSpace} JF_k(z)$$

  with JF_k the Jacobian matrix of F_k with $(F_k, z) = \mathcal{D}^k(F_0, z)$.

- Let $V \subset V(F_0)$ be a non-empty irreducible algebraic set. Then V is an *isosingular set* of F_0 if there exists a sequence $\{c_k\}_{k=1}^{\infty}$ such that V is an irreducible component of

  $$\overline{\{z \in V(F_0) \ : \ d_k(F_0, z) = c_k, k \in \mathbb{N}\}}.$$

- Let $V \subset V(F_0)$ be a non-empty irreducible algebraic set. Then $\operatorname{Iso}_{F_0}(V)$ is the unique isosingular set with respect to F_0 containing V such that $\operatorname{Iso}_{F_0}(V)$ and V have the same deflation sequence with respect to F_0.

- Let V be an isosingular set for F_0. The set of *singular points of* V with respect to F_0 is

  $$\operatorname{Sing}_{F_0}(V) = \left\{z \in V \ : \ \{d_k(F_0, z)\}_{k=0}^{\infty} \neq \{d_k(F_0, V)\}_{k=0}^{\infty}\right\}.$$

  Here, $d_k(F_0, V)$ is meant for a generic point in V.

- The *local dimension* of z with respect to F_0, denoted by $\dim_{F_0}(z)$, is the maximal dimension of the irreducible components of $V(F_0)$ containing z.

We next detail some particular results on isosingular sets which will be important to our methods going forward. The following theorem states that the singular points of an algebraic set are preserved under isosingular deflation. We use this result in the proof of Theorem 9.1.

Theorem 2.12. *(Hauenstein and Wampler, 2013, Theorem 5.9) Let V be an isosingular set for F_0 as in Definition 2.11. Then if $z \in V$ and $z \in \mathrm{Sing}(V(F_0))$ then $z \in \mathrm{Sing}_{F_0}(V)$.*

Finally, we have the following theorem which gives an isosingular deflation approach for constructing witness sets of the intersection of a known witness set with another algebraic set. We use this result in the proof of Theorem 8.4.

Theorem 2.13. *(Hauenstein and Wampler, 2017, Theorem 6.2) Given $g_1, \ldots, g_r \in \mathbb{C}[x]$, let Z be a union of irreducible components of $V(g_1, \ldots, g_r)$. Suppose $f_1, \ldots, f_s \in \mathbb{C}[y]$, $F(x,y) = \{g_1(x), \ldots, g_r(x), f_1(y), \ldots, f_s(y)\}, \Delta = \{(x,x) : x \in \mathbb{C}^n\}$, and $\pi(x,y) = x$. If $A \subset Z \cap V(f_1, \ldots, f_s)$ is an irreducible component, then there exists a nonempty Zariski open set $U \subset A$ such that for all $p \in U$, A is an irreducible component of $\pi\Big(\mathrm{Iso}_F((p,p)) \cap \Delta\Big)$.*

We provide the following illustrative example for the theorem.

Example 2.14. Let $g(x,y,z) := (x+y+z)y$ defining a witness system for $Z := V(x+y+z)$. Let $f(x,y,z) := y$ and note that $Z \cap V(f) = V(x+z, y)$ is irreducible. We construct

$$F_0(x,y,z,x',y',z') = [g(x,y,z) = (x+y+z)y, f(x',y',z') = y'].$$

Choose a generic witness point $p = (a, 0, -a) \in A = Z \cap V(f)$ for some fixed $a \in \mathbb{C}$. Then we compute the deflation sequence of F_0 at $(a, 0, -a, a, 0, -a)$ as $5, 3, 3, \ldots$ such that

$$\mathrm{Iso}_{F_0}((p,p)) = \{(b, 0, -b, c, 0, d) : b, c, d \in \mathbb{C}\}.$$

The polynomial system defining this 3-dimensional isosingular set is given by adding the 2×2 minors of the Jacobian of F_0 to F_0, giving

$$F_1(x,y,z,x',y',z') = \begin{bmatrix} (x+y+z)y \\ y' \\ y \\ x+2y+z \\ y \end{bmatrix}.$$

By Theorem 2.13, $Z \cap V(f)$ is an irreducible component of $V(F_1(x,y,z,x,y,z))$ and $G(x,y,z) := F_1(x,y,z,x,y,z)$ suffices as a witness system for $Z \cap V(f)$. We note that in this example, removing the redundancies in G would in fact show that $[x+z, y]$ is sufficient as a witness system for $Z \cap V(f)$.

3. Computation of Real Smooth Points – Equidimensional Case

This section contains our main results for the special case when the complex algebraic variety is equidimensional. In subsequent secions we consider the general case, where we use deformations and limits of algebraic sets.

Theorem 3.1. *Let $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ and assume that $V := V(f_1, \ldots, f_s) \subset \mathbb{C}^n$ is equidimensional of dimension $n - s$. Suppose that $g \in \mathbb{R}[x_1, \ldots, x_n]$ satisfies the following conditions:*

1. $\mathrm{Sing}(V) \cap \mathbb{R}^n \subset V(g)$*;*
2. $\dim (V \cap V(g)) < n - s$*.*

Then the set of points where g restricted to $V \cap \mathbb{R}^n$ attains its extreme values intersects each bounded connected component of $(V \setminus \mathrm{Sing}(V)) \cap \mathbb{R}^n$.

The proof of this theorem is based on the following lemma.

Lemma 3.2. *Let V be as in Theorem 3.1. Let $g \in \mathbb{R}[x_1, \ldots, x_n]$ such that* $\dim (V \cap V(g)) < n - s$*. Then, either $(V \setminus V(g)) \cap \mathbb{R}^n = \emptyset$ or g restricted to $V \cap \mathbb{R}^n$ attains a non-zero extreme value on each bounded connected component of $(V \setminus V(g)) \cap \mathbb{R}^n$.*

Proof. Assume that $(V \setminus V(g)) \cap \mathbb{R}^n \neq \emptyset$ and let C be a bounded connected component of the set $(V \setminus V(g)) \cap \mathbb{R}^n$. Since $C \not\subset V(g)$, there exists $x \in C$ with $g(x) \neq 0$. Let $\overline{C}$ be the Euclidean closure of C so that $\overline{C} \subset V \cap \mathbb{R}^n$ is closed and bounded, and g vanishes identically on $\overline{C} \setminus C$. By the extreme value theorem, g attains both a minimum and a maximum on $\overline{C}$. Since g is not identically zero on $\overline{C}$, either the minimum or the maximum value of g on $\overline{C}$ must be nonzero, so g attains a non-zero extreme value on C. □

Proof of Theorem 3.1. Assume that $(V \setminus \mathrm{Sing}(V)) \cap \mathbb{R}^n \neq \emptyset$. By Theorem 2.4, $\dim_{\mathbb{R}} V \cap \mathbb{R}^n = n - s$. By (2), $(V \setminus V(g)) \cap \mathbb{R}^n \neq \emptyset$. By (1), $(V \setminus V(g)) \cap \mathbb{R}^n \subset (V \setminus \mathrm{Sing}(V)) \cap \mathbb{R}^n$, so the bounded connected components of $(V \setminus V(g)) \cap \mathbb{R}^n$ are subsets of the bounded connected components of $(V \setminus \mathrm{Sing}(V)) \cap \mathbb{R}^n$. By Lemma 3.2, g restricted to $V \cap \mathbb{R}^n$ attains a non-zero extreme value on each bounded connected component of $(V \setminus V(g)) \cap \mathbb{R}^n$, thus yielding a point in every bounded connected component of $(V \setminus \mathrm{Sing}(V)) \cap \mathbb{R}^n$. □

Algorithm 5 in Section 11 computes real smooth points when $V(f_1, \ldots, f_s)$ is not equidimensional by using deformations and limits. However, the same algorithm can be used in the equidimensional case with input $f_1, \ldots, f_s$ and $\mathbf{a} = 0 \in \mathbb{R}^s$, i.e without deformation.

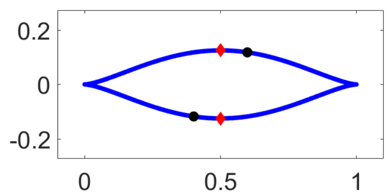


Figure 1: "Thom's lips"

Example 3.3. An example of a real curve with two singular cusps called "Thom's lips" defined by $f = y^2 - (x(1-x))^3$ is shown in Figure 1. An obvious choice of g which satisfies the conditions of Theorem 3.1 is $g = x(1 - x)$. Using Lagrange multipliers to optimize with respect to g results in two points $(0.5, \pm 0.125)$ plotted as red diamonds. Alternatively, the polynomial g can be constructed algorithmically (see Section 9) yielding, e.g., $g = 3(2x - 1)(x(1 - x))^2 + 2y$ which produces two points plotted as black circles, approximately $(0.5987, 0.1178)$ and $(0.4013, -0.1178)$.

Both yield a real smooth point on each of the two connected components of $(V \setminus \text{Sing}(V)) \cap \mathbb{R}^n$. We note that the first choice of g demonstrates that when Sing(V) is 0-dimensional, defining g as a product of a coordinate of these points is a simple way to satisfy the conditions of Theorem 3.1. The second choice of g demonstrates the general method described in Section 9 which works in every dimension.

4. Application to Kuramoto model

The Kuramoto model from Kuramoto (1975) is a dynamical system used to model synchronization amongst n coupled oscillators. The maximum number of equilibria (i.e. real solutions to steady-state equations) for $n \geq 4$ remains an open problem with details discussed in Coss et al. (2018). The following confirms the conjecture in Xin et al. (2016) for $n = 4$.

Theorem 4.1. *The maximum number of equilibria for the Kuramoto model with $n = 4$ oscillators is* 10.

The steady-state equations for the $n = 4$ Kuramoto model are

$$f_i(\theta; \omega) = \omega_i - \frac{1}{4} \sum_{j=1}^{4} \sin(\theta_i - \theta_j) = 0, \text{ for } i = 1, \ldots, 4$$

parameterized by the natural frequencies $\omega_i \in \mathbb{R}$. Since only the angle differences matter, one can assume $\theta_4 = 0$ and observe a necessary condition for equilibria is

$$0 = f_1 + f_2 + f_3 + f_4 = \omega_1 + \omega_2 + \omega_3 + \omega_4,$$

i.e., assume $\omega_4 = -(\omega_1 + \omega_2 + \omega_3)$. Substituting $s_i = \sin(\theta_i)$ and $c_i = \cos(\theta_i)$ yields

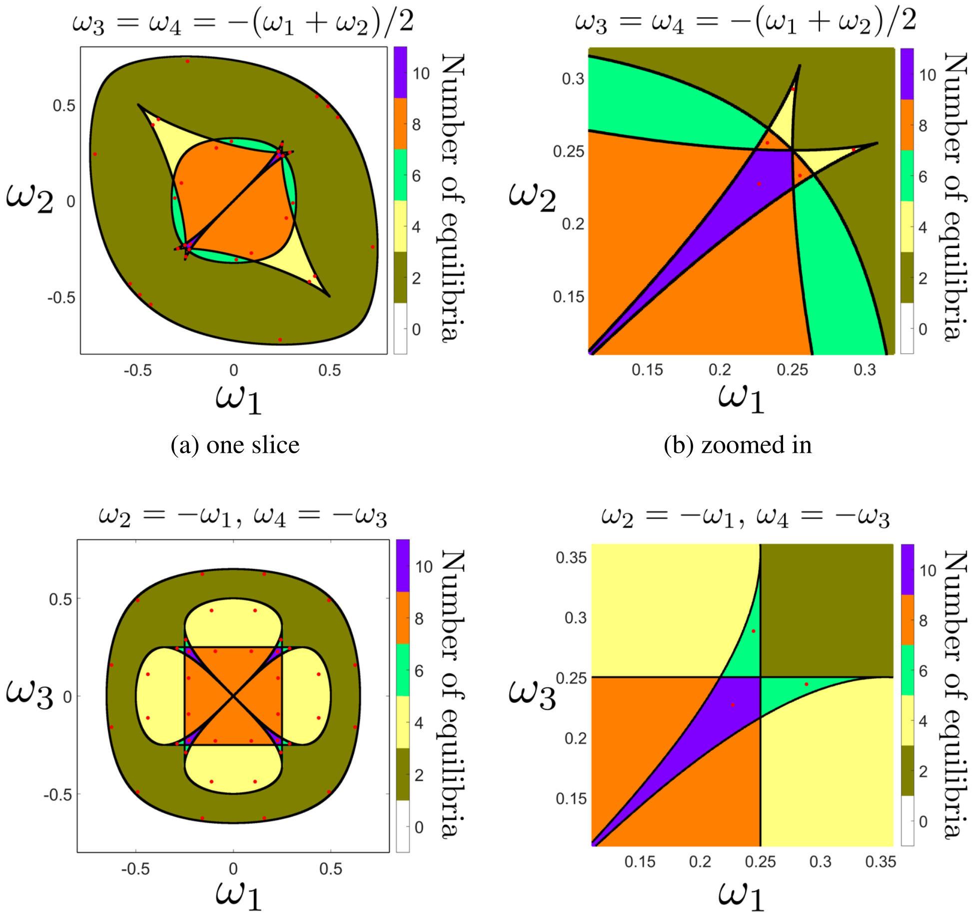
$$F(s, c; \omega) = \left\{\omega_i - \frac{1}{4} \sum_{j=1}^{4} (s_i c_j - s_j c_i), s_i^2 + c_i^2 - 1, \text{ for } i = 1, 2, 3\right\}$$

which is a polynomial system with variables $s = (s_1, s_2, s_3)$ and $c = (c_1, c_2, c_3)$, parameters $\omega = (\omega_1, \omega_2, \omega_3)$, and constants $s_4 = 0$ and $c_4 = 1$.

The goal is to compute the maximum number of isolated real solutions of $F = 0$ as ω varies over $\mathbb{R}^3$. Let $D(\omega)$ be the discriminant polynomial of the system F, a polynomial in ω of degree 48. The number of real solutions of F is constant in each connected component of $\mathbb{R}^3 \setminus V(D)$. Since it is easy to see that there can be no real solutions if $|\omega_i| \geq \frac{n-1}{n} = 0.75$, we need to compute at least one interior point in each of the bounded connected components of $\mathbb{R}^3 \setminus V(D)$. Applying Lemma 3.2 with $f = 0$ and $g = D$, i.e., by computing the real solutions of $\nabla D = 0$ and $D \neq 0$, accomplishes this task. Exploiting symmetry and utilizing `Bertini` (Bates et al.), `alphaCertified` (Hauenstein and Sottile (2012)), and `Macaulay2` (Grayson and Stillman) all solutions have been found and certified. In fact, this computation showed that all real critical points of D arose, up to symmetry, along two slices shown in Figure 2. A similar computation then counted the number of real solutions to $F = 0$ showing that the maximum number of equilibria is 10. All code used in these computations is available at `dx.doi.org/10.7274/r0-5c1t-jw53`.

5. Perturbations and Limits of Real Hypersurfaces

In this section, we begin to construct the tools for an algorithm using the same ideas as in Section 3, but in full generality so the results will hold when $V(f_1, \ldots, f_s)$ is not equidimensional, i.e. there are some components of dimension greater than $n - s$. To do this, a standard



(a) one slice (b) zoomed in

(c) other slice (d) zoomed in

Figure 2: Compact connected regions and critical points for the Kuramoto model with $n = 4$

approach (which can be seen in Safey El Din and Tsigaridas (2013) and Bannwarth and Safey El Din (2015)) is to perturb the defining polynomials of the algebraic set by infinitesimals to obtain an equidimensional variety, which has a limit as the infinitesimals approach zero that is equidimensional and contains the $n - s$ dimensional components of $V(f_1, \ldots, f_s)$.

