Certified predictor-corrector tracking for Newton homotopies

Jonathan D. Hauenstein* Alan C. Liddell, Jr. †

July 1, 2015

Abstract

We develop certified tracking procedures for Newton homotopies, which are homotopies for which only the constant terms are changed. For these homotopies, our certified procedures include using a constant predictor with Newton corrections, an Euler predictor with no corrections, and an Euler predictor with Newton corrections. In each case, the predictor is guaranteed to produce a point in the quadratic convergence basin of Newton's method. We analyze the complexity of a tracking procedure using a constant predictor with Newton corrections, with the number of steps bounded above by a constant multiple of the length of the path in the γ-metric. Examples are included to compare the behavior of these certified tracking methods.

Introduction

The ability to track solution paths defined by a homotopy has applications in many areas of science and engineering. Software for numerically solving polynomial systems by tracking solution paths, such as Bertini [1], HOM4PS-2 [11], and PHCpack [21], utilize heuristic computations in their tracking procedures. These software packages employ various predictor-corrector tracking methods [2], such as an RKF45 predictor and several Newton corrections. For NAG4M2 [12], scripts described in [3] can certifiably track paths using a constant predictor and one Newton correction. Some examples of other certified tracking methods using similar strategies include [4, 5, 16, 17]. Towards the goal of developing certifiable tracking procedures which utilize higher-order predictor-corrector strategies, we reduce the gap between the predictor-corrector strategies used in practice and the certifiable tracking methods by developing certifiable methods using an Euler predictor and Newton corrections for so-called Newton homotopies. The path tracking certificate is developed via Smale's α-theory [19].

For an analytic system $f : \mathbb{C}^N \to \mathbb{C}^N$ and a vector $v \in \mathbb{C}^N$, we consider Newton homotopies of the form

$$H(x, t) = f(x) + t \cdot v \tag{1}$$

For a given point $x_1 \in \mathbb{C}^N$, one can take $v = -f(x_1)$ and consider tracking the solution path of $H(x, t) = 0$ starting at $t = 1$ with $x = x_1$.

*Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN 46556 (hauenstein@nd.edu, `www.nd.edu/~jhauenst`). This author was partly supported by DARPA Young Faculty Award, DMS-1262428, NSF ACI-1460032, and Sloan Research Fellowship.

†Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN 46556 (aliddel1@nd.edu, `www.nd.edu/~aliddel1`). This author was partly supported by DMS-0943855.

Newton homotopies can be used for various computations. For example, Newton homotopies have been used to sample stationary points for potential energy landscapes [14] and for solving problems in economics [10]. Newton homotopies can also be used to perform monodromy loops. Such loops are useful in computing numerical irreducible decompositions [20] and monodromy groups of parameterized problems [13]. In these cases, the occurrence of a single error in tracking can lead to incorrect output. One can computationally *prove* theorems by using a certified tracking approach or an *a posteriori* certification scheme [7].

For $N \geq 1$ and $d \in (\mathbb{Z}_{\geq 1})^N$, let $\mathcal{H}_{N,d}$ be the space of polynomial systems $f : \mathbb{C}^N \to \mathbb{C}^N$ such that $\deg f_i = d_i$. Consider the probability distribution on $\mathcal{H}_{N,d}$ where the coefficients are independent complex Gaussian with mean 0 and variance 1 and take $x_1 = 0$. Then, with probability one, the path $\phi : [0,1] \to \mathbb{C}^N$ defined by $\phi(1) = 0$ and

$$H(\phi(t), t) = f(\phi(t)) - t \cdot f(0) \equiv 0$$

is continuous and smooth with $Df(\phi(t))$ being invertible for all $t \in [0,1]$. In particular, $\phi(0)$ is a nonsingular solution of $f(x) = 0$. Therefore, one obtains a numerical approximation of a solution of $f(x) = 0$ by approximately following the curve $\phi(t)$ from $t = 1$ to $t = 0$. Such a procedure for computing one solution of $f(x) = 0$ which, in the univariate case, i.e., $N = 1$, was used in [18].

Our contributions below include the certified tracking procedures `NewtonTracker`, `EulerTracker`, and `EulerNewtonTracker`. On the complexity side, our main contribution is Theorem 4 which shows that the number of predictor-corrector steps used by a modification of `NewtonTracker` is bounded above by $352\mathcal{L}$, where $\mathcal{L}$ is the length of the path in the γ-metric. In [15], the condition metric (or μ_{norm}-metric) was used to describe the complexity of path tracking for polynomial systems. Since, for nonlinear polynomial systems, μ_{norm} is an upper bound on γ, the γ-metric path length is shorter than the condition metric path length.

In Section 1, we review and summarize the necessary α-theory underpinning our results. In Section 2, we present `NewtonTracker` and describe our complexity bound in Section 3. In Section 4, we present `EulerTracker` and `EulerNewtonTracker`. We present examples in Section 5 demonstrating the practicality of these procedures, including comparisons with [3]. Appendix A contains tables of data.

1 Smale's α-theory

For an analytic system $f : \mathbb{C}^N \to \mathbb{C}^N$, α-theory provides sufficient conditions to prove that a point $x \in \mathbb{C}^N$ is in the quadratic convergence basin for Newton's method of some solution of $f = 0$. The following provides the necessary background information needed to develop our certified tracking algorithms with expanded details provided in [6, Ch. 8].

