Certifying reality of projections

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Abstract. Computational tools in numerical algebraic geometry can be used to numerically approximate solutions to a system of polynomial equations. If the system is well-constrained (i.e., square), Newton's method is locally quadratically convergent near each nonsingular solution. In such cases, Smale's alpha theory can be used to certify that a given point is in the quadratic convergence basin of some solution. This was extended to certifiably determine the reality of the corresponding solution when the polynomial system is real. Using the theory of Newton-invariant sets, we certifiably decide the reality of projections of solutions. We apply this method to certifiably count the number of real and totally real tritangent planes for instances of curves of genus 4.

Keywords: Certification, alpha theory, Newton's method, real solutions, numerical algebraic geometry

1 Introduction

For a well-constrained system of polynomial equations f, numerical algebraic geometric tools (see, e.g., [2,12]) can be used to compute numerical approximations of solutions of $f = 0$. These approximations can be certified to lie in a quadratic convergence basin of Newton's method applied to f using Smale's α-theory (see, e.g., [3, Chap. 8]). When the system f is real, α-theory can be used to certifiably determine if the true solution corresponding to an approximate solution is real [6]. That is, one can certifiably decide whether or not every coordinate of a solution is real from a sufficiently accurate approximation. It is often desirable in computational algebraic geometry to instead decide the reality of a projection of a solution of a real polynomial system. In this manuscript, we develop an approach for this situation using Newton-invariant sets [4].

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The paper is organized as follows. Section 2 provides a summary of Smale's α-theory and Newton-invariant sets. Section 3 provides our main results regarding certification of reality of projections. Section 4 applies the method to certifying real and totally real tritangents of various genus 4 curves.

2 Smale's alpha theory and Newton-invariant sets

Our certification procedure is based on the ability to certify quadratic convergence of Newton's method via Smale's α-theory (see, e.g., [3, Chap. 8]) and Newton-invariant sets [4]. This section summarizes these two items following [4].

Assume that $f\colon \mathbb{C}^n \to \mathbb{C}^n$ is an analytic map and consider the Newton iteration map $N_f\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$N_f(x) := \begin{cases} x - Df(x)^{-1}f(x) & \text{if } Df(x) \text{ is invertible,} \\ x & \text{otherwise,} \end{cases}$$

where $Df(x)$ is the Jacobian matrix of f at x. The map N_f is globally defined with fixed points $\{x \in \mathbb{C}^n \mid f(x) = 0 \text{ or } \operatorname{rank} Df(x) < n\}$. Hence, if $Df(x)$ is invertible and $N_f(x) = x$, then $f(x) = 0$.

One aims to find solutions of $f = 0$ by iterating N_f to locate fixed points. To that end, for each $k \geq 1$, define $N_f^k(x) := \underbrace{N_f \circ \cdots \circ N_f}_{k \text{ times}}(x)$.

Definition 1. *A point $x \in \mathbb{C}^n$ is an* approximate solution *of $f = 0$ if there exists $\xi \in \mathbb{C}^n$ such that $f(\xi) = 0$ and $\|N_f^k(x) - \xi\| \leq \left(\frac{1}{2}\right)^{2^k-1} \|x - \xi\|$ for each $k \geq 1$ where $\|\cdot\|$ is the Euclidean norm on $\mathbb{C}^n$. The point ξ is the* associated solution *to x and the sequence $\{N_f^k(x)\}_{k\geq 0}$ converges quadratically to ξ.*

Smale's α-theory provides sufficient conditions for x to be an approximate solution of $f = 0$ via data computable from f and x. We will use approximate solutions to determine characteristics of the corresponding associated solutions using Newton-invariant sets.

Definition 2. *A set $V \subset \mathbb{C}^n$ is called* Newton invariant *with respect to f if $N_f(v) \in V$ for every $v \in V$ and $\lim_{k\to\infty} N_f^k(v) \in V$ for every $v \in V$ such that this limit exists.*

For example, the set $V = \mathbb{R}^n$ is Newton invariant with respect to a real map f. The algorithm presented in Section 3 considers both the set of real numbers as well as other Newton-invariant sets to perform certification together with the following theorem derived from [3, Ch. 8] and [4].