First, we need the definition of Puiseux series.

Definition 5.1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and denote by $\mathbb{K}\langle\varepsilon\rangle$ the *field of Puiseux series* over $\mathbb{K}$, i.e.

$$\mathbb{K}\langle\varepsilon\rangle := \left\{ \sum_{i \geq i_0} a_i \varepsilon^{i/q} \ : \ i_0 \in \mathbb{Z}, q \in \mathbb{Z}_{>0}, a_i \in \mathbb{K} \right\}.$$

A Puiseux series $z = \sum_{i \geq i_0} a_i \varepsilon^{i/q} \in \mathbb{K}\langle\varepsilon\rangle$ is called *bounded* if $i_0 \geq 0$.

We note that the field of real Puiseux series $\mathbb{R}\langle\varepsilon\rangle$ is a real closed field, thus the intermediate value theorem holds (c.f. (Basu et al., 2006a, Theorem 2.11)).

We also note that for complex Puiseux series $\mathbb{C}\langle\varepsilon\rangle$, similar results hold when we replace real closed with algebraically closed (c.f. (Basu et al., 2006b, Theorem 2.92)).

We establish a notation for the concept of an *extension* of a semi-algebraic set $S \subset \mathbb{R}^n$ to $\mathbb{R}\langle\varepsilon\rangle$.

Definition 5.2. Given a semi-algebraic set $S \subset \mathbb{R}^n$, the *extension* of S to $\mathbb{R}\langle\varepsilon\rangle^n$, denoted $\mathrm{Ext}(S)$, is the semi-algebraic subset $S' \subset \mathbb{R}\langle\varepsilon\rangle^n$ defined by the same equations and inequalities as S, but considering their solutions in $\mathbb{R}\langle\varepsilon\rangle^n$. For a polynomial map $\varphi : S \to S'$ where $S \subset R^n$, $S' \subset \mathbb{R}^m$ semialgebraic sets, $\mathrm{Ext}(\varphi)$ denotes the map $\varphi' : \mathrm{Ext}(S) \to \mathrm{Ext}(S')$ defined by the same polynomials as ϕ.

Assume $f_1, \ldots, f_s \in \mathbb{R}[x]$ and $\varepsilon > 0$ a real infinitesimal. Let $F = f_1^2 + \cdots + f_s^2$ and note that $V(f_1, \ldots, f_s) \cap \mathbb{R}^n = V(F) \cap \mathbb{R}^n$. Consider $V_\varepsilon := V(F - \varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle$ as the perturbed version of the algebraic set $V(F)$. One reason for us to perturb our algebraic set is to obtain smoothness.

Lemma 5.3. *(Rouillier et al., 2000, Lemma 3.5) V_ε is a smooth hypersurface.*

The following result (Basu et al., 2006a, Proposition 12.36) states that semi-algebraicity is preserved as ε limits to 0.

Lemma 5.4. *Let $S \subset \mathbb{R}\langle\varepsilon\rangle^n$ be a semi-algebraic set. Then $\lim_{\varepsilon\to 0}(S)$ is a closed semi-algebraic set. Furthermore, if S is bounded and connected, then $\lim_{\varepsilon\to 0}(S)$ is connected.*

The next proposition on the limits of perturbed connected components of a real algebraic set appeared in the unpublished work Safey El Din and Tsigaridas (2013). We restate and prove it here for completeness.

Proposition 5.5. *Assume $f_1, \ldots, f_s \in \mathbb{R}[x]$, $F = f_1^2 + \cdots f_s^2$, and $\varepsilon > 0$ a real infinitesimal. Let $V_\varepsilon := V(F - \varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle^n$ and $V := \lim_{\varepsilon\to 0} V_\varepsilon \subset \mathbb{C}^n$. Suppose C is a connected component of $V \cap \mathbb{R}^n$. Then:*

1. *there exist connected components $C_{\varepsilon,1}, \ldots, C_{\varepsilon,l}$ of $V_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^n$ such that*
$$C = \bigcup_{i=1}^{l} \lim_{\varepsilon\to 0} C_{\varepsilon,i}$$
2. *if C is bounded by some open ball $B \subset \mathbb{R}^n$ and does not intersect the boundary of B, then $C_{\varepsilon,i}$ is bounded by and does not intersect the boundary of $Ext(B, \mathbb{R}\langle\varepsilon\rangle^n)$ for $1 \le i \le l$.*

Proof. Let $z \in C$. Then there exists a connected component $S \subset \mathbb{R}^n \setminus V(F)$ such that $z \in \overline{S}$. Then there exists some $p \in S$ such that $p \in B(z, r)$ for $r > 0$. We note that since F is nonnegative over $\mathbb{R}^n$, $F(p) > 0$. By (Basu et al., 2006a, Theorem 3.19), there exists a continuous semi-algebraic function $\gamma : [0, 1] \to \overline{S}$ with $\gamma(0) = z$, $\gamma(1) = p$ and $\gamma(t) \in S$ for all $t \in (0, 1]$. Note that we have $(F \circ \gamma)(0) = F(z) = 0$ and $(F \circ \gamma)(1) = F(p) > 0$.

Let $F' := \mathrm{Ext}(F)$ and $\gamma' := \mathrm{Ext}(\gamma)$, where Ext is the extension as in Definition 5.2. By the infinitesimal property of ε, $\mathrm{Ext}([0, 1])$ includes all Puiseux series with constant term in $[0, 1]$. The Intermediate Value Theorem applied to $F' \circ \gamma'$ gives some $t_\varepsilon \in \mathrm{Ext}([0, 1], \mathbb{R}\langle\varepsilon\rangle)$ such that $(F' \circ \gamma')(t_\varepsilon) = \varepsilon$. Let $z_\varepsilon := \gamma'(t_\varepsilon)$. Then $\lim_{\varepsilon\to 0}(z_\varepsilon) = z$. Let C_{z_ε} be the connected component of $V_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^n$ containing z_ε, and associate z to that component. Since there are finitely many connected components of $V_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^n$, as we run through all $z \in C$, a subset of these connected components are of the form C_{z_ε} with $\lim_{\varepsilon\to 0}(z_\varepsilon) = z$ for some $z \in C$. We denote these components of $V_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^n$ by $C_{\varepsilon,1}, \ldots, C_{\varepsilon,l}$. Clearly $C \subset \cup_{i=1}^{l} \lim_{\varepsilon\to 0} C_{\varepsilon,i}$.

Now suppose $z' \in \lim_{\varepsilon\to 0} C_{\varepsilon,i}$ for some $1 \leq i \leq l$. Then there exists some $z'_\varepsilon \in C_{\varepsilon,i}$ such that $\lim_{\varepsilon\to 0} z'_\varepsilon = z'$. We recall that $C_{\varepsilon,i}$ is associated to some $z \in C$, i.e. there exists some $z_\varepsilon \in C_{\varepsilon,i}$ such that $\lim_{\varepsilon\to 0} z_\varepsilon = z$. Since $C_{\varepsilon,i}$ is connected, there exists some continuous semi-algebraic function $\gamma : [0,1] \to C_{\varepsilon,i}$ such that $\gamma(0) = z'_\varepsilon$ and $\gamma(1) = z$ and $\Gamma := \gamma([0,1])$ is a connected semi-algebraic set. By the preservation of closed and boundedness under semi-algebraic mapping, Γ is closed and bounded and by Lemma 5.4, $\lim_{\varepsilon\to 0} \Gamma$ is connected. Furthermore, we note that $\lim_{\varepsilon\to 0} \Gamma \subset V \cap \mathbb{R}^n$, $\lim_{\varepsilon\to 0}(\gamma(0)) = z'$, and $\lim_{\varepsilon\to 0}(\gamma(1)) = z$. Hence $z' \in C$ and $\cup_{i=1}^{l} \lim_{\varepsilon\to 0} C_{\varepsilon,i} \subset C$, thus we have shown (1) as required.

Now suppose C is bounded by some ball $B \subset \mathbb{R}^n$ and does not intersect the boundary of B. Let $z_\varepsilon \in C_{\varepsilon,i}$ such that $\lim_{\varepsilon\to 0} z_\varepsilon \in C$. For sake of contradiction, suppose $z'_\varepsilon \in \overline{C_{\varepsilon,i} \setminus \mathrm{Ext}(B)}$. Since $C_{\varepsilon,i}$ is connected, there exists a continuous semi-algebraic function $\gamma : \mathrm{Ext}[0,1] \to C_{\varepsilon,i}$ such that $\gamma(0) = z_\varepsilon, \gamma(1) = z'_\varepsilon$ and $\Gamma := \gamma(\mathrm{Ext}[0,1])$ is a connected semi-algebraic set. The Intermediate Value Theorem applied to the polynomial defining the boundary of B composed by γ gives some $t_\varepsilon \in \mathrm{Ext}[0,1]$ such that $\gamma(t_\varepsilon)$ is in the boundary of $\mathrm{Ext}(B)$. Then $\lim_{\varepsilon\to 0} \gamma(t_\varepsilon)$ is in the boundary of B.

By the preservation of closed and boundedness under semi-algebraic mappings, Γ is closed and bounded. By Lemma 5.4, $\lim_{\varepsilon\to 0} \Gamma$ is connected. Then $\lim_{\varepsilon\to 0} \Gamma \subset C$ and $\lim_{\varepsilon\to 0} \gamma(t_\varepsilon) \in C$. However, this contradicts C intersecting the boundary of B. Thus $C_{\varepsilon,i}$ is bounded by (and does not intersect the boundary of) $\mathrm{Ext}(B, \mathbb{R}\langle\varepsilon\rangle^n)$. □

A natural question arises: Why it is necessary for us to perturb the sum of squares of the polynomials, rather than each polynomial separately, when we are working in this context over the real numbers? The following example illustrates what can go wrong over the reals, as opposed to the complex numbers.

Example 5.6. This example illustrates that for $s > 1$ with $V_\varepsilon := V(f_1 - a_1\varepsilon, \ldots, f_s - a_s\varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle^n$ and $V := \lim_{\varepsilon\to 0} V_\varepsilon \subset \mathbb{C}^n$, we may have connected components $C \subset V \cap \mathbb{R}^n$ that can only be extended to the complex part of V_ε, i.e. $C \not\subset \lim_{\varepsilon\to 0} (V_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^n)$.

Let $s = n = 2$, $f_1 = x_1^2 + x_2^2 - 1$, $f_2 = -(x_1-2)^2 - x_2^2 + 1$. Then with $V_\varepsilon := V(f_1 - \varepsilon, f_2 - \varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle^2$ we have

$$\left(\lim_{\varepsilon\to 0} V_\varepsilon\right) \cap \mathbb{R}^2 = \{(1,0)\}$$

is a single point, so this point is the only connected component. Note that any points in V_ε have to satisfy $f_1 = f_2$, so in particular they will correspond to points on the intersections of the graphs of f_1 and f_2. The graphs of f_1 and f_2 are the surfaces $P_1 := V(x_3 - x_1^2 - x_2^2 + 1)$ and $P_2 := V(x_3 + (x_1 - 2)^2 + x_2^2 - 1))$, respectively, which are two 3-dimensional parabolas.

Over the complex numbers, P_1 and P_2 intersect in two complex lines

$$V(x_2 \pm i(x_1 - 1), x_3 - 2x_1 + 2) \subset \mathbb{C}^3$$

so whenever $x_3 = \varepsilon$, i.e. $x_1 = \frac{\varepsilon}{2} + 1$, the points $\left(\frac{\varepsilon}{2} + 1, \pm i\frac{\varepsilon}{2}\right)$ are in V_ε and

$$\lim_{\varepsilon\to 0}\left(\frac{\varepsilon}{2} + 1, \pm i\frac{\varepsilon}{2}\right) = (1,0).$$

This also shows that

$$\lim_{\varepsilon\to 0}(V_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^n) = \emptyset.$$

In particular, over the reals P_1 is a convex parabola with vertex $(0,0,-1)$, P_2 is a concave parabola with vertex $(2,0,1)$, and they tangentially intersect at $(1,0,0)$. So for any real $\varepsilon \neq 0$ we have

$$P_1 \cap P_2 \cap V(x_3 - \varepsilon) \cap \mathbb{R}^3 = \emptyset.$$

Alternatively, for $F := f_1^2 + f_2^2$, if we define $V_\varepsilon := V(F - \varepsilon)$, then we proved above that

$$\left(\lim_{\varepsilon\to 0} V_\varepsilon\right) \cap \mathbb{R}^2 = \lim_{\varepsilon\to 0}\left(V_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^2\right).$$

6. Perturbations and Limits Polar Varieties

There are many different definitions of polar varieties; for a survey and comparison of different notions see Harris (2021). In this paper we reduce to the hypersurface case by taking the sum of squares of the given real polynomials. For this case, without loss of generality (in particular, when we apply a change of variables later in our paper), we can choose the appropriate number of partials to obtain a simplified definition of polar varieties. In practice, other notions of polar varieties may work better. We chose this presentation for its simplified notation and presentation, following the approach of Safey El Din and Tsigaridas (2013), for conciseness.

Definition 6.1. Let $F \in \mathbb{C}[x]$ be square-free and $V = V(F) \subset \mathbb{C}^n$. Consider the projections $\pi_i(x_1, \ldots, x_n) = (x_1, \ldots, x_i)$ for $i = 1, \ldots, n$. The *polar variety associated to* π_i of V is defined as

$$\operatorname{crit}(V, \pi_i) := V\left(F, \frac{\partial F}{\partial x_{i+1}}, \ldots, \frac{\partial F}{\partial x_n}\right) \subset \mathbb{C}^n \quad i = 1, \ldots, n,$$

based on how the polynomials defining this algebraic set correspond to the notion of critical points of a map.

We note that a significant difference in how this definition is stated compared to other notions of polar varieties is that it does not exclude the singular locus of an algebraic set V from the polar varieties associated to V. We will address the smoothness of V going forward via a change of variables and perturbations, so in fact it is natural to make this simplified modification for our context.

We use the following notation to perform a change of variables.

Definition 6.2. Let $F \in \mathbb{R}[x]$, $V := V(F) \subset \mathbb{C}^n$, and $A \in \mathrm{GL}_n(\mathbb{R})$. Then, we denote $F^A(x) := F(Ax)$, i.e. $V^A := V(F^A)$ is the image of V via the map $x \mapsto A^{-1}x$.

Next, we state some known results on polar varieties which will be used in the proofs of our algorithms. In particular, polar varieties provide a nice way for us to lower the complex dimension of an algebraic set without losing the real points in the set.

Theorem 6.3. *(Bank et al., 1997, Proposition 3) Let* $F \in \mathbb{R}[x]$ *be non-constant, square-free, and define a smooth algebraic set* $V := V(F) \subset \mathbb{C}^n$. *Then there exists a non-empty Zariski open set* $\mathcal{A} \subset \mathrm{GL}_n(\mathbb{C})$ *such that for all* $A \in \mathrm{GL}_n(\mathbb{R}) \cap \mathcal{A}$ *and* $1 \leq i \leq n, \operatorname{crit}(V^A, \pi_i)$ *is either empty or equidimensional of complex dimension* $i - 1$.