Let $Df(x)$ be the Jacobian matrix of f evaluated at x and consider $N_f : \mathbb{C}^N \to \mathbb{C}^N$ defined by

$$N_f(x) = \begin{cases} x - Df(x)^{-1} f(x) & Df(x) \text{ is invertible} \\ x & \text{otherwise.} \end{cases}$$

The map N_f defines a *Newton iteration* of f and, for $k \in \mathbb{N}$, the map

$$N_f^k = \underbrace{N_f \circ \cdots \circ N_f}_{k \text{ times}}$$

defines the k^{th} *Newton iteration* of f.

Definition 1 For an analytic system $f : \mathbb{C}^N \to \mathbb{C}^N$, a point $x \in \mathbb{C}^N$ is an *approximate solution* of $f = 0$ if there exists $\xi \in \mathbb{C}^N$ such that $f(\xi) = 0$ and

$$\|N_f^k(x) - \xi\| \leq \left(\frac{1}{2}\right)^{2^k - 1} \|x - \xi\|$$

for each $k \in \mathbb{N}$. In this case, the point ξ is called the *associated solution* of x and the sequence $\{N_f^k(x) \mid k \in \mathbb{N}\}$ *converges quadratically* to ξ.

If $Df(x)$ is invertible, the α-theoretic sufficient condition for certifying that x is an approximate solution of $f = 0$ is based on the following constants:

$$\begin{array}{rcl} \alpha(f, x) & = & \beta(f, x) \cdot \gamma(f, x), \\ \beta(f, x) & = & \|x - N_f(x)\|, \\ \gamma(f, x) & = & \sup\limits_{k \geq 2} \left\| \dfrac{Df(x)^{-1} D^k f(x)}{k!} \right\|^{\frac{1}{k-1}}. \end{array}$$

If $Df(x)$ is not invertible, we take $\beta(f, x) = 0$ and $\gamma(f, x) = \infty$. In this case, the product $0 \cdot \infty$ is defined based on $f(x)$, namely $\alpha(f, x) = 0$ if $f(x) = 0$ and $\alpha(f, x) = \infty$ if $f(x) \neq 0$.

The following version of Theorem 2 from [6, pg. 160] certifies that a given point x is an approximate solution.

Theorem 2 *If $f : \mathbb{C}^N \to \mathbb{C}^N$ is an analytic system and $x \in \mathbb{C}^N$ such that*

$$\alpha(f, x) \leq \alpha_0 = \frac{13 - 3\sqrt{17}}{4} \approx 0.1576707, \tag{2}$$

then x is an approximate solution of $f = 0$. Moreover, $\|x - \xi\| \leq 2\beta(f, x)$ where ξ is the associated solution of x.

2 Newton tracking

Traditional α-theoretic certified tracking schemes, such as [3, 4, 16], use a constant predictor with a corrector consisting of one Newton iteration. Here, we derive a certified tracking scheme that uses a constant predictor and a corrector consisting of potentially multiple Newton iterations per step. The following scheme ensures that the constant predictor lands within a quadratic convergence basin of Newton's method. Depending on local information, in may then be necessary to apply, say, K Newton iterations to attempt to minimize the total number of Newton iterations used throughout the entire tracking process. In particular, as K increases, the next stepsize can be made larger, but there is a diminishing return on the computational investment. Before deciding the number of Newton iterations at each step, we first provide an upper bound on the stepsize for the constant predictor to certifiably yield an approximate solution.

Theorem 3 *Let $f : \mathbb{C}^N \to \mathbb{C}^N$ be an analytic system, $v \in \mathbb{C}^N$, and consider the Newton homotopy $H(x, t) = f(x) + t \cdot v$. Suppose that $x^* \in \mathbb{C}^N$ and $t^* \in [0, 1]$ such that $Df(x^*)$ is invertible and $\alpha(H(\cdot, t^*), x^*) \leq \alpha_0$ where α_0 is defined in (2). If*

$$|\Delta t| \leq \frac{\alpha_0 - \alpha(H(\cdot, t^*), x^*)}{\gamma(f, x^*) \cdot \|Df(x^*)^{-1} \cdot v\|}, \tag{3}$$

then $\alpha(H(\cdot, t^* + \Delta t), x^*) \leq \alpha_0$, that is, x^* is an approximate solution of $H(x, t^* + \Delta t) = 0$.

Proof. Clearly, $\gamma(H(\cdot,t^*+\Delta t),x^*) = \gamma(H(\cdot,t^*),x^*) = \gamma(f,x^*)$. Moreover,

$$\begin{aligned}\beta(H(\cdot,t^*+\Delta t),x^*) &= \|Df(x^*)^{-1}H(x^*,t^*+\Delta t)\| \\ &= \|Df(x^*)^{-1}H(x^*,t^*) + Df(x^*)^{-1}\cdot v\cdot \Delta t\| \\ &\le \beta(H(\cdot,t^*),x^*) + \|Df(x^*)^{-1}\cdot v\|\cdot|\Delta t|.\end{aligned}$$

Multiplying both sides by $\gamma(f,x^*)$, we have

$$\alpha(H(\cdot,t^*+\Delta t),x^*) \le \alpha(H(\cdot,t^*),x^*) + \gamma(f,x^*)\cdot\|Df(x^*)^{-1}\cdot v\|\cdot|\Delta t|.$$