Theorem 1. *Let $f\colon \mathbb{C}^n \to \mathbb{C}^n$ be analytic, let $V \subset \mathbb{C}^n$ be Newton invariant with respect to f, let $x, y \in \mathbb{C}^n$ such that $Df(x)$ and $Df(y)$ are invertible, and let*

$$\alpha(f,x) := \beta(f,x) \cdot \gamma(f,x), \qquad \beta(f,x) := \|x - N_f(x)\| = \|Df(x)^{-1}f(x)\|,$$

$$\gamma(f,x) := \sup_{k\geq 2} \left\| \frac{Df(x)^{-1}D^k f(x)}{k!} \right\|^{\frac{1}{k-1}}, \qquad \delta_V(x) := \inf_{v\in V} \|x - v\|$$

where the norms are the corresponding vector and operator Euclidean norms.

1. *If* $4 \cdot \alpha(f,x) < 13 - 3\sqrt{17}$*, then* x *is an approximate solution of* $f = 0$*.*
2. *If* $100 \cdot \alpha(f,x) < 3$ *and* $20 \cdot \|x - y\| \cdot \gamma(f,x) < 1$*, then* x *and* y *are approximate solutions of* $f = 0$ *with the same associated solution.*
3. *Suppose that* x *is an approximate solution of* $f = 0$ *with associated solution* ξ*.*
   (a) $N_f(x)$ *is also an approximate solution with associated solution* ξ *and*

$$\|x - \xi\| \leq 2\beta(f,x) = 2\|x - N_f(x)\| = 2\|Df(x)^{-1}f(x)\|.$$

   (b) *If* $\delta_V(x) > 2\beta(f,x)$*, then* $\xi \notin V$*.*
   (c) *If* $100 \cdot \alpha(f,x) < 3$ *and* $20 \cdot \delta_V(x) \cdot \gamma(f,x) < 1$*, then* $\xi \in V$*.*

The value $\beta(f,x)$ is the Newton residual. When f is a polynomial system, $\gamma(f,x)$ is a maximum over finitely many terms and thus can be easily bounded above [11]. A similar bound for polynomial-exponential systems can be found in [5]. The value $\delta_V(x)$ is the distance between x and V. The special case of $V = \mathbb{R}^n$ was first considered in [6].

The following procedure from [4], which is based on Theorem 1, certifiably decides if the associated solution of a given approximate solution lies in a given Newton-invariant set V.

Procedure $b = $ **Certify**(f, x, δ_V)
Input A well-constrained analytic system $f\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\gamma(f,\cdot)$ can be computed (or bounded) algorithmically, a point $x \in \mathbb{C}^n$ which is an approximate solution of $f = 0$ with associated solution ξ such that $Df(\xi)^{-1}$ exists, and distance function δ_V for some Newton-invariant set V that can be computed algorithmically.
Output A boolean b which is `true` if $\xi \in V$ and `false` if $\xi \notin V$.
Begin
1. Compute $\beta := \beta(f,x)$, $\gamma := \gamma(f,x)$, $\alpha := \beta \cdot \gamma$, and $\delta := \delta_V(x)$.
2. If $\delta > 2\beta$, **Return** `false`.
3. If $100 \cdot \alpha < 3$ and $20 \cdot \delta \cdot \gamma < 1$, **Return** `true`.
4. Update $x := N_f(x)$ and go to Step 1.

3 Certification of reality

The systems under consideration are well-constrained polynomial systems

$$f(a, b_1, \ldots, b_k, c_1, \ldots, c_\ell, d_1, \ldots, d_\ell) = \begin{bmatrix} g(a) \\ p(a, b_i) \text{ for } i = 1, \ldots, k \\ p(a, c_i) \text{ for } i = 1, \ldots, \ell \\ p(a, d_i) \text{ for } i = 1, \ldots, \ell \end{bmatrix} \tag{1}$$

with variables $a \in \mathbb{C}^m$ and $b_r, c_s, d_t \in \mathbb{C}^q$, and polynomial systems $g\colon \mathbb{C}^m \to \mathbb{C}^u$ and $p\colon \mathbb{C}^{m+q} \to \mathbb{C}^w$ which have real coefficients such that

$$u \leq m \quad \text{and} \quad m + (k + 2\ell)q = u + (k + 2\ell)w. \tag{2}$$

The first condition in (2) yields that a is not over-constrained by g while the second condition provides that the whole system is well-constrained.