We note that in the above reference, the proof of this theorem consists of characterizing the set of matrices for which the result does not hold and showing that those matrices make up a Zariski closed set $\mathrm{GL}_n(\mathbb{C}) \setminus \mathcal{A}$, i.e. the complement of $\mathcal{A}$.

Corollary 6.4. *Let F and V be as in Theorem 6.3. Suppose ε is an infinitesimal and $V_\varepsilon := V(F-\varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle^n$ is a smooth algebraic set on the field of Puiseux series as in Definition 5.1. Then there exists a non-empty Zariski open set $\mathcal{A} \subset \mathrm{GL}_n(\mathbb{C}\langle\varepsilon\rangle)$ such that for all $A \in \mathrm{GL}_n(\mathbb{R}) \cap \mathcal{A}$ and $1 \le i \le n, \mathrm{crit}(V_\varepsilon^A, \pi_i)$ is either empty or equidimensional of complex dimension $i-1$.*

The proof of 6.4 follows from the following lemma.

Lemma 6.5. *Let $\mathcal{A} := \mathrm{GL}_n(\mathbb{C}\langle\varepsilon\rangle) \setminus V(Q)$ be a non-empty Zariski open subset of $\mathrm{GL}_n(\mathbb{C}\langle\varepsilon\rangle)$ defined by some polynomial $Q \in \mathbb{R}\langle\varepsilon\rangle[a_{i,j}]_{i,j=1}^n$. Then $\lim_{\varepsilon\to 0} \mathcal{A} \cap \mathrm{GL}_n(\mathbb{R})$ is also a non-empty Zariski open subset of $\mathrm{GL}_n(\mathbb{R})$.*

Proof. Since $\mathcal{A}$ is non-empty, $Q \neq 0$. Q is a polynomial in the variables $\{a_{i,j}\}_{i,j=1}^n$ with coefficients in $\mathbb{R}\langle\varepsilon\rangle$. We can assume, without loss of generality, that these coefficients are polynomials in $\varepsilon^{\frac{1}{q}}$ for some $q \in \mathbb{N}$ (by multiplying with a possible common denominator of these coefficients). Also, we can assume that Q has minimal degree in $\varepsilon^{\frac{1}{q}}$ among all such polynomials defining $\mathrm{GL}_n(\mathbb{C}\langle\varepsilon\rangle) \setminus \mathcal{A}$. Thus

$$Q = Q_0 + \varepsilon^{\frac{t}{q}} Q_1$$

where $t \in \mathbb{Z}^+$, $Q_0 \in \mathbb{R}[a_{i,j}]$, and $Q_1 \in \mathbb{R}[\varepsilon^{\frac{1}{q}}][a_{i,j}]$. If $Q_0 = 0$, then Q_1 has lower degree than Q in $\varepsilon^{\frac{1}{q}}$ and still defines $\mathrm{GL}_n(\mathbb{C}\langle\varepsilon\rangle) \setminus \mathcal{A}$, a contradiction. Therefore, $Q_0 \neq 0$ and

$$V(Q_0) = \lim_{\varepsilon\to 0} V(Q) \neq \mathrm{GL}_n(\mathbb{C})$$

so $\lim_{\varepsilon\to 0} \mathcal{A} \cap \mathrm{GL}_n(\mathbb{R})$ is also a non-empty Zariski open subset of $\mathrm{GL}_n(\mathbb{R})$. □

The statements of the next two propositions follow the approach of the unpublished work Safey El Din and Tsigaridas (2013), so we restate and prove them here.

Proposition 6.6. *Suppose $F \in \mathbb{R}[x]$ and $V(F) \cap \mathbb{R}^n$ is bounded. There exists a non-empty Zariski open set $O \in \mathrm{GL_n}(\mathbb{C})$ such that for $A \in O \cap GL_n(\mathbb{R})$, if $V^A = V(F^A)$ and $V_\varepsilon^A := V(F^A - \varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle^n$ for ε infinitesimal, then*
(i) *for all $1 \le i \le n$, $\mathrm{crit}(V_\varepsilon^A, \pi_i)$ is either empty or is smooth and equidimensional with complex dimension $i-1$;*
(ii) *for all $p \in V^A \cap \mathbb{R}^n$, $\pi_d^{-1}(\pi_d(p)) \cap (V^A \cap \mathbb{R}^n)$ is finite, where d is greater than or equal to the local real dimension of V^A at p.*

Proof. (i) First note that V_ε is smooth by Lemma 5.3. By Corollary 6.4, we obtain a non-empty Zariski open set $O_1 \in \mathrm{GL_n}(\mathbb{C}\langle\varepsilon\rangle)$ such that for $A \in O_1 \cap \mathrm{GL_n}(\mathbb{R})$, and $1 \le i \le n$ $\mathrm{crit}(V_\varepsilon^A, \pi_i)$ is either empty or equidimensional with complex dimension $i-1$.

(ii) Let $V := V(F)$. Since $V \cap \mathbb{R}^n$ is semi-algebraic, we can consider it as a union of connected components $C_1, \ldots, C_l$ with corresponding real dimension $d_1, \ldots, d_l$, as in Definition 2.2. Then the local real dimension of $V \cap \mathbb{R}^n$ at p is given by $\max_{p\in\overline{C_i}} d_i$, as in Definition 2.3.

Let V_i represent the Zariski closure of each C_i for $1 \le i \le l$. Then the corresponding complex dimensions of $V_1, \ldots, V_l$ are also $d_1, \ldots, d_l$. By a version of Noether's normalization lemma (see for example Logar (1989)), there exists a non-empty Zariski open set $O_{2,i} \in \mathrm{GL_n}(\mathbb{C})$ such that for $A \in O_{2,i} \cap \mathrm{GL_n}(\mathbb{R})$ and $q \in \mathbb{C}^{d_i}$, $\pi_{d_i}^{-1}(q) \cap V_i^A$ is finite. Then for $q \in \mathbb{R}^{d_i}$, $\pi_{d_i}^{-1}(q) \cap C_i$ is finite.

Let $p \in V^A \cap \mathbb{R}^n$ where $A \in O_2 = \cap_{i=1}^{l} O_{2,i}$. Suppose $d \geq \max_{p \in \overline{C_i}} d_i$. Then $\pi_d(p) \in \mathbb{R}^d$. For any $d_i = d$, taking O_2 as defined above guarantees $\pi_d^{-1}(\pi_d(p)) \cap (V^A \cap \mathbb{R}^n)$ is finite. Furthermore, for any d_i strictly less than d, $\pi_d^{-1}(\pi_d(p)) \cap (V^A \cap \mathbb{R}^n)$ is still finite for $A \in O_2$ because $\pi_d^{-1}(\pi_d(p)) \subset \pi_{d_i}^{-1}(\pi_{d_i}(p))$. Taking $O = O_1 \cap O_2$ completes the proof. □

Now we are ready to state the main result of this section, again emphasizing that it was originally proven in the unpublished paper Safey El Din and Tsigaridas (2013).

Proposition 6.7. *Let $F, g_1, \ldots, g_m \in \mathbb{R}[x]$ and let ε be infinitesimal. Suppose $F \geq 0$ on $\mathbb{R}^n$, $V(F) \cap \mathbb{R}^n$ is bounded, and suppose that **(i)** and **(ii)** from Proposition 6.6 hold with $A = I \in GL_n(\mathbb{R})$ for $V = V(F)$ and $V_\varepsilon := V(F - \varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle^n$. Then for $i = 0, \ldots, n$, $\lim_{\varepsilon \to 0} \mathrm{crit}(V_\varepsilon, \pi_i)$ is equidimensional of dimension $i - 1$ and for*

$$U := \{x \in \mathbb{R}^n \ : \ g_1(x) > 0, \ldots, g_m(x) > 0\} \subset \mathbb{R}^n$$

and $S := V(F) \cap U$ we have

$$\left(\lim_{\varepsilon \to 0} \mathrm{crit}(V_\varepsilon, \pi_i)\right) \cap U = S \quad \Leftrightarrow \quad \dim_{\mathbb{R}}(S) \leq i - 1.$$

Proof. ($\Rightarrow$) Fix some $0 \leq i \leq n$ and suppose

$$\left(\lim_{\varepsilon \to 0} \mathrm{crit}(V_\varepsilon, \pi_i)\right) \cap U = S.$$

If $S = \emptyset$, then $\dim_{\mathbb{R}}(S) = -1 \leq i - 1$ and we are done. So assume $S \neq \emptyset$. Since $S := V(F) \cap U$, $V(F) \cap \mathbb{R}^n \neq \emptyset$ and S is bounded by our initial assumption of $V(F) \cap \mathbb{R}^n$ being bounded.

By Lemma 6.9, for $1 \leq i \leq n$, $\mathrm{crit}(V_\varepsilon, \pi_i) \neq \emptyset$. Then by Proposition 6.6, $\mathrm{crit}(V_\varepsilon, \pi_i)$ is equidimensional of complex dimension $i - 1$. Then $\lim_{\varepsilon \to 0}(\mathrm{crit}(V_\varepsilon, \pi_i))$ has complex dimension $i - 1$. So the real dimension of $\lim_{\varepsilon \to 0}(\mathrm{crit}(V_\varepsilon, \pi_i))$ is $\leq i - 1$ and thus

$$\left(\lim_{\varepsilon \to 0} \mathrm{crit}(V_\varepsilon, \pi_i)\right) \cap U = S$$

has real dimension $\leq i - 1$.
($\Leftarrow$) Now suppose $\dim_{\mathbb{R}}(S) \leq i - 1$. Since $\mathrm{crit}(V_\varepsilon, \pi_i) \subset V_\varepsilon$, $\lim_{\varepsilon \to 0} V_\varepsilon = V(F)$, and $V(F) \cap U = S$,

$$\left(\lim_{\varepsilon \to 0} \mathrm{crit}(V_\varepsilon, \pi_i)\right) \cap U \subset S.$$

If $S = \emptyset$,

$$S \subset \left(\lim_{\varepsilon \to 0} \mathrm{crit}(V_\varepsilon, \pi_i)\right) \cap U$$

and we are done. So suppose $S \neq \emptyset$ and take $z = (z_1, \ldots, z_n) \in S$. Since $S = V(F) \cap U$, $z \in U$ and $z \in V(F) \cap \mathbb{R}^n$. Furthermore, the local real dimension of S at z is $\leq i - 1$, so the local real dimension of $V(F) \cap \mathbb{R}^n$ at z is also $\leq i - 1$.

Define F' as the function F where the first $i - 1$ coordinates have been evaluated at the first $i - 1$ coordinate values of z,i.e.

$$F' := F(x_i, \ldots, x_n) = F(z_1, \ldots, z_{i-1}, x_i, \ldots, x_n).$$

We note that since F is nonnegative over $\mathbb{R}^n$, F' is nonnegative over $\mathbb{R}^{n-i+1}$. Also define $z' := (z_i, \ldots, z_n)$, $V' := V(F')$ and $V'_\varepsilon := V(F' - \varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle^{n-i+1}$, and the canonical projection $\varphi_i(x) = x_i$ and the respective $\varphi'_i(x_i, \ldots, x_n) = x_i$. Note that z' is isolated in $V' \cap \mathbb{R}^{n-i+1}$ since $\pi_{i-1}^{-1}(\pi_{i-1}(z)) \cap V(F) \cap \mathbb{R}^n$ is finite by **(ii)** of Proposition 6.6.

Applying Lemma 6.8 to z' and V' we get some $z'_\varepsilon \in \mathrm{crit}(V'_\varepsilon, \varphi'_i)$ such that $\lim_{\varepsilon\to 0} z'_\varepsilon = z' = (z_i, \ldots, z_n)$. Define $z_\varepsilon := (z_1, \ldots, z_{i-1}, z'_\varepsilon)$ and $V^*_\varepsilon = V_\varepsilon \cap \mathrm{Ext}(\pi_{i-1}^{-1}(z_1, \ldots, z_{i-1}))$, where Ext is the extension from Definition 5.2. Then $z_\varepsilon \in \mathrm{crit}(V^*_\varepsilon, \varphi_i)$. By Lemma 6.10, $z_\varepsilon \in \mathrm{crit}(V_\varepsilon, \pi_i)$. Since $\lim_{\varepsilon\to 0} z_\varepsilon = z$,

$$S \subset \Big(\lim_{\varepsilon\to 0} \mathrm{crit}(V_\varepsilon, \pi_i)\Big) \cap U.$$

□

The following lemmas were used in the proof of the above proposition.

Lemma 6.8. *Assume $f_1, \ldots, f_s \in \mathbb{R}[x], F = f_1^2 + \cdots f_s^2$, and $\varepsilon > 0$ a real infinitesimal. Let $V_\varepsilon := V(F - \varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle^n$ and $V := \lim_{\varepsilon\to 0} V_\varepsilon \subset \mathbb{C}^n$. Suppose $z \in V \cap \mathbb{R}^n$ and there exists a neighborhood $B(z, r) \subset \mathbb{R}^n$ for some $r > 0$ such that $B(z, r) \cap V \cap \mathbb{R}^n$ is a finite set. Then there exists $z_\varepsilon \in \mathrm{crit}(V_\varepsilon, \pi_1)$ such that $\lim_{\varepsilon\to 0} z_\varepsilon = z$.*

Proof. Since $B(z, r) \cap V \cap \mathbb{R}^n$ is a finite set, there exists some $r' > 0$ such that z is the only point in $B(z, r') \cap V \cap \mathbb{R}^n$. So $\{z\}$ is a bounded connected component of $V \cap \mathbb{R}^n$. Then by Proposition 5.5, there exist connected components $C_{\varepsilon,1}, \ldots, C_{\varepsilon,l}$ of $V_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^n$ such that $\{z\} = \cup_{i=1}^{l} \lim_{\varepsilon\to 0} C_{\varepsilon,i}$ and $C_{\varepsilon,i}$ is bounded by and does not intersect the boundary of $\mathrm{Ext}(B(z, r')) \subset \mathbb{R}\langle\varepsilon\rangle^n$ for $1 \le i \le l$.