By (3), we have that $\alpha(H(\cdot,t^*+\Delta t),x^*) \le \alpha_0$. □

After using the constant predictor with Δt satisfying (3), one then applies $K \ge 0$ Newton iterations to the system $H(x,t^*+\Delta t) = 0$ starting at x^*. Intuitively, one expects that $\alpha_0 - \alpha(H(\cdot,t^*+\Delta t),N_f^K(x^*))$ will increase and converge quadratically to α_0 as K increases whereas $\gamma(H(\cdot,t^*),N_f^K(x^*))\cdot\|Df(N_f^K(x^*))^{-1}\cdot v\|$ should remain roughly constant. Taking these as assumptions along with assuming that

$$\alpha(H(\cdot,t^*+\Delta t),N_f^K(x^*)) = \frac{\alpha(H(\cdot,t^*+\Delta t),x^*)}{2^{2^K-1}},$$

we want to determine $K \ge 0$ that minimizes the total number of Newton iterations per length of the step. For simplicity, we abbreviate $\alpha = \alpha(H(\cdot,t^*+\Delta t),x^*)$. Thus, we want to maximize $K \ge 0$ such that

$$\frac{K+1}{\alpha_0 - \alpha/2^{2^K-1}} < \frac{j+1}{\alpha_0 - \alpha/2^{2^j-1}}$$

for all $j = 0,1\ldots,K-1$. We note the additional one in the numerator arises since one Newton iteration must be performed to compute α. When $j = K-1$, we need

$$\frac{K+1}{\alpha_0 - \alpha/2^{2^K-1}} < \frac{K}{\alpha_0 - \alpha/2^{2^{K-1}-1}}.$$

This is equivalent to

$$\alpha_0 < \alpha\cdot\left(\frac{K+1}{2^{2^{K-1}-1}} - \frac{K}{2^{2^K-1}}\right).$$

Since $\alpha \le \alpha_0$, we must have $K = 1$ or $K = 2$. With this, our analysis suggests

$$K = \begin{cases} 2 & \text{if } 4\alpha_0/5 < \alpha(H(\cdot,t^*+\Delta t),x^*) \le \alpha_0 \\ 1 & \text{if } 2\alpha_0/3 < \alpha(H(\cdot,t^*+\Delta t),x^*) \le 4\alpha_0/5 \\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

That is, the number of Newton corrections performed depends on $\alpha(H(\cdot,t^*+\Delta t),x^*)$.

This suggests the procedure `NewtonTracker`, with each step yielding an approximate solution of $H(x,t^*+\Delta t) = 0$ via Theorem 3.

Procedure 1 `NewtonTracker`

Input: An analytic function $f : \mathbb{C}^N \to \mathbb{C}^N$, a vector $v \in \mathbb{C}^N$, and a point $x_1 \in \mathbb{C}^N$ with $\alpha(f(\cdot) + v, x_1) \le \alpha_0$.

Output: A point $x_0 \in \mathbb{C}^N$ with $\alpha(f, x_0) \le \alpha_0$.

Define $H(x,t) := f(x) + t \cdot v$.
Initialize $x^* := x_1$ and $t^* := 1$.
while $t^* > 0$ **do**
    if $\alpha(H(\cdot, t^*), x^*) > 4\alpha_0/5$ **then**
        Set $x^* := N^2_{H(\cdot,t^*)}(x^*)$.
    else if $\alpha(H(\cdot, t^*), x^*) > 2\alpha_0/3$ **then**
        Set $x^* := N_{H(\cdot,t^*)}(x^*)$.
    Compute $\Delta t := \dfrac{\alpha_0 - \alpha(H(\cdot, t^*), x^*)}{\gamma(f, x^*) \cdot \|Df(x^*)^{-1} \cdot v\|}$.
    if $\Delta t > t^*$ **then**
        Set $t^* := 0$.
    else
        Set $t^* := t^* - \Delta t$.
return $x_0 := x^*$.

3 Complexity analysis for Newton tracking

For a Newton homotopy $H(x,t) = f(x) + t \cdot v$, we have $\|\dot{x}(t)\| = \|Df(x(t))^{-1} \cdot v\|$ along a solution path $x(t)$. Since the stepsizes for `NewtonTracker` are inversely proportional to the product $\gamma(f, x^*) \cdot \|\dot{x}(t)\|$, the length of the path in the γ-metric, namely

$$\int_0^1 \gamma(f, x(t)) \cdot \|\dot{x}(t)\| dt = \int_0^1 \gamma(f, x(t)) \cdot \|Df(x(t))^{-1} \cdot v\| dt, \tag{5}$$

is directly related to the number of predictor-corrector steps used when tracking the path. In Theorem 4, we make this statement precise by bounding the number of predictor-corrector steps in terms of the length in the γ-metric for the following modification of the `NewtonTracker` procedure, called `ModifiedNewtonTracker`.

There are two differences between `NewtonTracker` and `ModifiedNewtonTracker`. The first is that `ModifiedNewtonTracker` enforces an upper bound, $\alpha_0/6$ in this case, on the value of $\alpha(H(\cdot, t^*), x^*)$ before one performs a step. If z^* is the associated solution of x^* with respect to $H(x, t^*) = 0$ such that $Df(z^*)$ is invertible, then one could simply replace this while loop with a fixed number of Newton iterations to enforce this upper bound. Thus, we can still consider the complexity of `ModifiedNewtonTracker` in terms of the number of predictor-corrector steps.