Example 1. To illustrate the setup, we consider an example with $m = 3$, $k = 0$, $\ell = 1$, $q = 1$, $u = 1$, and $w = 2$ so that (2) holds, resulting in a well-constrained system of 5 polynomials in 5 variables. Namely, we consider

$$f(a,c,d) = \begin{bmatrix} g(a) \\ \hline p(a,c) \\ \hline p(a,d) \end{bmatrix} = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 - 1 \\ \hline a_1 + (1-c^2)(a_2 c + a_3 c^2) \\ a_1(3c^2-1) + a_2(2c^5 - 4c^3 + 2c - 1) \\ \hline a_1 + (1-d^2)(a_2 d + a_3 d^2) \\ a_1(3d^2-1) + a_2(2d^5 - 4d^3 + 2d - 1) \end{bmatrix}.$$

Since the polynomial system f in (1) has real coefficients, we can use Theorem 1 with $V = \mathbb{R}^n$ where $n = m + (k+2\ell)q = u + (k+2\ell)w$ to certifiably determine if all coordinates of the associated solution are simultaneously real.

Example 2. Let f be the polynomial system with real coefficients considered in Ex. 1 with Newton-invariant set $V = \mathbb{R}^5$. For the points P_1 and P_2, respectively:

$$\left(\frac{1543}{8003} + \frac{\sqrt{-1}}{530485174}, \frac{-34488}{50521} - \frac{\sqrt{-1}}{190996265}, \frac{32768}{46489} - \frac{\sqrt{-1}}{310964547}, \frac{6713}{18120} + \frac{4777\sqrt{-1}}{19088}, \frac{6713}{18120} - \frac{4538\sqrt{-1}}{18133}\right),$$
$$\left(\frac{18245}{111912} - \frac{\sqrt{-1}}{772703930}, \frac{15244}{38793} - \frac{\sqrt{-1}}{307556791}, \frac{27099}{29944} - \frac{\sqrt{-1}}{155308656}, \frac{-44817}{40271} - \frac{\sqrt{-1}}{372454657}, \frac{8603}{8149} + \frac{\sqrt{-1}}{608134511}\right),$$

alphaCertified [6] computed the following information:

j	upper bound of $\alpha(f, P_j)$	$\beta(f, P_j)$	upper bound of $\gamma(f, P_j)$	$\delta_{\mathbb{R}^5}(P_j)$
1	$1.32 \cdot 10^{-5}$	$2.05 \cdot 10^{-8}$	$6.40 \cdot 10^2$	0.35
2	$2.38 \cdot 10^{-4}$	$1.47 \cdot 10^{-8}$	$1.63 \cdot 10^2$	$7.98 \cdot 10^{-9}$

Item 1 of Theorem 1 yields that both points P_1 and P_2 are approximate solutions of $f = 0$. Suppose that ξ_1 and ξ_2, respectively, are the corresponding associated solutions. Items 3b and 3c, respectively, provide that $\xi_1 \notin \mathbb{R}^5$ and $\xi_2 \in \mathbb{R}^5$.

Rather than consider all coordinates simultaneously, the following shows that we can certifiably decide the reality of some of the coordinates.

Theorem 2. *For f as in (1), the set*

$$V = \left\{(a, b_1, \ldots, b_k, c_1, \ldots, c_\ell, \mathrm{conj}(c_1), \ldots, \mathrm{conj}(c_\ell)) \in \mathbb{R}^m \times (\mathbb{R}^q)^k \times (\mathbb{C}^q)^{2\ell}\right\} \quad (3)$$

is Newton invariant with respect to f where conj() *denotes complex conjugate.*

Proof. Suppose that $v = (a, b_1, \ldots, b_k, c_1, \ldots, c_\ell, d_1, \ldots, d_\ell) \in V$ such that the Jacobian matrix $Df(v)$ is invertible. Let $\Delta v = Df(v)^{-1} f(v)$ and write

$$\Delta v = \left[\, \Delta a^T \; \Delta b_1^T \cdots \Delta b_k^T \; \Delta c_1^T \cdots \Delta c_\ell^T \; \Delta d_1^T \cdots \Delta d_\ell^T \,\right]^T.$$