Then for $1 \le i \le l$, we have $C_{\varepsilon,i} \subset \mathrm{Ext}(B(z, r'))$ and is closed and bounded. By the extreme value theorem, $C_{\varepsilon,i} \cap \mathrm{crit}(V_\varepsilon, \pi_1) \neq \emptyset$. Since $\lim_{\varepsilon\to 0} C_{\varepsilon,i} = \{z\}$, all $z_\varepsilon \in C_{\varepsilon,i} \cap \mathrm{crit}(V_\varepsilon, \pi_1) \neq \emptyset$ are such that $\lim_{\varepsilon\to 0} z_\varepsilon = z$, and we are done. □

Lemma 6.9. *Assume $f_1, \ldots, f_s \in \mathbb{R}[x], F = f_1^2 + \cdots f_s^2$, and $\varepsilon > 0$ a real infinitesimal. Let $V_\varepsilon := V(F - \varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle^n$ and $V := \lim_{\varepsilon\to 0} V_\varepsilon \subset \mathbb{C}^n$ with $V \cap \mathbb{R}^n$ nonempty and bounded. Then $\mathrm{crit}(V_\varepsilon, \pi_i)$ is nonempty and intersects each bounded connected components of $V_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^n$ for all $1 \le i \le n$.*

Proof. Since $V \cap \mathbb{R}^n$ is nonempty, there exists some nonempty connected component $C \subset V \cap \mathbb{R}^n$. Let $z \in C$. Since $V \cap \mathbb{R}^n$ is bounded, there exists some $r > 0$ such that the $C \subset B(z, r) \subset \mathbb{R}^n$ and C does not intersect the boundary of $B(z, r)$. So by Proposition 5.5, there exist connected components $C_{\varepsilon,1}, \ldots, C_{\varepsilon,l}$ of $V_\varepsilon \cap \mathbb{R}\langle\varepsilon\rangle^n$ such that $C = \cup_{j=1}^{l} \lim_{\varepsilon\to 0} C_{\varepsilon,j}$ and $C_{\varepsilon,j}$ is bounded by and does not intersect the boundary of $\mathrm{Ext}(B(z, r)) \subset \mathbb{R}\langle\varepsilon\rangle^n$ for $1 \le j \le l$. Thus, for $1 \le j \le l$, we have $C_{\varepsilon,j}$ is closed and bounded. Hence, by Lemma 6.8, $C_{\varepsilon,j} \cap \mathrm{crit}(V_\varepsilon, \pi_1) \neq \emptyset$. Since by definition $\mathrm{crit}(V_\varepsilon, \pi_1) \subset \mathrm{crit}(V_\varepsilon, \pi_i)$, we are done. □

Lemma 6.10. *Let $F \in \mathbb{C}[x]$ and $\alpha = (\alpha_1, \ldots, \alpha_{i-1}) \in \mathbb{C}^i$. Suppose $V_{i,\alpha}$ is the algebraic set $V(F) \cap \pi_{i-1}^{-1}(\alpha)$ and φ_i is the projection defined by $\varphi_i(x) = x_i$. Then*

$$\mathrm{crit}(V_{i,\alpha}, \varphi_i) \subset \mathrm{crit}(V(F), \pi_i).$$

Proof. We recall that by definition

$$\mathrm{crit}(V(F), \pi_i) = V\Big(F, \frac{\partial F}{\partial x_{i+1}}, \ldots, \frac{\partial F}{\partial x_n}\Big).$$

By how we have defined $V_{i,\alpha}$, $\mathrm{crit}(V_{i,\alpha}, \varphi_i)$ is the algebraic set defined by the polynomials $F, x_1 - \alpha_1, \ldots, x_{i-1} - \alpha_{i-1}$ and the maximal minors of the Jacobian matrix of the polynomials. Then in fact,

$$\mathrm{crit}(V_{i,\alpha}, \varphi_i) = V\Big(F, x_1 - \alpha_1, \ldots, x_{i-1} - \alpha_{i-1}, \frac{\partial F}{\partial x_{i+1}}, \ldots, \frac{\partial F}{\partial x_n}\Big)$$

and we are done. □

7. Shifting from Infinitesimals to Complex Perturbations

In this section, we establish results in order to formulate our algorithms so they can be implemented not only purely symbolically, but also in a numerical algebraic geometry context. Here we track our perturbed set to its limit variety by employing homotopy continuation while our perturbation constant follows a complex arc towards zero.

To this end, we first shift from the paradigm of real infinitesimals to arbitrarily small real numbers, as established by the following result from real algebraic geometry.

Theorem 7.1. *(Basu et al., 2006a, Proposition 3.17) A result holds over $\mathbb{R}\langle\varepsilon\rangle$ if and only if there exists some $e_0 \in \mathbb{R}$ such that it also holds for all $e \in (0, e_0) \cap \mathbb{R}$.*

For our purposes, we also want to establish that we are able to make this switch in terms of witness set computations, which are done over the complex numbers. Therefore, the results in this section can be formulated in terms of a more general perturbation setup, based on the following result from Faugére et al. on perturbing the defining polynomials of an algebraic set.

Lemma 7.2. *(Faugère et al., 2008, Lemma 1) Let $f_1, \ldots, f_s \in \mathbb{R}[x]$ and fix $l \leq s$ and $\{i_1, \ldots, i_l\} \subset \{1, \ldots, s\}$. Then there exists a Zariski closed subset $\mathcal{A} \times \mathcal{E} \subset \mathbb{C}^s \times \mathbb{C}$ such that for all $\mathbf{a} := (a_1, \ldots, a_s) \in \mathbb{R}^s \setminus \mathcal{A}$ and $e \in \mathbb{R} \setminus \mathcal{E}$, the ideal generated by the polynomials $f_{i_1} - ea_{i_1}, \ldots, f_{i_l} - ea_{i_l}$ is a radical equidimensional ideal and $V(f_{i_1} - ea_{i_1}, \ldots, f_{i_l} - ea_{i_l})$ is either empty or smooth of dimension $n - l$.*

Remark 7.3. We will use these more general deformations in Sections 7, 8 and 9, where the results are valid over $\mathbb{C}$. In the previous sections and in Section 11 we need to use a more specialized perturbation of the polar varieties for our results to hold over the reals. See Example 5.6 why the more general deformations may not work over the reals.

Using Lemma 7.2, we define a genericity assumption which holds over the complex numbers.

Definition 7.4. Consider polynomials $f_1, \ldots, f_s \in \mathbb{R}[x]$ and point $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{R}^s$. We say that $f_1, \ldots f_s$ and $\mathbf{a}$ satisfy Assumption `(A)` if

`(A)`: There exists $e_0 > 0$ such that for all $0 < e \leq e_0$, the polynomials $f_1 - ea_1, \ldots, f_s - ea_s$ generate a radical equidimensional ideal and $V_e^{\mathbf{a}} := V(f_1 - ea_1, \ldots, f_s - ea_s)$ is smooth and has dimension $n - s$.

Assumption `(A)` in Definition 7.4 guarantees the existence of some $e_0 > 0$; however, in practice this number can be arbitrarily small. Instead of trying to compute an e_0 that works for a given system $f_1, \ldots, f_s$, the next result shows that we can choose a generic $\xi \in \mathbb{C}$ with $|\xi| = 1$ to replace e_0 with ξ and e with $t\xi$, where $t \in (0, 1]$.

Proposition 7.5. *Let $f_1, f_2, \ldots, f_s \in \mathbb{R}[x]$, $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{R}^s$ and let ε be infinitesimal. Assume that $V_\varepsilon^{\mathbf{a}} := V(f_1 - \varepsilon a_1, \ldots, f_s - \varepsilon a_s) \subset \mathbb{C}\langle\varepsilon\rangle^n$ is smooth and equidimensional of dimension $n-s$. Then for all but finitely many $\xi \in \mathbb{C}$ with $|\xi| = 1$ and for all $t \in (0,1]$, $V_{t\xi}^{\mathbf{a}} := V(f_1 - t\xi a_1, \ldots, f_s - t\xi a_s) \subset \mathbb{C}^n$ is smooth and equidimensional of dimension $n-s$ and in that case we have*

$$\lim_{\varepsilon\to 0} V_\varepsilon^{\mathbf{a}} = \lim_{t\to 0} V_{t\xi}^{\mathbf{a}}.$$

Proof. First, we show that for all but a finite number of choices of $\xi \in \mathbb{C}$, $V_\xi^{\mathbf{a}} = V(f_1 - \xi a_1, \ldots, f_s - \xi a_s)$ is smooth by proving that the set of "bad" choices ξ is a proper Zariski closed subset of $\mathbb{C}$, thus finite. Note that from our assumptions on $V_\varepsilon^{\mathbf{a}}$ we get that $f_1, \ldots, f_s$ and $\mathbf{a}$ satisfy Assumption `(A)` for some $\mathbf{e}_0 > 0$. Consider the ideal using new variables x_0, z and $\lambda_1, \ldots, \lambda_s$:

$$\begin{aligned} I := \langle f_1^{(h)} - a_1 z x_0^{\deg(f_1)}, \ldots, f_s^{(h)} - a_s z x_0^{\deg(f_s)} \rangle \\ + \langle (\lambda_1 \nabla(f_1) + \ldots + \lambda_s \nabla(f_s))^{(h)} \rangle. \end{aligned}$$

Here $g^{(h)}$ denotes the homogenization of $g \in \mathbb{R}[x_1, \ldots, x_n]$ by the variable x_0 and ∇ is the differential operator in the variables $x_1, \ldots, x_n$. Thus I is bi-homogeneous in the variables $(\lambda_1, \ldots \lambda_s)$ and $(x_0, \ldots, x_n)$. Then the projection of $X(I) \subset \mathbb{P}^n \times \mathbb{P}^s \times \mathbb{C}$ onto $\mathbb{C}$ is a Zariski closed subset of $\mathbb{C}$, and since e_0 is not in the projection, the projection is not $\mathbb{C}$, thus a finite set Z. Clearly, for $\xi \in \mathbb{C} \setminus Z$ and for all $p \in V_\xi^{\mathbf{a}}$, the Jacobian of $f_1 - \xi a_1, \ldots, f_s - \xi a_s$ at p has rank s, thus $V_\xi^{\mathbf{a}}$ is smooth and equidimensional of dimension $n-s$. This also implies that for all but finitely many $\xi \in \mathbb{C}$ with $|\xi| = 1$ and for all $t \in (0,1]$ we have that $V_{t\xi}^{\mathbf{a}} = V(f_1 - t\xi a_1, \ldots, f_s - t\xi a_s)$ is smooth and equidimensional.

Fix $\xi \in \mathbb{C} \setminus Z$ with $|\xi| = 1$ so $V_{t\xi}^{\mathbf{a}}$ is smooth and equidimensional. To prove the second claim, let $L_1, \ldots, L_{n-s} \in \mathbb{C}[x]$ be linear polynomials such that $\mathcal{L} := V(L_1, \ldots, L_{n-s})$ is a generic linear space of codimension $n-s$ which intersects both $\lim\limits_{\varepsilon\to 0} V_\varepsilon^{\mathbf{a}}$ and $\lim\limits_{t\to 0} V_{t\xi}^{\mathbf{a}}$ transversely. By our assumptions, both $V_\varepsilon^{\mathbf{a}} \cap \mathcal{L} \subset \mathbb{C}\langle\varepsilon\rangle^n$ and $V_{t\xi}^{\mathbf{a}} \cap \mathcal{L} \subset \mathbb{C}^n$ are finite for any fixed $t \in (0,1]$.

Then since $\mathcal{L}$ does not depend on either ε or t,

$$\begin{aligned} \lim_{\varepsilon\to 0}\left(V_\varepsilon^{\mathbf{a}} \cap \mathcal{L}\right) &= \lim_{\varepsilon\to 0} V_\varepsilon^{\mathbf{a}} \cap \mathcal{L} \quad \text{and} \\ \lim_{t\to 0}\left(V_{t\xi}^{\mathbf{a}} \cap \mathcal{L}\right) &= \lim_{t\to 0} V_{t\xi}^{\mathbf{a}} \cap \mathcal{L}. \end{aligned}$$

Since $\mathcal{L}$ is a generic linear space which intersects both $\lim\limits_{\varepsilon\to 0} V_\varepsilon^{\mathbf{a}}$ and $\lim\limits_{t\to 0} V_{t\xi}$ transversely, $\lim_{\varepsilon\to 0} V_\varepsilon^{\mathbf{a}} \cap \mathcal{L} = \lim_{t\to 0} V_{t\xi}^{\mathbf{a}} \cap \mathcal{L}$ implies $\lim_{\varepsilon\to 0} V_\varepsilon^{\mathbf{a}} = \lim_{t\to 0} V_{t\xi}^{\mathbf{a}}$.

So it is sufficient to prove that

$$\lim_{\varepsilon\to 0}\left(V_\varepsilon^{\mathbf{a}} \cap \mathcal{L}\right) = \lim_{t\to 0}\left(V_{t\xi}^{\mathbf{a}} \cap \mathcal{L}\right)$$

to achieve the desired result.

Let $H \subset \mathbb{R}[x, \varepsilon]$ be the system

$$H := H(x, \varepsilon) = [f_1 - \varepsilon a_1, \ldots, f_s - \varepsilon a_s, L_1, \ldots, L_{n-s}].$$

Let $S \subset \mathbb{C}\langle\varepsilon\rangle^n$ be the finite set of bounded solutions of $H = 0$, where bounded is as defined for Puiseux series in Definition 5.1. Then for all $x(\varepsilon) \in S$, let $\lim_{\varepsilon\to 0} x(\varepsilon) = x_0 \in \mathbb{C}^n$. Furthermore, by the definition of H, $\lim_{\varepsilon\to 0} S = \lim_{\varepsilon\to 0}\left(V_\varepsilon^{\mathbf{a}} \cap \mathcal{L}\right)$.

Using Puiseux's Theorem (e.g., see (Fischer, 2001, Chap. 7) and (Sommese and Wampler, 2005, § 10.2, Thm. A.3.2, Cor. A.3.3)), one can ensure that each of the finitely many Puiseux series under consideration has a positive radius of convergence. In particular, since $\varepsilon > 0$ is a real infinitesimal and $x(\varepsilon) \in S$ is bounded, each $x(\varepsilon) \in S$ has an interval of convergence $(0, e_x) \subset \mathbb{R}$ for some $e_x > 0$. Choose $e_0 > 0$ such that $e_0 < \min_{x \in S} e_x$. Now we make a switch, and instead of considering $x(\varepsilon) \in S$ an element $\mathbb{C}\langle\varepsilon\rangle^n$, we consider x as a function $\mathbb{C} \to \mathbb{C}^n$ which is well-defined for $z \in \mathbb{C}$ with $|z| \leq e_0$. Abusing the notation, we denote by x both the Puiseux series and the corresponding complex function.

Recall that if a pair $(x^*, z^*) \in \mathbb{C}^n \times \mathbb{C}$ has the property that $H(x^*, z^*) = 0$ and $\det JH(x^*, z^*) = 0$, where JH is the Jacobian matrix of H with respect to the x variables, then z^* is a critical point and x^* is a branch point for for $H(x, z) = 0$. Let C denote the set of all critical points of $H(x, z) = 0$. Then, since $|S| < \infty$, we know $|C| < \infty$.

Now let $z \in C$. Then there exists some $\xi_z \in \mathbb{C}$ with $|\xi_z| = 1$ such that for $t \in \mathbb{R}$, the path $\xi_z t$ passes through z, so that $x(t\xi_z) \in \mathbb{C}^n$ has some branching point. Let $Z = \{\xi_z : z \in C\}$, a subset of the unit circle in $\mathbb{C}$. Since $|C| < \infty$, $|Z| < \infty$. Then, for any $\xi \in \mathbb{C} \setminus Z$ with $|\xi| = 1$, we have that $x(t\xi) \in \mathbb{C}^n$ for $t \in (0, 1]$ does not pass through branching points. Since $\mathbb{C} \setminus Z$ is Zariski dense in $\mathbb{C}$, the same holds for generic $\xi \in \mathbb{C}$ with $|\xi| = 1$.