The second is that the stepsize Δt is selected so that the proof of Theorem 3 yields

$$\alpha(H(\cdot, t^* + r), x^*) \le \alpha_0/3 \quad \text{for} \quad |r| \le |\Delta t|.$$

These changes imply that

$$\Delta t \cdot \gamma(f, x^*) \cdot \|Df(x^*)^{-1} \cdot v\| \ge \alpha_0/6,$$

which produces the following complexity result.

Procedure 2 `ModifiedNewtonTracker`

Input: An analytic function $f : \mathbb{C}^N \to \mathbb{C}^N$, a vector $v \in \mathbb{C}^N$, and a point $x_1 \in \mathbb{C}^N$ with $\alpha(f(\cdot) + v, x_1) \leq \alpha_0$.
Output: A point $x_0 \in \mathbb{C}^N$ with $\alpha(f, x_0) \leq \alpha_0$.

1: Define $H(x, t) := f(x) + t \cdot v$.
2: Initialize $x^* := x_1$ and $t^* := 1$.
3: **while** $t^* > 0$ **do**
4:     **while** $\alpha(H(\cdot, t^*), x^*) > \alpha_0/6$ **do**
5:         Update $x^* := N_{H(\cdot,t^*)}(x^*)$.
6:     Compute $\Delta t := \dfrac{\alpha_0/3 - \alpha(H(\cdot, t^*), x^*)}{\gamma(f, x^*) \cdot \|Df(x^*)^{-1} \cdot v\|}$.
7:     **if** $\Delta t > t^*$ **then**
8:         Set $t^* := 0$.
9:     **else**
10:         Set $t^* := t^* - \Delta t$.
11: **return** $x_0 := x^*$.

Theorem 4 *Let $f : \mathbb{C}^N \to \mathbb{C}^N$ be analytic, $v \in \mathbb{C}^N$, and $H(x, t) = f(x) + t \cdot v$. Suppose that the* `ModifiedNewtonTracker` *procedure terminates when tracking a homotopy path $x(t)$ for $0 \leq t \leq 1$ where Df is invertible along the path. If $\mathcal{L}$ is the length of the path in the γ-metric given in (5), then the number of predictor-corrector steps needed to obtain an approximate solution x_0 for $f = 0$ is bounded above by $352\mathcal{L}$.*

Proof. Suppose that `ModifiedNewtonTracker` took P steps with, say,

$$1 = t_0 > t_1 > \cdots > t_{P-1} > t_P = 0.$$

For $i = 0, \ldots, P-1$, define $\Delta t_i = t_i - t_{i+1} > 0$ and let y_i be the approximate solution of $H(x, t_i) = 0$ computed after the loop in Line 4 is completed. We know

$$\mathcal{L} = \sum_{i=0}^{P-1} \int_{t_{i+1}}^{t_i} \gamma(f, x(t)) \cdot \|Df(x(t))^{-1} \cdot v\| dt \geq \sum_{i=0}^{P-1} \Delta t_i \min_{t_{i+1} \leq t \leq t_i} \gamma(f, x(t)) \cdot \|Df(x(t))^{-1} \cdot v\|.$$

Fix $0 \leq i \leq P - 1$ and suppose that $t \in [t_{i+1}, t_i]$. Define

$$v_i(t) = \|y_i - x(t)\| \cdot \gamma(f, x(t)) \quad \text{and} \quad u_i(t) = \|y_i - x(t)\| \cdot \gamma(f, y_i).$$

Assuming that $v_i(t)$ and $u_i(t)$ are less than $1 - \sqrt{2}/2$. Proposition 3 of [6, Ch. 8] provides

$$\gamma(f, y_i) \leq \frac{\gamma(f, x(t))}{(1 - v_i(t))(1 - 4v_i(t) + 2v_i(t)^2)}$$

and

$$\gamma(f, x(t)) \leq \frac{\gamma(f, y_i)}{(1 - u_i(t))(1 - 4u_i(t) + 2u_i(t)^2)}.$$

Since $\|y_i - x(t)\| \leq 2\beta(H(\cdot, t), y_i)$, we know

$$u_i(t) \leq 2\alpha(H(\cdot, t), y_i) \leq 2\alpha_0/3 < 1 - \sqrt{2}/2$$

which implies

$$v_i(t) \le \frac{2\alpha_0/3}{(1-u_i(t))(1-4u_i(t)+2u_i(t)^2)} \le \frac{41}{210} < 1-\sqrt{2}/2.$$

Therefore,

$$\gamma(f,x(t)) \ge \gamma(f,y_i)(1-v_i(t))(1-4v_i(t)+2v_i(t)^2).$$

Similarly, Lemma 2 of [6, Ch. 8] provides that

$$\begin{aligned} \|Df(y_i)^{-1}\cdot v\| &\le \|Df(y_i)^{-1}Df(x(t))\|\cdot\|Df(x(t))^{-1}\cdot v\| \\ &\le \frac{(1-v_i(t))^2}{1-4v_i(t)+2v_i(t)^2}\|Df(x(t))^{-1}\cdot v\|. \end{aligned}$$

Hence,

$$\begin{aligned} \gamma(f,x(t))\cdot\|Df(x(t))^{-1}\cdot v\| &\ge \frac{(1-4v_i(t)+2v_i(t)^2)^2}{1-v_i(t)}\gamma(f,y_i)\cdot\|Df(y_i)^{-1}\cdot v\| \\ &\ge \frac{4}{37}\gamma(f,y_i)\cdot\|Df(y_i)^{-1}\cdot v\|. \end{aligned}$$