Since f has real coefficients, we know that

$$\mathrm{conj}(\Delta v) = \mathrm{conj}(Df(v)^{-1} f(v)) = Df(\mathrm{conj}(v))^{-1} f(\mathrm{conj}(v)).$$

Since $v \in V$, $\mathrm{conj}(v) = (a, b_1, \ldots, b_k, d_1, \ldots, d_\ell, c_1, \ldots, c_\ell) \in V$. Based on the structure of f, it immediately follows that

$$\mathrm{conj}(\Delta v) = Df(\mathrm{conj}(v))^{-1} f(\mathrm{conj}(v)) = \left[\, \Delta a^T \; \Delta b_1^T \cdots \Delta b_k^T \; \Delta d_1^T \cdots \Delta d_\ell^T \; \Delta c_1^T \cdots \Delta c_\ell^T \,\right]^T.$$

Hence, $\operatorname{conj}(\Delta a) = \Delta a$, $\operatorname{conj}(\Delta b_i) = \Delta b_i$, and $\operatorname{conj}(\Delta c_j) = \Delta d_j$. Thus, it immediately follows that $N_f(v) = v - \Delta v \in V$.

The remaining condition in Defn. 2 follows from the fact that V is closed. □

All that remains to utilize **Certify** is to provide a formula for δ_V.

Proposition 1. *For any* $x = (a, b_1, \ldots, b_k, c_1, \ldots, c_\ell, d_1, \ldots, d_\ell) \in \mathbb{C}^{m+(k+2\ell)q}$ *and* V *as in (3),*

$$\delta_V(x) = \frac{1}{2}\left\| \begin{array}{l} (a - \operatorname{conj}(a), b_1 - \operatorname{conj}(b_1), \ldots, b_k - \operatorname{conj}(b_k), \\ \quad c_1 - \operatorname{conj}(d_1), \ldots, c_\ell - \operatorname{conj}(d_\ell), d_1 - \operatorname{conj}(c_1), \ldots, d_\ell - \operatorname{conj}(c_\ell)) \end{array} \right\|. \quad (4)$$

Proof. The projection of $x = (a, b_1, \ldots, b_k, c_1, \ldots, c_\ell, d_1, \ldots, d_\ell)$ onto V is

$$\begin{array}{l} v = \frac{1}{2}(a + \operatorname{conj}(a), b_1 + \operatorname{conj}(b_1), \ldots, b_k + \operatorname{conj}(b_k), \\ \qquad c_1 + \operatorname{conj}(d_1), \ldots, c_\ell + \operatorname{conj}(d_\ell), d_1 + \operatorname{conj}(c_1), \ldots, d_\ell + \operatorname{conj}(c_\ell)). \end{array}$$

Thus, $\delta_V(x) = \|x - v\|$ which simplifies to (4). □

Example 3. For the polynomial system f considered in Ex. 1, Theorem 2 provides that $V = \{(a, c_1, \operatorname{conj}(c_1)) \in \mathbb{R}^3 \times \mathbb{C} \times \mathbb{C}\}$ is Newton invariant with respect to f. Let ξ_1 be the associated solution of the first point P_1 from Ex. 1. Since $\delta_V(P_1) = 8.88 \cdot 10^{-9}$, we know $\xi_1 \in V$ using the data from Ex. 2 together with Item 3c of Theorem 1, i.e., the first three coordinates of ξ_1 are real and the last two coordinates are complex conjugates of each other. Hence, $\xi_1 \in V \setminus \mathbb{R}^5$.

4 Tritangents

We conclude by applying this new certification method to a problem from real algebraic geometry considered in [8,9]. A *smooth space sextic* is a nonsingular algebraic curve $C \subset \mathbb{P}^3$ which is the intersection of a quadric surface Q and cubic surface Γ. The curve C is a curve of degree 6 and genus 4, and every hyperplane of $\mathbb{P}^3$ intersects C in exactly 6 points (counting multiplicities). The problem considered in [8,9] concerns counting the number of hyperplanes which are tangent to C at all points of intersection.

Definition 3. *A plane* $H \subset \mathbb{P}^3$ *is a* tritangent plane *for* C *if every point in* $C \cap H$ *has even intersection multiplicity.*

In the generic case, each tritangent plane intersects C in 3 points, each with multiplicity 2, and there are a total of 120 complex tritangent planes. For simplicity, we henceforth restrict our attention to the generic case. Each of the 120 tritangent planes can be categorized as either totally real, real, or nonreal.