So let $\xi \in \mathbb{C} \setminus Z$ with $|\xi| = 1$ and $H_\xi \subset \mathbb{C}^{n+1}$ be the homotopy defined by the system

$$H_\xi := H_\xi(x, t) = [f_1 - t\xi a_1, \ldots, f_s - t\xi a_s, L_1 \ldots, L_{n-s}].$$

The limit points of the solutions of H_ξ are $\lim_{t\to 0}\left(V^{\mathbf{a}}_{t\xi} \cap \mathcal{L}\right)$. Let $T \subset \mathbb{C}^n$ be the roots of $H_\xi(x, 1)$. Then $|T| = |V^{\mathbf{a}}_\varepsilon \cap \mathcal{L}| < \infty$. Furthermore, by the above argument, the homotopy paths for H_ξ are exactly described by the points in $V^{\mathbf{a}}_\varepsilon \cap \mathcal{L} \subset \mathbb{C}\langle\varepsilon\rangle^n$ by replacing ε with $t\xi$. Hence,

$$\lim_{\varepsilon\to 0}\left(V^{\mathbf{a}}_\varepsilon \cap \mathcal{L}\right) = \lim_{t\to 0}\left(V^{\mathbf{a}}_{t\xi} \cap \mathcal{L}\right).$$

□

The following illustrates why we take $\xi \in \mathbb{C} \setminus \mathbb{R}$, a process generally known as the "gamma trick," e.g., see (Sommese and Wampler, 2005, Chap. 7)

Example 7.6. Let $f(x) = x^3 - 3x^2 + 2x$, $a = 1$, and $H_\xi(x, t) = f(x) - t\xi$. For $\xi = 1$, since $H_\xi(x, 0) = x(x - 1)(x - 2) = 0$ has three real solutions, $H_\xi(x, 1) = f(x) - 1 = 0$ has one real solution, and $H_\xi(x, t)$ has real coefficients, there must be a value of $t \in (0, 1)$ such that $H_1(x, t) = 0$ has a singular solution, namely $t \approx 0.3849$. However, for, say, $\xi = 1 + \sqrt{-1}$, then $H_\xi(x, t) = 0$ has three nonsingular solutions for all $t \in [0, 1]$. A similar statement holds for $\xi \in \mathbb{C} \setminus \mathbb{R}$ with algebraic probability one.

Proposition 7.5 gives a proof of correctness for Witness Points in Limits Algorithm 2 which computes a *witness point set* (as in Definition 2.9) of a limit with algebraic probability one. In Step (iv) of Algorithm 2 we use numerical homotopy continuation method, see Bates et al. (2013) for more details.

8. Deflated Witness Systems for Limits

In this section, we apply some of the main ideas from numerical algebraic geometry following Bates et al. (2013) to complex algebraic sets obtained as limits of perturbed positive dimensional algebraic set. We will use the notion of a *deflated witness system*:

Algorithm 2 WITNESSPOINTSINLIMITS

Input: $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$, $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{R}^s$ and $L = \{L_1, \ldots, L_{n-s}\} \subset \mathbb{R}[x_1, \ldots, x_n]$ generic linear polynomials.

Output: flag =TRUE if $V(f_1 - a_1 e, \ldots, f_s - a_s e) \cap V(L)$ is 0-dimensional for all sufficiently small $e > 0$ and the finite set of points in $W := \lim_{e \to 0+} (V(f_1 - a_1 e, \ldots, f_s - a_s e) \cap V(L))$, flag =FALSE otherwise and $W = \emptyset$.

1. Loop
   (i) Choose generic $\xi \in \mathbb{C}$ with $|\xi| = 1$.
   (ii) Define $H_\xi(x,t) := [f_1 - t\xi a_1, \ldots, f_s - t\xi a_s, L_1, \ldots, L_{n-s}]$.
   (iii) If $|V(H_\xi(x,1))| = \infty$, exit loop and return flag = FALSE, $W = \emptyset$.
   (iv) Compute $\lim_{t\to 0} V(H_\xi(x,t))$ via a homotopy continuation, starting at $t = 1$.
   (v) If no branch points were hit during homotopy tracking, exit loop and return flag = TRUE, $W = \lim_{t\to 0} V(H_\xi(x,t))$.

Definition 8.1. Let $V \subset \mathbb{C}^n$ be an equidimensional algebraic set and (F, L, W) a witness set for V as in Definition 2.9. Then if each irreducible component of V has multiplicity one with respect to F, F is called a *deflated witness system* and (F, L, W) is a *deflated witness set* for V.

When trying to compute a deflated witness system for a variety defined by a limit, difficulties that arise are that the limit points may be singular, arising from multiple paths converging to the same limit point, or that the witness system $f = (f_1, \ldots, f_s)$ for the original algebraic set $V(f)$ is not a witness system for $\lim_{t\to 0} V^{\mathbf{a}}_{t\xi} = \lim_{t\to 0} V(f_1 - a_1 t\xi, \ldots, f_s - a_s t\xi)$. This is demonstrated in the following example.

Example 8.2. Consider $f(x,y) = (xy, xy - x) \subset \mathbb{C}[x,y]$ and $\mathbf{a} = (1,0)$. Then for $\xi = 1$ we get $V^{\mathbf{a}}_{t\xi} = V(xy - t, xy - x)$, so for all fixed $t \in (0,1]$ we get $V^{\mathbf{a}}_{t\xi} = \{(t,1)\}$, so $\lim_{t\to 0} V^{\mathbf{a}}_{t\xi} = \{(0,1)\}$. But $V(f(x,y)) = V(x)$ is not a witness system for $\{(0,1)\}$. We show that applying a straightforward isosingular deflation to f does not provide a deflated witness system for $\lim_{t\to 0} V^{\mathbf{a}}_{t\xi} = \{(0,1)\}$. We compute

$$Jf(x,y) = \begin{bmatrix} y & x \\ y-1 & x \end{bmatrix}.$$

For $p = (0,1)$, $\operatorname{rank}(Jf(0,1)) = 1$, so the determinant of $Jf(0,1)$. Thus isosingular deflation gives $F = [xy, xy - x, x]$. But $V(F) = V(x)$, so F is not a witness system for $\{(0,1)\}$.

To overcome these difficulties, we will use Theorem 2.13 to compute a deflated witness system for the intersection of $V(f_1 - a_1 t\xi, \ldots, f_s - a_s t\xi) \cap V(t) \subset \mathbb{C}^{n+1}$ (considering t as an extra complex variable, but $\xi \in \mathbb{C}$ is fixed). The DEFLATED WITNESS SYSTEM ALGORITHM 3 computes a deflated witness system for irreducible components of a variety defined as a limit.

Example 8.3 (Example 8.2 cont)**.** We apply Algorithm 3 for this example. We have to do isosingular deflation of $F_0(x,y,t) := (xy - t, xy - x)$, as in Steps (1) and (2). Here, we compute

$$JF_0(x,y,t) = \begin{bmatrix} y & x & -1 \\ y-1 & x & 0 \end{bmatrix}.$$

Algorithm 3 DeflatedWitnessSystem

Input: $f_1, \ldots, f_s \in \mathbb{R}[x]$, $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{R}^s$, and $p \in V := \lim_{e \to 0+} V(f_1 - a_1 e, \ldots, f_s - a_s e)$, a generic point on a unique irreducible component V_p of V.

Output: A deflated witness system $G \subset \mathbb{R}[x]$ for V_p.

1. Define $F_0(x,t) := (f_1 - a_1 t, \ldots, f_s - a_s t) \in \mathbb{R}[x,t]^s$ and $q := (p, 0) \in \mathbb{R}^{n+1}$.
2. $F :=$ IsosingularDeflation(F_0, q). `// See (Hauenstein and Wampler, 2013, Alg. 6.3)`
3. Define $G_0(x) := F(x, 0)$.
4. $G :=$ IsosingularDeflation(G_0, p).`// See (Hauenstein and Wampler, 2013, Alg. 6.3)`
5. Return G.

For $q = (0,1,0)$, $\mathrm{rank}(JF_0(0,1,0)) = 1$. We get in Step (2) a deflated system $F(x,y,t) = [xy, xy - x, x, y - 1]$. Since $G_0(x,y) := F$ is already deflated for $p = (0,1)$, $G = G_0$ is the output.

Theorem 8.4. *Let $f_1, \ldots, f_s$, $\mathbf{a}$, and p as in the input of* Algorithm 3. *Then G, computed by* Algorithm 3, *satisfies the output specifications.*

Proof. Since V_p is an irreducible component of V, there exists an irreducible component $Z \subset V(F_0(x,t)) \subset \mathbb{C}^{n+1}$ such that $V_p \times \{0\}$ is an irreducible component of $Z \cap V(t)$ which is an intersection. Hence, one can apply the isosingular deflation approach applied to intersections in Theorem 2.13. Although Theorem 2.13 would deflate $H_0(x,t,t') := (F_0(x,t), t')$ at $q' := (p,0,0)$, the simplicity of the intersection together with t' contained in H_0 easily shows that one obtains an equivalent deflation as deflating $F_0(x,t)$ at $q = (p,0)$, resulting in the system $F(x,t)$. Therefore, V_p must be an irreducible component of $V(F(x,0))$ so $G_0(x) := F(x,0)$ is a witness system for V_p. Since G_0 need not be a deflated witness system for V_p, one deflates G_0 at p to yield a deflated witness system G for V_p. □

9. Computation of g

The final key tool required to compute a real smooth point on every bounded connected component of an algebraic set V is a "well-chosen" polynomial g that satisfies the conditions of Theorem 3.1, i.e., $\mathrm{Sing}(V) \cap \mathbb{R}^n \subset V(g)$ and $\dim(V \cap V(g)) < \dim(V)$. There exist symbolic methods to compute such a g for an equidimensional variety V with $\dim(V) = n - s$. For example, (Safey El Din et al., 2018, Lemma 4.3) computes the defining equation $\omega \in \mathbb{R}[x_1, \ldots, x_{n-s+1}]$ of the projection $\overline{\pi_{n-s+1}(V^A)}$ of a generic linear transformation V^A of V such that this projection is a hypersurface. Then, g can be taken to be one of the partial derivatives of ω. This idea could be extended to the case when V is not equidimensional using infinitesimal deformations and limits (c.f., Safey El Din and Tsigaridas (2018)). In our Computation of g Algorithm 4, we provide a new approach based on isosingular deflation, as discussed in Subsection 2.6, which computes several g's depending on the isosingular deflation sequence of the irreducible components.

Algorithm 4 COMPUTATIONOFG

Input: $f_1, \ldots, f_s \in \mathbb{R}[x]$, $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{R}^s$.

Output: $\left\{\left(g_j, (G_j, L, W_j)\right) \;:\; j = 1, \ldots, r\right\}$ such that for all $i \neq j \in \{1, \ldots, r\}$, $V_e^{\mathbf{a}} := V(f_1 - a_1 e, \ldots, f_s - a_s e)$ and $V := \lim_{e \to 0+} V_e^{\mathbf{a}}$:

(i) $g_j \in \mathbb{R}[x]$, $G_j, L \subset \mathbb{R}[x]$, and $W_j \subset V$.

(ii) (G_j, L, W_j) is a deflated witness set of some $V_j \subset V$, where V_j is a union of irreducible components of V;

(iii) $V = \bigcup_{j=1}^r V_j$

(iv) $\mathrm{Sing}(V_j) \subseteq V(g_j)$

(v) $\dim(V_j \cap V(g_j)) < n - s$

(vi) $\dim(V_i \cap V_j) < n - s$ and $V_i \cap V_j \subseteq V(g_j)$.

1. Loop

   (a) Choose a generic system $L \subset \mathbb{R}[x]$ of $n - s$ linear polynomials.

   (b) $(\text{flag}, W) :=$ WitnessPointsInLimits$(\{f_1, \ldots, f_s\}, \mathbf{a}, L)$. `// See Algorithm 2`

   (c) If flag =TRUE, exit loop.

2. Set $j := 1$.

3. Loop

   (a) Pick some $p \in W$.

   (b) $W_j := \{p\}$.

   (c) Update $W := W \setminus \{p\}$.

   (d) $G_j :=$ DeflatedWitnessSystem$(\{f_1, \ldots, f_s\}, \mathbf{a}, p)$. `// See Algorithm 3`
   `//` $G_j \subset \mathbb{R}[x]$ `is a witness system for the irreducible component` $V_p \subset V$ `containing` p `such that` $f_1, \ldots, f_s \in G_j$, $G_j(p) = 0$ `and` $\operatorname{rank} JG_j(p) = s$.

   (e) For all $p' \in W$
   If $G_j(p') = 0$ and $\operatorname{rank} JG_j(p') = s$, then
   Update $W_j := W_j \cup \{p'\}$ and $W := W \setminus \{p'\}$.

   (f) Compute $g_j(x) := \det(M(x))$, where M is a generic rational linear combination of all $s \times s$ submatrices of $JG_j(x)$.

   (g) If $W \neq \emptyset$, increment $j := j + 1$.

Theorem 9.1. *Let $f_1, \ldots, f_s$, $\mathbf{a}$, $V_e^{\mathbf{a}}$, and V be as in the input and output specifications of* ALGORITHM 4. *Assume that $V_e^{\mathbf{a}} := V(f_1 - a_1 e, \ldots, f_s - a_s e)$ satisfies Assumption* (A). *Then* ALGORITHM 4 *is correct.*

Proof. By our assumption on the genericity of L and Assumption (A), W is finite and each point $p \in W$ is a generic point of a unique irreducible components V_p of V containing p. Based on the output of Algorithm 3, assume that for any $p \in W$, in Step (3d) we compute $G_j \subset \mathbb{R}[x]$ such that the irreducible component $V_p \subset V$ containing p is an irreducible component of $V(G_j)$, $f_1, \ldots, f_s \in G_j$, $G_j(p) = 0$ and rank $JG_j(p) = s$. Then, $G_j \subset \mathbb{R}[x]$ computed in Step (3d) deflates all generic points of V_p. Step (4) adds all other points from W which are deflated by G_j. In particular, every other point on V_p contained in W will be added to W_j. Hence, (G_j, L, W_j) is a deflated witness set for a union of irreducible components of V, denoted by V_j, proving (ii). Since $\bigcup_j W_j = W$, we also get $\bigcup_j V_j = V$, which proves (iii). If $y \in \mathrm{Sing}(V_j)$, then $\mathrm{rank}(JG_j(y)) < s$ so all $s \times s$ minors of $JG_j(y)$ vanish. Hence, $g_j(y) = \det(M(y)) = 0$ proving (iv). Conversely, for any $p' \in W_j$, some $s \times s$ minor of $JG_j(p')$ does not vanish at p'. Since g_j is a generic choice of combinations of all such minors, $g_j(p') \neq 0$ for all $p' \in W_j$. By Assumption (A), $V = \lim_{e \to 0} V_e^{\mathbf{a}}$ is equidimensional of dimension $n - s$, so for all $p' \in W$, $\dim V_{p'} = n - s$. Since g_j does not vanish identically on $V_{p'}$ for any $p' \in W_j$, we get $\dim(V_j \cap V(g_j)) < n - s$, proving (v).

To prove the first claim in (vi), note that each V_i is a union of $(n-s)$-dimensional irreducible components of V and sample points from the irreducible components of V are uniquely assigned to one W_j. Then for $i \neq j$, V_i and V_j cannot share an irreducible component, so their intersection is lower dimensional.