Therefore,

$$\mathcal{L} \ge \frac{4}{37}\sum_{i=0}^{P-1}\Delta t_i\cdot\gamma(f,y_i)\cdot\|Df(y_i)^{-1}\cdot v\| \ge \frac{4\cdot P\cdot\alpha_0}{37\cdot 6}$$

which shows that $P \le \frac{111\mathcal{L}}{2\alpha_0} \le 352\mathcal{L}$. □

4 Euler-Newton tracking

In `NewtonTracker`, the stepsize was determined so that the constant predictor would yield an approximate solution. In this section, we determine a stepsize to guarantee that the Euler predictor will yield an approximate solution. To that end, we first demonstrate that this is equivalent to having the point arising after using a constant predictor and performing one Newton iteration is an approximate solution. This immediately yields that the certifiable stepsize for an Euler predictor can be at least as large as the constant predictor. In addition to showing a small increase in stepsize in Theorem 5, this shows that one can certifiably track a path of a Newton homotopy using only Euler predictions.

4.1 Equivalence of Newton and Euler

Let $f:\mathbb{C}^N\to\mathbb{C}^N$ be an analytic system, $v\in\mathbb{C}^N$, and consider the Newton homotopy $H(x,t) = f(x)+t\cdot v$. Suppose that $x^*\in\mathbb{C}^N$ and $t^*\in[0,1]$ such that $H(x^*,t^*)=0$ and $Df(x^*)$ is invertible. For $\Delta t\neq 0$, the resulting point using a constant predictor, moving from t^* to $t^*+\Delta t$, followed by a Newton iteration at $t^*+\Delta t$ is

$$N_{H(\cdot,t^*+\Delta t)}(x^*) = x^* - Df(x^*)^{-1}H(x^*,t^*+\Delta t) = N_{H(\cdot,t^*)}(x^*) - Df(x^*)^{-1}\cdot v\cdot\Delta t. \quad (6)$$

Now, we consider the same setup using a Euler predictor without a corrector. To that end, we first define an Euler predictor step which arises from the approximation:

$$H(x+\Delta x, t+\Delta t) \approx H(x,t) + Df(x)\cdot\Delta x + v\cdot\Delta t.$$

Upon setting the right-hand side equal to zero and solving for Δx, we have that the *Euler prediction* of H from t^* to $t^* + \Delta t$ is

$$\begin{array}{rcccl} E_{H,t^*,\Delta t}(x^*) & = & x^* - Df(x^*)^{-1}(H(x^*,t^*) + v \cdot \Delta t) & = & x^* - Df(x^*)^{-1}H(x^*, t^* + \Delta t) \\ & = & N_{H(\cdot,t^*+\Delta t)}(x^*). & & \end{array}$$

That is, for Newton homotopies, a constant predictor with one Newton iteration is the same as using an Euler predictor with no corrections.

4.2 Certified tracking using Euler and Newton

The key distinction between what is presented here and `NewtonTracker` is the requirement on the prediction step. In `NewtonTracker`, the constant predictor is constrained to yield a point in the quadratic convergence basin and then applies Newton iterations to move closer to the corresponding solution. In the two tracking procedures below, `EulerTracker` and `EulerNewtonTracker`, the Euler predictor is constrained to certifiably produce a point in the quadratic convergence basin followed by Newton iterations to move closer to the corresponding solution. Thus, by the equivalence provided above and the results presented in the preceding sections, we show how to use α-theory to certify that the result of a Newton iteration applied to a point x^* is an approximate solution where x^* need not *a priori* be an approximate solution.

Theorem 5 *Let $f : \mathbb{C}^N \to \mathbb{C}^N$ be an analytic system, $v \in \mathbb{C}^N$, and consider the Newton homotopy $H(x,t) = f(x) + t \cdot v$. Suppose that $x^* \in \mathbb{C}^N$ and $t^* \in [0,1]$ such that $Df(x^*)$ is invertible and $\alpha(H(\cdot,t^*),x^*) \le \alpha_0$ where α_0 is defined in (2). If*

$$|\Delta t| \le \frac{e_0 - \alpha(H(\cdot,t^*),x^*)}{\gamma(f,x^*) \cdot \|Df(x^*)^{-1} \cdot v\|} \quad \text{where} \quad e_0 = \frac{1 + 4\sqrt{\alpha_0} - \sqrt{8\alpha_0 + 8\sqrt{\alpha_0} + 1}}{4\sqrt{\alpha_0}} \approx 0.161405 \qquad (7)$$

then $\alpha(H(\cdot,t^*+\Delta t), E_{H,t^*,\Delta t}(x^*)) \le \alpha_0$, that is, $E_{H,t^*,\Delta t}(x^*) = N_{H(\cdot,t^*+\Delta t)}(x^*)$ is an approximate solution of $H(x, t^* + \Delta t) = 0$.