Definition 4. *A tritangent plane* H *is* real *if it can be expressed as the solution set of a linear equation with real coefficients and* nonreal *otherwise. A real tritangent plane is* totally real *if each point in* $C \cap H$ *is real.*

Example 4. The smooth space sextic curve $C \subset \mathbb{P}^3$ equal to

$$\{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 \mid x_0^2 + x_0x_3 = x_1x_2,\ x_0x_2(x_0 + x_1 + x_3) = x_3(x_1^2 - x_2^2 + x_3^2)\}$$

has 16 real tritangents, 7 of which are totally real, and 104 nonreal tritangents.

4.1 Counting real and totally real tritangents

Gross and Harris [7] prove that the number of real tritangents of a genus 4 curve is either 0, 8, 16, 24, 32, 64 or 120. This number depends only on the topological properties of the real part of the curve, as summarized in Table 4.2.

Example 5. Since the curve C in Ex. 4 has 16 real tritangents, it follows from [7] that the real part of C consists of two connected components.

In contrast, *totally* real tritangents reflect the *extrinsic* geometry of the real part of the curve. Indeed, Kummer [9] recently obtained bounds on the number of totally real tritangents for each real topological type. We will use our certification procedure to prove results that help close the gaps between the theoretical bounds and instances which have actually been realized.

To that end, we formulate a well-constrained parameterized polynomial system of the form (1) as follows. For a generic smooth space sextic $C = Q \cap \Gamma \subset \mathbb{P}^3$, let q and c be quadric and cubic polynomials that define Q and Γ, respectively. By assuming the coordinates are in general position, we solve in affine space by setting the first coordinate equal to 1. In particular, we are seeking $a \in \mathbb{C}^3$, $x_1, x_2, x_3 \in \mathbb{C}^3$, and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^2$ such that

$$f(h, x_1, \lambda_1, x_2, \lambda_2, x_3, \lambda_3) = \begin{bmatrix} p(h, x_1, \lambda_1) \\ p(h, x_2, \lambda_2) \\ p(h, x_3, \lambda_3) \end{bmatrix} = 0 \quad \text{with} \quad p(h, x_i, \lambda_i) = \begin{bmatrix} H(X_i) \\ q(X_i) \\ c(X_i) \\ \begin{bmatrix} \nabla_x H(X_i) \\ \nabla_x q(X_i) \\ \nabla_x c(X_i) \end{bmatrix} \Lambda_i \end{bmatrix} \tag{5}$$

where $H = [1, h] \in \widehat{\mathbb{P}}^3$, $X_i = [1, x_i] \in \mathbb{P}^3$, $\Lambda_i = [1, \lambda] \in \mathbb{P}^2$, and $\nabla_x \zeta([1, x]) \in \mathbb{C}^3$ is the gradient of ζ with respect to x. In particular, f is a system of 18 polynomials in 18 variables with the first 3 polynomials in p enforcing that $X_i \in C \cap H$ and the last 3 polynomials providing that H is tangent to C at X_i. The values of k and ℓ from (1) are dependent on the number of real points in $C \cap H$. A real tritangent H will either have three or one real points in $C \cap H$ corresponding, respectively to totally real tritangents ($k = 3$ and $\ell = 0$) and real tritangents that are not totally real ($k = \ell = 1$).

Remark 1. For generic quadric q and cubic c, the condition $f = 0$ in (5) has $120 \cdot 3! = 720$ isolated solutions where the factor $3! = 6$ corresponds to trivial reorderings. By selecting one point in each orbit, (5) can be used as a parameter homotopy [10], where the parameters are the coefficients of q and c, to compute tritangents for generic smooth space sextic curves.

4.2 Computational results

In the following, we utilize Bertini [1] to numerically approximate the tritangents via a parameter homotopy following Remark 1. After heuristically classifying the tritangents as either totally real, real, or nonreal, we use the results from Section 3 applied to f in (5) to certify the results using alphaCertified [6]. More computational details for applying our approach to the examples that follow can be found at `http://dx.doi.org/10.7274/R0DB7ZW2`. The reported timings are based on using either one (in serial) or all 64 (in parallel) cores of a 2.4GHz AMD Opteron Processor 6378 with 128 GB RAM.