To prove the second claim in (vi) we use Theorem 2.12 as follows. Let $y \in V_i \cap V_j$. Suppose that X is an irreducible component of V_i and Y is an irreducible component of V_j such that $y \in X \cap Y$. Let $\xi \in \mathbb{C}$ be generic with $|\xi| = 1$, t a complex variable, and denote $f_\xi^{\mathbf{a}} = f_\xi^{\mathbf{a}}(x, t) := (f_1 - a_1 t\xi, \ldots, f_s - a_s t\xi)$. Then, $X \times \{0\}$ and $Y \times \{0\}$ are irreducible varieties of $\mathbb{C}^{n+1}$ and both are subsets of $V(f_\xi^{\mathbf{a}}) \subset \mathbb{C}^{n+1}$. Therefore, each is contained in a unique isosingular set of $f_\xi^{\mathbf{a}}$ denoted by $\mathrm{Iso}_{f_\xi^{\mathbf{a}}}(X \times \{0\})$ and $\mathrm{Iso}_{f_\xi^{\mathbf{a}}}(Y \times \{0\})$, respectively. Let $F_i(x, t)$ and $F_j(x, t)$ be their corresponding deflated witness systems, respectively. If $F_i = F_j$, i.e. the two isosingular sets of $f_\xi^{\mathbf{a}}$ are the same, then $\mathrm{Iso}_{F_j(x,0)}(X) \neq \mathrm{Iso}_{F_j(x,0)}(Y)$ (otherwise $X = Y$) so $y \in \mathrm{Sing}_{F_j(x,0)}(Y)$. Note that by the Deflated Witness System Algorithm 3, $G_j(x)$ is the deflation of $F_j(x, 0)$ at a generic point of V_j. This implies by Theorem 2.12 that $y \in \mathrm{Sing}_{G_j}(Y)$ and $g_j(y) = 0$.

If $F_i \neq F_j$, then $(y, 0)$ is in the intersection of two different isosingular sets so $(y, 0)$ has a different deflation sequence than a generic point in $Y \times \{0\}$, i.e., $(y, 0) \in \mathrm{Sing}_{f_\xi^{\mathbf{a}}}(Y \times \{0\})$. By Theorem 2.12, we have that $(y, 0) \in \mathrm{Sing}_{F_j}(Y \times \{0\})$. Denoting the Jacobian by $J := JF_j(x, t)$, we have that $\mathrm{rank} J(y, 0) < s$ with $\mathrm{rank} J(y', 0) = s$ for all generic $y' \in Y$. Consider $J' := JF_j(x, 0)$. (i.e. column of J corresponding to ∂t removed). Note that $Jf(x)$ is a submatrix of J', since $f \subset F_j(x, 0)$. If $\mathrm{rank} J'(y') = s$ for generic $y' \in Y$, then $G_j = F_j(x, 0)$, $y \in \mathrm{Sing}_{G_j}(Y)$, and $g_j(y) = 0$. If $\mathrm{rank} J'(y') < s$ for generic $y' \in Y$, we claim that $\mathrm{rank} J'(y) < \mathrm{rank} J'(y')$ for generic $y' \in Y$. First note that both $\mathrm{rank} Jf(y) \leq s - 1$ and $\mathrm{rank} Jf(y') \leq s - 1$ for $f = (f_1, \ldots, f_s)$, so without loss of generality (after maybe some Gaussian elimination on these Jacobian matrices), we assume that $\nabla f_1(y) = \nabla f_1(y') = 0$. Note that the ∂t column of $J = JF_j(x, t)$ has the only possibly non-zero constant entries in the rows corresponding to $f_1 - a_1 t\xi, \ldots, f_s - a_s t\xi$. Then for a generic $y' \in Y$ we have $\mathrm{rank} J'(y') = s - 1$, since $\mathrm{rank} J(y', 0) = s$, thus among all $s \times s$ minors of $J(y', 0)$ some has to be non-zero, and the only possible non-zeros are the ones that are a_1 times the $(s-1) \times (s-1)$ minors of $J'(y')$, thus we must have $a_1 \neq 0$ and $\mathrm{rank} J'(y') = s - 1$. On the other hand, the $s \times s$ minors of $J(y, 0)$ contain all $(s-1) \times (s-1)$ minors of $J'(y)$ times a_1, so all these minors of $J'(y)$ must be zero. This implies that $\mathrm{rank} J'(y) < s - 1$. Thus, $\mathrm{rank} J'(y) < \mathrm{rank} J'(y')$. In particular, $y \in \mathrm{Sing}_{F_j(x,0)}(Y)$ and by Theorem 2.12, $y \in \mathrm{Sing}_{G_j}(Y)$ which implies that $g_j(y) = 0$. This proves (vi), and the theorem. □

One advantage of the approach using isosingular deflation is that, in many problems, the number of iterations in the deflation process is a small constant (zero or one). In this case, the degrees of the polynomials in the output of both Deflated Witness Set Algorithm 3 and Computation of g Algorithm 4 are comparable to the maximal degree of the input polynomials $f_1, \ldots, f_s$. On the other hand, the degree of the polynomial $\omega \in \mathbb{R}[x_1, \ldots, x_{n-s+1}]$ computed in the symbolic approach in (Safey El Din et al., 2018, Lemma 4.3) mentioned at the beginning of this section is the degree of V bounded by the product of the degrees of the input polynomials. Nonetheless, the disadvantage of our approach is that in the worst case, we need as many iterations in the deflation as the multiplicity of the points and this may result polynomials with higher degree than the degree of ω in (Safey El Din et al., 2018, Lemma 4.3). We have the following bound on the degree of g as a function on the number of iterations in the deflation:

Proposition 9.2. *Let $f = (f_1, \ldots, f_s)$ and $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{R}^s$ such that $V_e^{\mathbf{a}} := V(f_1 - a_1 e, \ldots, f_s - a_s e)$ satisfies Assumption* (A). *Let $D := \max_{i=1}^{s}\{\deg(f_i)\}$ and fix $p \in V := \lim_{e\to 0} V_e^{\mathbf{a}}$. If* Algorithm 3 *takes k iterations of the isosingular deflation to output $G \subset \mathbb{R}[x]$, the degrees of the polynomials in G are bounded by $s^k D$. Furthermore, if $g(x) := \det(M(x)) \in \mathbb{R}[x]$ where $M(x)$ is a $s \times s$ submatrix of $JG(x)$, then $\deg(g) \leq s^{k+1} D$.*

Proof. The first claim follows from the fact that each iteration of the deflation algorithm adds the minors of the Jacobian of the polynomials in the previous iteration, and these minors have size less than s. Thus, the degrees of polynomials added to the system in each iteration are at most s times the degrees of the polynomials in the previous iteration. The second claim follows from the first. □

10. Finite Critical Points of g

In this section, we establish a key result characterizing when a function g will have a finite number of critical points over an algebraic set. This is an adaptation of Theorem 36 and Lemma 37 from Hong et al. (2020).

Definition 10.1. Given $f_1, \ldots, f_s, g \in \mathbb{R}[x]$. We say that $x \in \mathbb{C}^n$ is a *critical point* of g for $V(f_1, \ldots, f_s)$ if $x \in V(f_1, \ldots, f_s)$ and

$$\nabla g(x) \in \operatorname{span}_{\mathbb{C}} (\nabla f_1(x), \ldots, \nabla f_s(x)),$$

where ∇ denotes the gradient operation.

We give the following example to illustrate the possibility of a g having infinite critical points over a smooth V, to motivate why the rest of this section is necessary.

Example 10.2. Let $f = x^2 + 4y^2 - 4xy + 2$ and $g = x$. Then there are an infinite number critical points of g over the algebraic set $V(f)$, defined by $(x, y) = \left(x, \frac{x}{2}\right)$.

We need the following corollary of Sard's theorem from (Sommese and Wampler, 2005, Theorem A.6.1). It uses the notion of *quasi-projective* sets, which are the intersections of a Zariski-open and a Zariski-closed subset inside some projective space. Let X_{reg} denote the set of smooth points in X.

Theorem 10.3. *Let $f(x)$ denote a system of n algebraic functions on an irreducible quasiprojective set X. Then there is a Zariski openset $U \subset \overline{f(X)} \subset \mathbb{C}^n$ such that for $y \in U$, $V(f(x) - y) \cap X_{\text{reg}}$ is smooth of dimension equal to the corank of f, i.e.* $\dim X - \dim \overline{f(X)}$. *Moreover, the Jacobian matrix of f is of rank equal to* $\dim X - \dim \overline{f(X)}$ *at all points of $V(f(x) - y) \cap X_{\text{reg}}$.*

Theorem 10.4. *Let $f_1,\dots,f_s \in \mathbb{R}[x]$ and assume that $V(f_1,\dots,f_s) \subset \mathbb{C}^n$ is a smooth equidimensional algebraic set of dimension $n-s$. Let $g_0 \in \mathbb{R}[x]$. Then there exists a Zariski closed proper subset $\mathcal{S}$ of $\mathbb{C}^n$ with* $\dim(\mathcal{S}) < n$ *such that for all $c = (c_1,\dots,c_n) \in \mathbb{R}^n \setminus \mathcal{S}$ the polynomial*

$$g := g_0 \cdot \Big((x_1-c_1)^2 + \cdots + (x_n-c_n)^2 + 1\Big) \in \mathbb{R}[x]$$

has finitely many critical points for $V(f_1,\dots,f_s)$ where g does not vanish.

Proof. Let V be an irreducible component of $V(f_1,\dots,f_s)$. By our assumptions, $\dim(V) = n-s$ and V is smooth. We will prove that g has finitely many critical points for $V(f_1,\dots,f_s)$ that lie in $V \setminus V(g)$, and since this will be true for all irreducible components of $V(f_1,\dots,f_s)$, we get the claim of the theorem.

We can assume that

$$\dim(V \cap V(g)) < n-s$$

otherwise, since V is irreducible, $V \subset V(g)$ and there is nothing to prove.

To simplify the notation, define for $c = (c_1,\dots,c_n) \in \mathbb{R}^n$

$$U_c(x) := (x_1-c_1)^2 + \cdots + (x_n-c_n)^2 + 1.$$

Then

$$\nabla g(x) = U_c(x)\nabla g_0(x) + g_0(x)\nabla U_c(x)$$

Thus, a point $x \in V$ is a critical point of g for $V(f_1,\dots,f_s)$ if and only if

$$U_c(x)\nabla g_0(x) + g_0(x)\nabla U_c(x) \ \in\ \operatorname{span}_{\mathbb{C}}\left(\nabla f_1(x),\dots,\nabla f_s(x)\right),$$

This implies that $x \in V$ is a critical point of g for $V(f_1,\dots,f_s)$ such that $g(x) \neq 0$ if and only if there exists $\lambda = (\lambda_1,\dots,\lambda_s) \in \mathbb{C}^s$ such that

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \frac{U_c(x)}{g_0(x)} \begin{bmatrix} \partial_{x_1} g_0(x) \\ \vdots \\ \partial_{x_n} g_0(x) \end{bmatrix} + 2 \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \lambda_1 \begin{bmatrix} \partial_{x_1} f_1(x) \\ \vdots \\ \partial_{x_n} f_1(x) \end{bmatrix} \cdots - \lambda_s \begin{bmatrix} \partial_{x_1} f_s(x) \\ \vdots \\ \partial_{x_n} f_s(x) \end{bmatrix}.$$

Let

$$W := \{(x,t,\lambda) \in V \times \mathbb{C}^{s+1} \mid g(x) \neq 0, t \neq 0\}$$

and define $p_i : W \to \mathbb{C}$ for $i = 1,\dots,n$,

$$p_i(x,t,\lambda) := t\partial_{x_i} g_0(x) + 2x_i - \lambda_1 \partial_{x_i} f_1(x) - \cdots - \lambda_s \partial_{x_i} f_s(x).$$

Thus, $x \in V \setminus V(g)$ is a critical point of g for $V(f_1,\dots,f_s)$ if and only if there exists $(t,\lambda) \in \mathbb{C}^{s+1}$ such that (x,t,λ) satisfies

$$t = \frac{U_c(x)}{g_0(x)} \quad \text{and} \quad p_i(x,t,\lambda) = c_i \quad i = 1,\dots,n.$$

First we prove that for $p = (p_1,\dots,p_n) : W \to \mathbb{C}^n$, p is dominant. For all $x^* \in V$ and for $t = \lambda_1 = \cdots = \lambda_s = 0$ we have

$$J_W p(x^*,0,0) = [2 \cdot I_{n-s} | \nabla g_0(x^*) | -Jf(x^*)],$$

where $J_W p$ is the Jacobian of p in a local parametrization of W at $(x^*, 0, 0)$. By our assumtion on V, $\mathrm{rank} Jf(x^*) = s$, thus $\mathrm{rank} J_W p(x^*, 0, 0) \geq n$. This implies that the image of p is n-dimensional, thus p is dominant. Since W inherits the irreducibility and and smoothness of V, we get that $\overline{p(W)} = \mathbb{C}^n$.

We can apply Theorem 10.3 for p, so there exists a Zariski closed subset $\mathcal{S}$ of $\mathbb{C}^n$ and such that for all $c \in \mathbb{C}^n \setminus \mathcal{S}$ for $W_1 := \{(x, t, \lambda) \in W \mid p(x, t, \lambda) = c\}$ we have

$$\dim(W_1) = \dim(W) - n = 1$$

using that $\dim(W) = n - s + s + 1 = n + 1$ by our assumption that $\dim(V \cap V(g)) < n - s$.

Fix $c \in \mathbb{R}^n \setminus \mathcal{S}$. Next we show that that

$$\dim \{(x, t, \lambda) \in W_1 \ : \ U_c(x) - tg_0(x) = 0\} = 0.$$

If the above dimension is not 0 then 0 is a critical value of the function $q(x, t) := U_c(x) - tg_0(x) \ : \ W_1 \to \mathbb{C}$. If we have such a critical value, then there exists $(x^*, t^*, \lambda^*) \in W_1$ such that $\nabla q(x^*, t^*) = 0$, i.e.

$$[\partial_{x_1} q(x^*, t^*), \ldots \partial_{x_n} q(x^*, t^*), g_0(x^*)] = [0, \ldots, 0, 0]\,.$$

Thus we must have $g_0(x^*) = 0$. However, $W_1 \subset W$, so for $(x^*, t^*, \lambda^*) \in W_1$ we have $g_0(x^*) \neq 0$, a contradiction.

This implies that for any $c \in \mathbb{R}^n \setminus \mathcal{S}$, the solution set of $p_i(x, t, \lambda) = c_i$ for $i = 1, \ldots, n$ and the equation $t = \frac{U_c(x)}{g_0(x)}$ is a zero dimensional subset $Z \subset W$. The set $\{x : (x, t, \lambda) \in Z\}$ is the finite set of critical points of g for $V(f_1, \ldots, f_s)$ in $V \setminus V(g)$.

□

11. Computation of Real Smooth Points - General Case

After introducing all necessary theory and subroutines for our purposes, now we are ready to return to our main topic, computing smooth points on general real algebraic varieties.

We first define two genericity assumptions, informed by our previous results, in particular Proposition 6.6 and Theorem 10.4. Recall that $\mathrm{crit}(V, \pi_i)$ is the polar variety of the algebraic set V with respect to the projection π_i as in Definition 6.1.

Definition 11.1. Consider polynomial $F \in \mathbb{R}[x]$ with $V(F) \cap \mathbb{R}^n$ bounded, matrix $A \in \mathrm{GL_n}(\mathbb{R})$. Define $V^A = V(F^A)$ and $V_e^A := V(F^A - e) \subset \mathbb{C}^n$ for some constant $e > 0$. We say that F and A satisfy Assumption `(B)` if:

- `(1)`: there exists $e_0 > 0$ such that for all $0 < e \leq e_0$ and all $1 \leq i \leq n$, $\mathrm{crit}(V_e^A, \pi_i)$ is either empty or is smooth and equidimensional with complex dimension $i - 1$;
- `(2)`: for all $p \in V^A \cap \mathbb{R}^n$, $\pi_d^{-1}(\pi_d(p)) \cap (V^A \cap \mathbb{R}^n)$ is finite, where d is greater than or equal to the local real dimension of V^A at p;

Definition 11.2. Consider polynomials $F, g \in \mathbb{R}[x]$ and constant $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$. Define $V_e := V(F - e) \subset \mathbb{C}^n$ for some constant $e > 0$. We say that F, g and c satisfy Assumption `(C)` if:

(C): There exists $e_0 > 0$ such that for all $0 < e \le e_0$, all $1 \le i \le n$, the polynomial

$$\overline{g} := g \cdot \left((x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + 1\right) \in \mathbb{R}[x]$$

has finitely many critical points for the polar variety $\mathrm{crit}(V_e, \pi_i)$ where g does not vanish.