Proof. Define $y(\Delta t) = E_{H,t^*,\Delta t}(x^*) = x^* - Df(x^*)^{-1}(H(x^*,t^*) + v \cdot \Delta t)$. First, we need to ensure that $Df(y(\Delta t))$ is invertible. By Lemma 2(a) of [6, Ch. 8], $Df(y(\Delta t))$ is invertible if $\|x^* - y(\Delta t) \cdot \|\gamma(f,x^*) < c_0 := 1 - \sqrt{2}/2$. Since $\alpha(H(\cdot,t^*),x^*) \le \alpha_0$, we enforce

$$|\Delta t| < \frac{c_0 - \alpha_0}{\gamma(f,x^*) \cdot \|Df(x^*)^{-1} \cdot v\|}$$

so that

$$\begin{array}{rcl} \|x^* - y(\Delta t)\|\gamma(f,x^*) & \le & \gamma(f,x^*)(\|Df(x^*)^{-1}H(x^*,t^*)\| + |\Delta t| \cdot \|Df(x^*)^{-1} \cdot v\|) \\ & = & \alpha(H(\cdot,t^*),x^*) + |\Delta t| \cdot \gamma(f,x^*) \cdot \|Df(x^*)^{-1} \cdot v\| \\ & < & \alpha_0 + (c_0 - \alpha_0) = c_0. \end{array}$$

Define $u = \|x^* - y(\Delta t)\| \cdot \gamma(f,x^*) < c_0 < 1$. By Lemma 2(b) of [6, Ch. 8],

$$\|Df(y(\Delta t))^{-1}Df(x^*)\| \le \frac{(1-u)^2}{1 - 4u + 2u^2}.$$

Since the constant and linear terms of a Taylor series expansion of $H(y(\Delta t), t^* + \Delta t)$ centered at (x^*, t^*) vanish, we have

$$\begin{aligned}
\beta(H(\cdot, t^* + \Delta t), y(\Delta t)) &= \|Df(y(\Delta t))^{-1} H(y(\Delta t), t^* + \Delta t)\| \\
&\leq \|Df(y(\Delta t))^{-1} Df(x^*)\| \sum_{k=2}^{\infty} \left\| \frac{Df(x^*)^{-1} D^k f(x^*)}{k!} \right\| \cdot \|x^* - y(\Delta t)\|^k \\
&\leq \frac{(1-u)^2 \|x^* - y(\Delta t)\|}{1 - 4u + 2u^2} \sum_{k=2}^{\infty} u^{k-1} \\
&= \frac{(1-u)u^2}{(1-4u+2u^2)\gamma(f, x^*)}.
\end{aligned}$$

Also, by Proposition 3 of [6, Ch. 8],

$$\gamma(H(\cdot, t^* + \Delta t), y(\Delta t)) = \gamma(f, y(\Delta t)) \leq \frac{\gamma(f, x^*)}{(1 - 4u + 2u^2)(1 - u)}.$$

Thus,

$$\alpha(H(\cdot, t^* + \Delta t), y(\Delta t)) \leq \left(\frac{u}{1 - 4u + 2u^2} \right)^2.$$

This shows that if $u \leq (1 - 4u + 2u^2)\sqrt{\alpha_0}$, then $\alpha(H(\cdot, t^* + \Delta t), y(\Delta t)) \leq \alpha_0$.

Consider the univariate polynomial $p(z) = (1 - 4z + 2z^2)\sqrt{\alpha_0} - z$. Since $p(0) > 0$ and e_0, as defined in (7), is the smallest positive solution of $p(z) = 0$, the result follows provided that $u \leq e_0 < c_0$. Since

$$u \leq \alpha(H(\cdot, t^*), x^*) + |\Delta t| \cdot \gamma(f, x^*) \cdot \|Df(x^*)^{-1} v\|,$$

we know $u \leq e_0$ by (7). □

Remark 6 This proof depends on Lemma 2(a) of [6, Ch. 8] to show that $Df(y(\Delta t))$ is invertible. This places a constraint on the stepsize that would also limit the use of higher-order predictors provided that one follows a similar proof strategy. We explore the limit of this constraint, namely

$$\|x^* - y(\Delta t)\| \cdot \gamma(f, x^*) < c_0 = 1 - \sqrt{2}/2 < 2e_0.$$

Thus, if one would instead use a higher-order method to determine $y(\Delta t)$, one has

$$\|x^* - y(\Delta t)\| = |\Delta t| \cdot \|Df(x^*)^{-1} \cdot v\| + O(|\Delta t|^2).$$

Up to first order, this constraint yields

$$|\Delta t| < \frac{c_0}{\gamma(f, x^*) \cdot \|Df(x^*)^{-1} \cdot v\|} < \frac{2e_0}{\gamma(f, x^*) \cdot \|Df(x^*)^{-1} \cdot v\|}.$$

Theorem 5 suggests the following certified tracker, `EulerTracker`, which uses only Euler predictions.

Following similar simplifying assumptions as posed in Section 2, the following Euler-Newton tracking algorithm, `EulerNewtonTracker`, uses the same Newton correction strategy described in (4).

Remark 7 All of our certified tracking procedures are described based on the use of exact arithmetic. However, they could be implemented in finite-precision arithmetic using so-called robust α-theory (see [6, Ch. 8]). Continued fractions could be used to develop a height-reducing robust tracking strategy.

Procedure 3 `EulerTracker`

Input: An analytic function $f : \mathbb{C}^N \to \mathbb{C}^N$, a vector $v \in \mathbb{C}^N$, and a point $x_1 \in \mathbb{C}^N$ with $\alpha(f(\cdot) + v, x_1) \leq \alpha_0$.

Output: A point $x_0 \in \mathbb{C}^N$ with $\alpha(f, x_0) \leq \alpha_0$.