Example 6. For $i = 1, 2$, let $C_i \subset \mathbb{P}^3$ be defined by $q_i = c_i = 0$ where

$$q_1(x) = q_2(x) = x_0x_3 - x_1x_2$$
$$c_1(x) = (25x_0^3 - 24x_0^2x_1 - 89x_0^2x_2 - 55x_0^2x_3 - 14x_1^3 - 31x_1^2x_2 + 86x_1x_2x_3 + 74x_2^2x_3 - 45x_2x_3^2 - 62x_3^3)/100$$
$$c_2(x) = (89x_0^3 - 41x_0^2x_1 - 87x_0x_1^2 - 26x_0x_2^2 - 25x_1^2x_2 + 42x_1^2x_3 + 56x_1x_2^2 + 87x_2^3 - 67x_2x_3^2 - 42x_3^3)/100.$$

We first use a parameter homotopy in Bertini following Remark 1 to numerically approximate the solutions of $f = 0$ in (5). Each of these instances took approximately 45 seconds in serial and 1.5 seconds in parallel to compute all numerical solutions to roughly 50 correct digits. Converting to rational numbers and applying alphaCertified to each instance shows that all numerical approximations computed by Bertini are approximate solutions in roughly 33 minutes using rational arithmetic with serial processing.

First, we certify that we have indeed computed 120 distinct tritangents up to the action of reordering. This is accomplished by comparing the pairwise distances between the h coordinates corresponding to the tritangent hyperplane with the known error bound 2β from Item 3a of Theorem 1. In both of our examples, $2\beta < 10^{-54}$ while the pairwise distances were larger than 10^{-2} showing that 120 distinct tritangents were computed as expected.

Second, we compare the size of the imaginary parts of the h coordinates with the error bound 2β to certifiably determine which are nonreal tritangets. For both cases, this proves that there are 104 nonreal tritangents leaving 16 tritangents requiring further investigation.

Third, we apply **Certify** with $V = \mathbb{R}^{18}$ to certifiably determine the number of totally real tritangents. This proves that C_1 and C_2 have exactly 0 and 16 totally real tritangents, respectively.

The only remaining item is to show that the 16 tritangents for C_1 are real which follows from our new results in Section 3. We reorder the intersection points so that the first one has the smallest imaginary part and apply **Certify** with V as in (3) where $k = 1$ and $\ell = 1$, i.e., one real intersection point and a pair of complex conjugate intersection points.

In summary, these computations prove that both C_1 and C_2 have 16 real tritangents, where none and all of these 16 are totally real, respectively.

Example 6 provides two new instances of results that had not been realized in [8]. Combining these two examples together with results from [8,9] shows that any number between 0 and 16 totally real tritangents can be realized for a smooth sextic curve which has 16 real tritangents. In Table 4.2 we summarize the theoretical bounds from [9] for the number of totally real tritangents, together with the values that are realized in [8] and our computations (including the computations we describe below). In particular, the bold numbers show new results we obtained using our certification approach. Only 4 open cases remain to be realized or shown to be impossible: 120 real tritangents with between 80 and 83 totally real tritangents.

# real tritangents [7]	# connected real components	dividing type?	range of # totally real [9]	realized # totally real ([8] & our results)
0	0	No	[0,0]	[0,0]
8	1	No	[0,8]	[0,8]
16	2	No	[0,16]	[**0**,**16**]
24	3	Yes	[0,24]	[0,**24**]
32	3	No	[8,32]	[**8**,32]
64	4	No	[32,64]	[**32**,**64**]
120	5	Yes	[80,120]	[84,120]

Table 1. Summary of results for tritangents of genus 4 curves with **bold** numbers showing the new results obtained using our certification approach.

Typically, our computations to generate these results started with the Cayley cubic $c = -x_0^2x_2 + x_0^2x_3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_3 - x_3^3$ and selected quadrics q which intersected various real components of the Cayley cubic surface Γ defined by c. We then randomly perturbed all of the coefficients of q and c to locally explore the surrounding area of the parameter space of the selected instance. As in Ex. 6, Bertini was used to compute numerical approximations of the solutions with certification provided by alphaCertified.

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