The following theorem and corresponding proof establish the correctness of the main algorithm of the paper, Real Smooth Point Algorithm 5.

Theorem 11.3. *Fix $n, i, f_1, \ldots, f_s$ as in the input of* Algorithm 5 *such that $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ is bounded. Assume $A \in GL_n(\mathbb{R})$ such that A and $F = f_1^2 + \cdots + f_s^2$ satisfy Assumption* (B) *as in Definition 11.1. Also, for each $j = 1, \ldots, r$, in Step (4) of* Algorithm 5 *we assume that F^A, g_j and c satisfy Assumption* (C) *as in Definition 11.2. Then* Algorithm 5 *is correct. Furthermore, let Z the output of* Algorithm 5*. If $Z = \emptyset$, then $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ has no connected components of dimension $i - 1$. If $Z \ne \emptyset$, then $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ has some connected components of dimension at least $i - 1$.*

Proof. Fix $1 \le i \le n$. Using the notation $V_e^A := V(F^A - e)$, by Assumption (B), $\mathrm{crit}(V_e^A, \pi_i)$ is smooth and equidimensional of dimension $i - 1$ for all sufficiently small $e > 0$. Only locally in this proof we use the simplified notation

$$V := \lim_{e \to 0} \mathrm{crit}(V_e^A, \pi_i) \subset \mathbb{C}^n$$

without designating its dependence on i or A, otherwise the indices would become too involved. By Proposition 5.5 (over $\mathbb{C}$ instead of $\mathbb{R}$), the set V is a Zariski closed set that is either equidimensional of dimension $i - 1$ or empty. Assume that $\{(g_j, (G_j, L, W_j)) : j = 1, ..., r\}$ satisfies output specifications (i)-(vi) of Algorithm 4. Fix $j \in \{1, \ldots, r\}$ and let $V_j \subset V$ be the union of irreducible components of V with witness set (G_j, L, W_j). First we establish that U_j defined in Step (4a) is finite. We note that $\mathrm{crit}(V_e^A, \pi_i)$ is smooth and equidimensional for all sufficiently small $e > 0$. If $|U_j| = \infty$ then we redefine g_j with a generic $c \in \mathbb{R}^n$. Using Assumption (C), we get that the redefined U_j is finite and the loop will terminate.

Next, since $\dim(V_j \cap V(g_j)) < i - 1$ by (v) in Algorithm 4, either $(V_j \setminus V(g_j)) \cap \mathbb{R}^n = \emptyset$ or for each bounded connected component C of $V_j \cap \mathbb{R}^n$ where g_j is not identically zero, there exists $z \in U_j \cap C$ such that $g_j(z) \ne 0$. Suppose $(V_j \setminus V(g_j)) \cap \mathbb{R}^n \ne \emptyset$. Let $C_1, \ldots, C_t \subset V_j \cap \mathbb{R}^n$ be the bounded connected components of $V_j \cap \mathbb{R}^n$ where g_j is not identically zero. Fix $m \in \{1, \ldots, t\}$. Since each C_m is compact, the distance from C_m to C_k is positive for each $m \ne k$. Also, for all sufficiently small e, $V_e^A \cap \mathbb{R}^n$ is also compact. Since $C_m \subset V \cap \mathbb{R}^n$ is compact, Proposition 5.5 shows that there exist connected components $C_{m,1}^{(e)}, \ldots, C_{m,s_m}^{(e)}$ of $V_e^A \cap \mathbb{R}^n$ for all sufficiently small $e > 0$ such that $C_m = \bigcup_{l=1}^{s_m} \lim_{e \to 0^+} C_{m,l}^{(e)}$, each $C_{m,l}^{(e)}$ is bounded, and since C_m and C_j has positive distance for $m \ne j$, also by Proposition 5.5 we have that

$$\cup_{l=1}^{s_m} C_{m,l}^{(e)} \cap \cup_{l=1}^{s_j} C_{j,l}^{(e)} = \emptyset$$

for all $j \ne m$. For each $l = 1, \ldots, s_m$, let $\mathcal{S}_{m,l}^{(e)} := \pi_x(V(L^{(j)})) \cap C_{m,l}^{(e)}$. By Lemma 3.2, $\mathcal{S}_{m,l}^{(e)} \ne \emptyset$ and it contains all points in $C_{m,l}^{(e)}$ where g_j takes its extreme values. Let $\mathcal{S}_m := \bigcup_{l=1}^{s_m} \lim_{e \to 0} \mathcal{S}_{m,l}^{(e)}$. Since $\mathcal{S}_{m,l}^{(e)}$ is bounded for all sufficiently small e, none of the limit points escape to infinity. Suppose that for all $z \in \mathcal{S}_m$ we have $g_j(z) = 0$. Since C_m is compact, by the Extreme Value Theorem, g_j attains both a minimum and a maximum on C_m. Since g_j is not identically zero on C_m, either the minimum or the maximum value of g_j on C_m must be nonzero. Let $z^* \in C_m$ such that $|g_j(z^*)| > 0$. Let $z_e^* \in C_{m,l}^{(e)}$ for some $l = 1, \ldots, s_m$ such that $\lim_{e\to 0} z_e^* = z^*$. Then for any $z \in \mathcal{S}_m$, if $z_e \in \mathcal{S}_m^{(e)}$ such that $\lim_{e\to 0} z_e = z$, then for sufficiently small e we have that $|g_j(z_e^*)| > |g_j(z_e)|$ by $\lim_{e\to 0} g_j(z_e) = g_j(z) = 0$. Since $\mathcal{S}_m$ is finite, we can choose a common e_0 value for all $z \in \mathcal{S}_m$ so that if $0 < e < e_0$ then $|g_j(z_e^*)| > |g_j(z_e)|$ for all $z_e \in \mathcal{S}_m^{(e)}$. Thus, $\mathcal{S}_m^{(e)}$ could not contain all points of $C_{m,l}^{(e)}$ for $l = 1, \ldots, s_i$ where g_j takes its extreme values, a contradiction. This proves $\pi_x\left(\lim_{e\to 0} V(L_e^{(j)})\right) \cap C_m$ contains a point $z \in C_m$ such that $g_j(z) \neq 0$, i.e. $T_j \cap C_m \neq \emptyset$.

Next, let $Z_j = T_j \cap V_j$ and $Z = \bigcup_{j=1}^r Z_j$ as in Steps (4) and (5). Since $V = \bigcup_{j=1}^r V_j$ and for each $j = 1, \ldots, r$, $\mathrm{Sing}(V_j) \subset V(g_j)$, $V_k \cap V_j \subset V(g_j)$ for all $k \neq j$ by (iii)-(vi) in Algorithm 4, these points are smooth in $V_j \cap \mathbb{R}^n$, and also smooth in $V \cap \mathbb{R}^n$. Thus if $Z \neq \emptyset$, by Theorem 2.4 and Proposition 6.7, $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ must have dimension $\geq i - 1$ connected components. Conversely, if $V(f_1, \ldots, f_s) \cap \mathbb{R}^n = V(F) \cap \mathbb{R}^n$ has a bounded connected component of dimension $i - 1$, then by Proposition 6.7 we have $V(F) \cap \mathbb{R}^n = V$, so there exists $j \in \{1, \ldots, r\}$ such that $V_j \cap \mathbb{R}^n$ has a bounded connected component of dimension $i-1$. By Theorem 2.4, this component has real smooth points. In fact, these real smooth points form a semi-algebraic set that has also dimension $i - 1$. However, since $\dim\left(V_j \cap V(g_j)\right) < i - 1$, g_j does not vanish on all real smooth points of this component, but it vanishes on the singular points. By the above argument $T_j \cap V_j$ must contain points where g_j is not zero, thus Z_j and Z are not empty. □

Example 11.4. Consider $f_1, f_2 \in \mathbb{R}[x_1, x_2, x_3]$ where

$$f_1 = (x^2 + 1)(x^2 + y^2 + z^2 - 1) \text{ and } f_2 = (x^2 + 1)(x + y + z - 1).$$

Clearly, $V(f_1, f_2)$ is not equidimensional, but $V(f_1, f_2) \cap \mathbb{R}^3$ is compact of dimension 1. With $\mathbf{a} = (1, 1)$, the limit variety V is a curve with two irreducible components: $V_1 = V(x^2 + y^2 + z^2 - 1, x + y + z - 1)$ and $V_2 = V(x^2 + 1, x^2 + y^2 + z^2 - x - y - z)$. We utilize Algorithm 5 to compute a smooth point on this real curve. Using $g_1 = x - y$ and $g_2 = x(2y - 1)$, respectively, one obtains $S_1 = \{(1 \pm \sqrt{3}, 1 \mp \sqrt{3}, 1)/3\}$ consisting of two smooth points on $V_1 \cap \mathbb{R}^3$ and $S_2 = \emptyset$.

Using Proposition 9.2, we can bound the number of homotopy paths followed in Step (3) in the Real Smooth Point Algorithm 5, which is the bottleneck of our method. Note that the number of iterations r is at most $\deg(V) \leq D^n$ where $D := \max_{i=1}^s\{\deg(f_i)\}$. Thus the Membership Test Algorithm 1 utilized in Step (4) of the Real Smooth Point Algorithm 5 follows at most $|W_j| = \deg(V_j) \leq \deg(V) \leq D^n$ homotopy paths.

Algorithm 5 REALSMOOTHPOINT

Input: $f = (f_1, \ldots, f_s) \subset \mathbb{R}[x_1, \ldots, x_n]$, $i \in \{1, \ldots, n\}$, $n \geq 2$.

Output: $Z \subset \mathbb{R}^n$, a finite set containing smooth points in each $(i-1)$-dimensional bounded connected component of $V_{\mathbb{R}}(f)$

1. Define $F := f_1^2 + \cdots + f_s^2$ and $e_1 := (1, 0, \ldots, 0)$.
2. Choose generic $A \in GL_n(\mathbb{R})$.
3. $\{(g_1, D_1), \ldots, (g_r, D_r)\} :=$ ComputationOfG$\Big(F^A, \frac{\partial F^A}{\partial x_{i+1}}, \ldots, \frac{\partial F^A}{\partial x_n}, \mathbf{e}_1\Big)$. `// See Algorithm 4`
   `// ` D_j `is a deflated witness set for some` V_j `a union of irreducible components of` $\lim_{e\to 0^+} \mathrm{crit}(V_e^A, \pi_i)$ `where` $\mathrm{crit}(V_e^A, \pi_i) := V(F^A - e, \frac{\partial F}{\partial x_{i+1}}, \ldots, \frac{\partial F}{\partial x_n})$ `for` e `a parameter.`
4. For $j = 1, \ldots, r$
   (a) Loop
   $$L^{(j)} := \left\{F^A, \frac{\partial F^A}{\partial x_{i+1}}, \ldots, \frac{\partial F^A}{\partial x_n}\right\} \cup \left\{\frac{\partial g_j}{\partial x_k} + \lambda_0 \frac{\partial F^A}{\partial x_k} + \sum_{t=i+1}^{n} \lambda_t \frac{\partial^2 F^A}{\partial x_t \partial x_k} : k = 1, \ldots, n\right\}.$$
   `// ` $L^{(j)}$ `is the Lagrange multiplier system in variables` $x_1, \ldots, x_n, \lambda_0, \lambda_1, \ldots, \lambda_s$.
   (flag, U_j) :=WitnessPointsInLimits$\Big(L^{(j)}, \mathbf{e}_1, \emptyset\Big)$. `// See Algorithm 2`
   If flag =TRUE, exit loop.
   Choose generic $c \in \mathbb{R}^n$.
   $\overline{g_j} := g_j \cdot \Big((x_1 - c_1)^2 + \cdots (x_n - c_n)^2 + 1\Big)$.
   Restart loop with $g_j := \overline{g_j}$.
   (b) Compute $T_j := \pi_x(U_j) \setminus V(g_j) \cap \mathbb{R}^n$. `// ` π_x `projection onto` x `coordinates`
   (c) Set $Z_j := \emptyset$.
   (d) For each $p \in T_j$
   If MembershipTest$\Big(p, D_j\Big)$ = TRUE, then `// See Algorithm 1`
   $Z_j := Z_j \cup \{p\}$.
5. Return $Z := \bigcup_{j=1}^r Z_j$.

Corollary 11.5. *Fix $i \in \{1, \ldots, n\}$. Let $f_1, \ldots, f_s \in \mathbb{R}[x], A \in GL_n(\mathbb{R})$ such that A and $F = f_1^2 + \cdots + f_s^2$ satisfy Assumption* (B)*. Assume that for some fixed $j \in \{1, \ldots, r\}$, the zero-dimensional polynomial system $L^{(j)}$ as in* ALGORITHM 5 *Step (4a) is zero-dimensional. Then, the number of complex roots of $L^{(j)}$ is bounded by* $\deg(g_j)^n (2D)^{n-i+1} \leq (n-i+1)^{(k_j+1)n} (2D)^{2n-i+1}$, *where $D := \max_{i=1}^s \{\deg(f_i)\}$, assuming that* $\deg(g_j) \geq 2D$, *and k_j is the number of iterations of the isosingular deflation needed to compute G_j using* ALGORITHM 3.

Proof. In Step 3 we input $n - i + 1$ polynomials of degrees at most $2D$ to Algorithm 4. By Proposition 9.2 with $s = n - i + 1$, we get that $\deg(g_j) \leq (n - i + 1)^{k_j+1}(2D)$ for all $j = 1, \ldots, r$.

Then the defining equations of $L^{(j)}$ include $n - i + 1$ polynomials of degree at most $2D$ and n polynomials of degree at most $\deg(g_j)$ (assuming that $\deg(g_j) \geq 2D$). The Bezout bound for the number of common roots of $L^{(j)}$ gives the claim of the Corollary. □

12. Numerical Real Dimension Algorithm

Our real dimension algorithm is as follows.

Algorithm 6 NumericalRealDimension

Input: $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ not all zero, such that $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ is bounded and $n \geq 2$.

Output: The real dimension of $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$.

1. Let $i := n$.
2. Loop
   (a) $S :=$ RealSmoothPoint$(f_1, \ldots, f_s, i)$. `// See Algorithm 5`
   `//` $S \subset \mathbb{R}^n$ `contains smooth points in` $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$.
   (b) If $S \neq \emptyset$, exit loop and return $i - 1$.
   (c) Increment $i := i - 1$.
   (d) If $i = 0$, exit loop and return -1.