Define $H(x,t) := f(x) + t \cdot v$.
Initialize $x^* := x_1$ and $t^* := 1$.
while $t^* > 0$ **do**
    Compute $\Delta t := \dfrac{e_0 - \alpha(H(\cdot, t^*), x^*)}{\gamma(f, x^*) \cdot \|Df(x^*)^{-1} \cdot v\|}$.
    if $\Delta t > t^*$ **then**
        Set $\Delta t := t^*$.
    Update $x^* := E_{H,t^*,-\Delta t}(x^*)$ and then $t^* := t^* - \Delta t$.
return $x_0 := x^*$.

Procedure 4 `EulerNewtonTracker`

Input: An analytic function $f : \mathbb{C}^N \to \mathbb{C}^N$, a vector $v \in \mathbb{C}^N$, and a point $x_1 \in \mathbb{C}^N$ with $\alpha(f(\cdot) + v, x_1) \leq \alpha_0$.

Output: A point $x_0 \in \mathbb{C}^N$ with $\alpha(f, x_0) \leq \alpha_0$.

Define $H(x,t) := f(x) + t \cdot v$.
Initialize $x^* := x_1$ and $t^* := 1$.
while $t^* > 0$ **do**
    if $\alpha(H(\cdot, t^*), x^*) > 4\alpha_0/5$ **then**
        Set $x^* := N^2_{H(\cdot,t^*)}(x^*)$.
    else if $\alpha(H(\cdot, t^*), x^*) > 2\alpha_0/3$ **then**
        Set $x^* := N_{H(\cdot,t^*)}(x^*)$.
    Compute $\Delta t := \dfrac{e_0 - \alpha(H(\cdot, t^*), x^*)}{\gamma(f, x^*) \cdot \|Df(x^*)^{-1} \cdot v\|}$.
    if $\Delta t > t^*$ **then**
        Set $\Delta t := t^*$.
    Update $x^* := E_{H,t^*,-\Delta t}(x^*)$ and then $t^* := t^* - \Delta t$.
return $x_0 := x^*$.

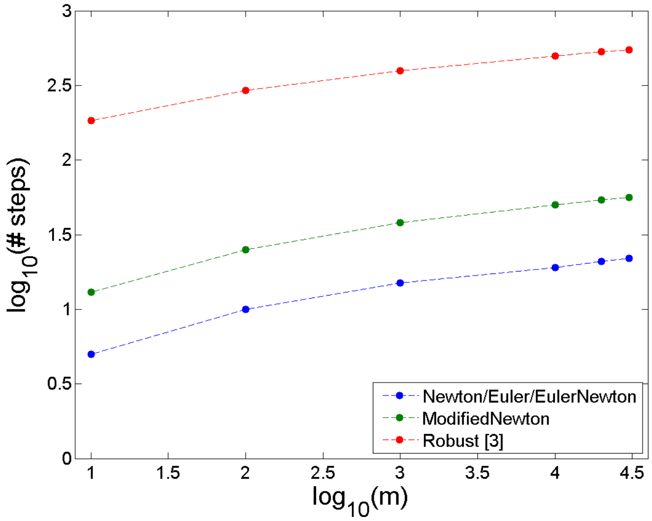

Figure 1: Plot of the number of steps used by various tracking methods

5 Performance

5.1 Univariate quadratics

A collection of quadratic homotopies in $\mathbb{P}^1$ was considered in [3, § 9.1]. After dehomogenizing and replacing t with $1-t$ since we track from $t=1$ to $t=0$, these quadratic Newton homotopies have the form

$$H(x,t) = x^2 - (1+m) + m \cdot t,$$

where $m > -1$ and start at $x_1 = 1$ when $t = 1$.

Our first collection of examples are for the selected values of $m \geq 10$ that are also presented in [3, Table 3]. Table 1 presents the number of steps taken by `NewtonTracker`, `ModifiedNewtonTracker`, `EulerTracker`, and `EulerNewtonTracker` which is graphically represented in Figure 1. For these examples, `NewtonTracker`, `EulerTracker`, and `EulerNewtonTracker` all took the same number of steps, which is due to the small relative difference between the constants α_0 and e_0. However, the cost for `NewtonTracker` to do this was due to the use of two Newton iterations after each predictor step (except at the end). The Euler predictions were all relatively accurate so that `EulerTracker` could take large steps and `EulerNewtonTracker` did not need to perform any Newton iterations. Thus, in terms of the total number of Newton and Euler iterations, the two Euler-based methods were identical and roughly one-half of the cost of `NewtonTracker`. For `ModifiedNewtonTracker`, only one Newton correction was used after each prediction. The stepsize restriction needed to simplify the complexity analysis was more costly in terms of steps taken than the other three new methods, but still considerably less than the approach of [3].

The key measure for the number of steps in our tracking methods is the length of the path in the γ-metric presented in (5). For $m > -1$, since $f(x) = x^2 - (1+m)$ and $v = m$, we have

$$\gamma(f,x) = \frac{1}{2|x|} \quad \text{and} \quad \|Df(x)^{-1} \cdot v\| = \frac{|m|}{2|x|}.$$

Thus, the length of the the path $x(t) = \sqrt{1 + m - m \cdot t}$ in the γ-metric is

$$\mathcal{L}(m) = \int_0^1 \frac{|m|}{4x(t)^2} dt = \frac{|m|}{4} \int_0^1 \frac{1}{1 + m - m \cdot t} dt = \frac{|\ln(m+1)|}{4}.$$

Since $\mathcal{L}(m)$ is logarithmic in $m+1$, the number of steps for `ModifiedNewtonTracker` is also logarithmic in $m+1$ by Theorem 4. Due to the relationship between the other three tracking methods and `ModifiedNewtonTracker`, one expects this relationship to hold for all of them. Table 2 demonstrates this by showing that the number of steps taken per unit length of the path in the γ-metric is roughly constant as m increases.