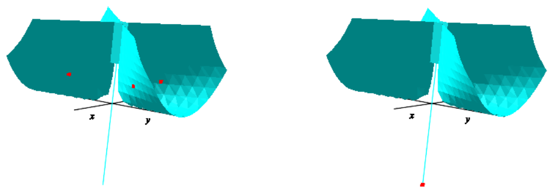
Theorem 12.1. *Let $n \geq 2$, $f_1, \ldots, f_s \in \mathbb{R}[x]$ such that $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ is bounded. Assume that the conditions of Theorem 11.3 are satisfied for $1 \leq i \leq n$. Then* Algorithm 6 *is correct.*

Proof. By assumption, Theorem 11.3 gives the correctness of Real Smooth Point Algorithm 5 in Step (2a). We prove by induction on $n - i < n$ that we have the following loop invariant in Step (2): $\dim(V(f_1, \ldots, f_s) \cap \mathbb{R}^n) \leq i - 1$. This is true when $n - i = 0$. Assume it is true for $0 \leq n - i < n$, i.e. $\dim(V(f_1, \ldots, f_s) \cap \mathbb{R}^n) \leq i - 1$. In Step (2b) if $S \neq \emptyset$ then by Theorem 11.3, $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ has some connected components of dimension at least $i - 1$. By the inductive hypothesis, we get that this dimension must be equal to $i - 1$, so that is the real dimension of $V(f_1, \ldots, f_s)$ that we return and exit the loop. Otherwise, we have $S = \emptyset$. We claim that in this case if $i > 1$ then $\dim(V(f_1, \ldots, f_s) \cap \mathbb{R}^n) < i - 1$. Since $S = \emptyset$, Theorem 11.3 implies that there are no connected components of dimension $i - 1$ in $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$. Again, by the inductive hypothesis, we get that $\dim(V(f_1, \ldots, f_s) \cap \mathbb{R}^n) < i - 1$, maintaining the loop invariant. In particular, if $i = 1$ this implies that $V(f_1, \ldots, f_s) \cap \mathbb{R}^n = \emptyset$, i.e. the dimension is -1 by convension. □

Example 12.2. The Whitney umbrella is a real algebraic set consisting of a 2-dimensional umbrella-like surface with a 1-dimensional handle along the z-axis and defined by $f_1 = x^2 - y^2 z$. Since the surface is not compact, we add $f_2 = x^2 + y^2 + z^2 + w^2 - 4$ following Proposition 2.7. $g = x$ satisfies the requirements of Theorem 3.1 and results in the red smooth points shown in Figure 3(a), confirming the real dimension is two.

To instead determine the local real dimension of the handle of the umbrella, we localize our computations by taking $f_2 = x^2 + y^2 + (z + 1)^2 + w^2 - \frac{1}{4}$. As expected, an optimization using

$g = z + 1$ results in no smooth real points. We return to Step 2 and add the equation defining our next polar variety, namely $f_3 = 2y^3 - 4yz^2 - 4yz$. Optimizing the new system with respect to $g = z + 1$ yields the smooth real point on the handle shown in Figure 3(b), confirming the real dimension of the handle is one.



(a) Dim. 2 Smooth Points (b) Dim. 1 Smooth Point

Figure 3: Whitney umbrella

13. Implementation on a Benchmark Family of Problems

A benchmark family that appears in the papers Bannwarth and Safey El Din (2015) and Lairez and Safey El Din (2021) are hypersurfaces $V(f_n) \subset \mathbb{C}^n$ for $n \geq 3$ such that

$$f_n(x) = \left(\sum_{j=1}^n x_j^2\right)^2 - 4\sum_{j=1}^n \left(x_j x_{j+1}\right)^2 \tag{2}$$

where $x_{n+1} = x_1$. Since f_n is homogeneous, one knows $\dim V(f_n) \cap \mathbb{R}^n = \dim(V(f_n, s_n) \cap \mathbb{R}^n) + 1$ where $s_n = \sum_{j=1}^n x_j^2 - 1$ in which $V(f_n, s_n) \cap \mathbb{R}^n$ is compact. The cases $3 \leq n \leq 6$ were solved in Bannwarth and Safey El Din (2015), which were improved in Lairez and Safey El Din (2021) where they consider the cases $3 \leq n \leq 8$. Here we consider the cases $3 \leq n \leq 9$. All code used in these computations is available at `dx.doi.org/10.7274/r0-5c1t-jw53` with the timings reported using `Bertini` (Bates et al.) on an AMD Opteron 6378 2.4 GHz processor using one (serial) or 64 (parallel) cores.

For $n = 3$ with $g = \partial f_3/\partial x_1$, one obtains smooth points on $V(f_3) \cap \mathbb{R}^3$ thereby showing $\dim V(f_3) \cap \mathbb{R}^3 = 2$ in about a second in serial.

For $n = 4$, $V(f_4)$ has multiplicity 2 with respect to f_4 since

$$f_4(x_1, x_2, x_3, x_4) = \left(x_1^2 - x_2^2 + x_3^2 - x_4^2\right)^2.$$

Trivially, a deflated witness system for $V(f_4)$ is $G = x_1^2 - x_2^2 + x_3^2 - x_4^2$. For $g = x_1x_2$, one obtains smooth points on $V(f_4) \cap \mathbb{R}^4$ showing $\dim V(f_4) \cap \mathbb{R}^4 = 3$ in about a second in serial.

For $n = 5, \ldots, 9$, with $g = \partial f_n/\partial x_1$, one does not obtain smooth points on $V(f_n) \cap \mathbb{R}^n$ showing $\dim V(f_n) \cap \mathbb{R}^n < n - 1$. Therefore, one can move down the dimensions searching for real smooth points using perturbed polar varieties, similarly to Step (2) of Algorithm 6. Nonsingular real points are first found at dimension 2, i.e., $\dim V(f_n) \cap \mathbb{R}^n = 2$. In fact, at dimension 2, the polar variety contains various irreducible components of degree 2 and testing one is enough to confirm the existence of a smooth real point. Table 1 lists the total computation time using parallel processing.

Table 1: Summary of benchmark problem (2) for $5 \leq n \leq 9$

n	$\dim V(f_n) \cap \mathbb{R}^n$	Time (min)
5	2	3.63
6	2	5.73
7	2	34.81
8	2	159.81
9	2	2675.25

Acknowledgments

This paper is dedicated to our friend and coauthor Agnes Szanto who passed away during revisions. The authors thank Mohab Safey El Din, Elias Tsigaridas, and Hoon Hong for many discussions regarding real algebraic geometry and the anonymous reviewers for their excellent comments and suggestions. This research was partly supported by NSF grants CCF-1812746 (Hauenstein) and CCF-1813340 (Szanto and Harris).

References

Aubry, P., Rouillier, F., Safey El Din, M., 2002. Real solving for positive dimensional systems. J. Symbolic Comput. 34, 543–560.

Bank, B., Giusti, M., Heintz, J., Lecerf, G., Matera, G., Solernó, P., 2015. Degeneracy loci and polynomial equation solving. Foundations of Computational Mathematics 15, 159–184.

Bank, B., Giusti, M., Heintz, J., Mandel, R., Mbakop, G.M., 1997. Polar varieties and efficient real equation solving: the hypersurface case, in: Proceedings of the 3rd International Conference on Approximation and Optimization in the Caribbean (Puebla, 1995), Benemérita Univ. Autón. Puebla, Puebla. p. 13.

Bank, B., Giusti, M., Heintz, J., Pardo, L.M., 2004. Generalized polar varieties and an efficient real elimination procedure. Kybernetika (Prague) 40, 519–550.

Bank, B., Giusti, M., Heintz, J., Pardo, L.M., 2009. On the intrinsic complexity of point finding in real singular hypersurfaces. Inform. Process. Lett. 109, 1141–1144.

Bank, B., Giusti, M., Heintz, J., Safey El Din, M., Schost, E., 2010. On the geometry of polar varieties. Applicable Algebra in Engineering, Communication and Computing 21, 33–83.

Bannwarth, I., Safey El Din, M., 2015. Probabilistic algorithm for computing the dimension of real algebraic sets, in: ISSAC'15—Proceedings of the 2015 ACM International Symposium on Symbolic and Algebraic Computation. ACM, New York, pp. 37–44.

Basu, S., Pollack, R., Roy, M.F., 2006a. Algorithms in real algebraic geometry. volume 10 of *Algorithms and Computation in Mathematics*. Second ed., Springer-Verlag, Berlin.

Basu, S., Pollack, R., Roy, M.F., 2006b. Computing the dimension of a semi-algebraic set. Journal of Mathematical Sciences 134, 2346–2353.

Bates, D.J., Hauenstein, J.D., Sommese, A.J., Wampler, C.W., . Bertini: Software for numerical algebraic geometry. Available at `bertini.nd.edu`.

Bates, D.J., Hauenstein, J.D., Sommese, A.J., Wampler, C.W., 2013. Numerically solving polynomial systems with Bertini. volume 25 of *Software, Environments, and Tools*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

Becker, E., Neuhaus, R., 1993. Computation of real radicals of polynomial ideals, in: Computational algebraic geometry (Nice, 1992). Birkhäuser Boston, Boston, MA. volume 109 of *Progr. Math.*, pp. 1–20.

Bernardi, A., Blekherman, G., Ottaviani, G., 2018. On real typical ranks. Boll. Unione Mat. Ital. 11, 293–307.

Chen, C., Davenport, J.H., May, J.P., Moreno Maza, M., Xia, B., Xiao, R., 2013. Triangular decomposition of semi-algebraic systems. J. Symbolic Comput. 49, 3–26.

Collins, G.E., 1975. Quantifier elimination for real closed fields by cylindrical algebraic decompostion, in: Brakhage, H. (Ed.), Automata Theory and Formal Languages, Springer Berlin Heidelberg, Berlin, Heidelberg. pp. 134–183.

Coss, O., Hauenstein, J.D., Hong, H., Molzahn, D.K., 2018. Locating and counting equilibria of the Kuramoto model with rank-one coupling. SIAM J. Appl. Algebra Geom. 2, 45–71.

Elliott, J., Schost, É., 2019. Bit complexity for critical point computation in smooth and compact real hypersurfaces. ACM Communications in Computer Algebra 53, 114–117.

Faugère, J.C., Moroz, G., Rouillier, F., Safey El Din, M., 2008. Classification of the perspective-three-point problem, discriminant variety and real solving polynomial systems of inequalities, in: ISSAC'08 – Proceedings of the 2008 ACM International Symposium on Symbolic and Algebraic Computation, ACM, New York, NY, USA. pp. 79–86.

Fischer, G., 2001. Plane algebraic curves. volume 15 of *Student Mathematical Library*. American Mathematical Society, Providence, RI. Translated from the 1994 German original by Leslie Kay.

Friedland, S., 2012. On the generic and typical ranks of 3-tensors. Linear Algebra and its Applications 436, 478–497.

Grayson, D.R., Stillman, M.E., . Macaulay2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2/`.

Grigor′ev, D.Y., Vorobjov, Jr., N.N., 1988. Solving systems of polynomial inequalities in subexponential time. J. Symbolic Comput. 5, 37–64.

Harris, K.E., 2021. Real Algebraic Geometry with Numerical Homotopy Methods. Ph.D. thesis. North Carolina State University.

Hauenstein, J.D., 2013. Numerically computing real points on algebraic sets. Acta Applicandae Mathematicae 125, 105–119.

Hauenstein, J.D., Sottile, F., 2012. Algorithm 921: alphaCertified: certifying solutions to polynomial systems. ACM Trans. Math. Software 38, Art. 28, 20.

Hauenstein, J.D., Wampler, C.W., 2013. Isosingular sets and deflation. Found. Comput. Math. 13, 371–403.

Hauenstein, J.D., Wampler, C.W., 2017. Unification and extension of intersection algorithms in numerical algebraic geometry. Appl. Math. Comput. 293, 226–243.

Hong, H., Rohal, J., Din, M.S.E., Schost, E., 2020. Connectivity in semi-algebraic sets i. `arXiv:2011.02162`.

Krick, T., Pardo, L.M., Sombra, M., 2001. Sharp estimates for the arithmetic Nullstellensatz. Duke Math. J. 109, 521–598.

Kruskal, J.B., 1989. Rank, decomposition, and uniqueness for 3-way and N-way arrays, in: Multiway data analysis (Rome, 1988). North-Holland, Amsterdam, pp. 7–18.

Kuramoto, Y., 1975. Self-entrainment of a population of coupled non-linear oscillators. Lect. Notes Phys. 39, 420–422.

Lairez, P., Safey El Din, M., 2021. Computing the dimension of real algebraic sets, in: ISSAC 2021 - 46th International Symposium on Symbolic and Algebraic Computation, ACM, New York. pp. 257–264.

Logar, A., 1989. A computational proof of the Noether normalization lemma. Applied Algebra, Algebraic Algorithms and Error-Correcting Codes , 259–273doi:`10.1007/3-540-51083-4_65`.

Ma, Y., Wang, C., Zhi, L., 2016. A certificate for semidefinite relaxations in computing positive-dimensional real radical ideals. J. Symbolic Comput. 72, 1–20.

Marshall, M., 2008. Positive polynomials and sums of squares. volume 146 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI.

Neuhaus, R., 1998. Computation of real radicals of polynomial ideals. II. J. Pure Appl. Algebra 124, 261–280.

Rouillier, F., Roy, M.F., Safey El Din, M., 2000. Finding at least one point in each connected component of a real algebraic set defined by a single equation. Journal of Complexity 16, 716 – 750.

Safey El Din, M., 2007. Testing sign conditions on a multivariate polynomial and applications. Math. Comput. Sci. 1, 177–207.

Safey El Din, M., Schost, E., 2003. Polar varieties and computation of one point in each connected component of a smooth algebraic set, in: Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation, ACM, New York. pp. 224–231.

Safey El Din, M., Spaenlehauer, P.J., 2016. Critical point computations on smooth varieties: degree and complexity bounds, in: Proceedings of the 2016 ACM International Symposium on Symbolic and Algebraic Computation, ACM, New York. pp. 183–190.

Safey El Din, M., Tsigaridas, E., 2013. A probabilistic algorithm to compute the real dimension of a semi-algebraic set. CoRR abs/1304.1928. URL: `http://arxiv.org/abs/1304.1928`, `arXiv:1304.1928`.

Safey El Din, M., Tsigaridas, E., 2018. Personal communication. ICERM - Nonlinear Algebra Program.

Safey El Din, M., Yang, Z.H., Zhi, L., 2018. On the complexity of computing real radicals of polynomial systems, in: ISSAC'18—Proceedings of the 2018 ACM International Symposium on Symbolic and Algebraic Computation. ACM, New York, pp. 351–358.

Safey El Din, M., Yang, Z.H., Zhi, L., 2019. Computing real radicals and S-radicals of polynomial systems. Journal of

Symbolic Computation .
Sommese, A.J., Wampler, II, C.W., 2005. The numerical solution of systems of polynomials. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ. Arising in engineering and science.
Spang, S.J., 2008. A zero-dimensional approach to compute real radicals. Comput. Sci. J. Moldova 16, 64–92.
Vorobjov, N., 1999. Complexity of computing the local dimension of a semialgebraic set. J. Symbolic Comput. 27, 565–579.
Wang, F., 2016. Computation of Real Radical Ideals by Semidefinite Programming and Iterative Methods. Ph.D. thesis. University of Western Ontario.
Wu, W., Reid, G., 2013. Finding points on real solution components and applications to differential polynomial systems, in: ISSAC 2013—Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, ACM, New York. pp. 339–346.
Xin, X., Kikkawa, T., Liu, Y., 2016. Analytical solutions of equilibrium points of the standard Kuramoto model: 3 and 4 oscillators, in: 2016 American Control Conference (ACC), pp. 2447–2452.
Zeng, G., 1999. Computation of generalized real radicals of polynomial ideals. Sci. China Ser. A 42, 272–280.