As m increases without bound, the path in both the Euclidean metric and γ-metric increases without bound. However, as m decreases to -1, the path heads towards the singular solution at the origin. The length of the path in the Euclidean metric is bounded whereas it is unbounded in the γ-metric. Our second set of examples considers selected values of m which approach -1. Table 3 shows the number of steps and total number of Newton and Euler iterations for selected values of m approaching -1 with Table 4 showing that the number of steps per unit length in the γ-metric is roughly constant. In this case, due to the curvature of the path approaching the singularity, the values of α after the Euler predictions were larger than the above examples, but not large enough for `EulerNewtonTracker` to need to perform additional Newton iterations. However, just as above, `NewtonTracker` required two Newton iterations at every step except the last. The added cost of these extra Newton iterations allowed for overall slightly fewer steps than the Euler prediction methods, but still more total iterations.

5.2 Random polynomial systems

For dense polynomial systems $f : \mathbb{C}^N \to \mathbb{C}^N$ where all of the coefficients are selected randomly, we considered tracking the solution path defined by the Newton homotopy $H(x,t) = f(x) - t \cdot f(x_1)$ where $x_1 = 0$.

In our experiment, for selected $N \geq 1$ and $d \geq 2$, we tracked the corresponding solution path of the Newton homotopy starting at the origin for 100 random systems consisting of N degree d polynomials in N variables where the coefficients were independent complex Gaussian with mean 0 and variance 1. Table 5 shows the average number of steps as well as average number of Newton and Euler iterations used by our four certified tracking procedures and the approach of [3] for $1 \leq N \leq 3$ and $2 \leq d \leq 5$, which is graphically represented in Figure 2. Table 6 contains data on our four tracking procedures for other values of N and d. In our experiments, `NewtonTracker` and `ModifiedNewtonTracker` performed two and one Newton iterations per step, respectively, except for the last step. Moreover, `EulerNewtonTracker` did not utilize any Newton iterations so that it was equivalent to `EulerTracker`.

Finally, we consider the logarithm of average number of steps where the base is the input size, namely the number of coefficients $N \cdot \binom{N+d}{d}$, which is presented in Table 7 This table provides initial computational evidence that our certified algorithms could be polynomial in the input size. For example, suppose that the average number of steps is bounded above by a number of the form

$$S(N,d) := C \cdot \left(N \cdot \binom{N+d}{d} \right)^k,$$

where C and k are universal constants. Therefore,

$$\log_{N \cdot \binom{N+d}{d}} S(N,d) = k + \log_{N \cdot \binom{N+d}{d}} C.$$

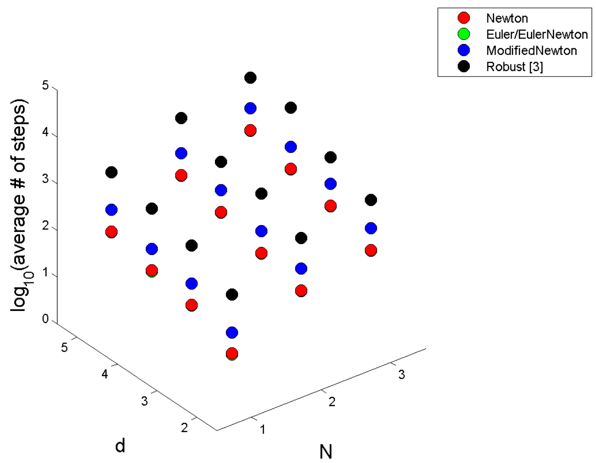

Figure 2: Plot of the number of steps used by various tracking methods

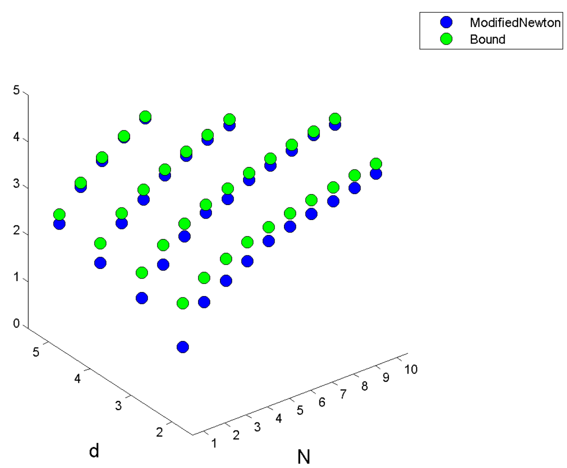

Figure 3: Comparison of $\log_{10}$(average number of steps using `ModifiedNewtonTracker`) and $\log_{10}(S(N, d))$ where $C = 150$ and $k = 0.6$

For small values of N and d, the logarithm of C may be quite large compared with k which helps to explain the general decrease in the values presented in Table 7 as N and d increase. Figure 3 compares the average number of steps for `ModifiedNewtonTracker` with $S(N, d)$ for $C = 150$ and $k = 0.6$.

References

[1] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Bertini: Software for Numerical Algebraic Geometry. Available at `bertini.nd.edu`.

[2] D.J. Bates, J.D. Hauenstein, and A.J. Sommese. Efficient path tracking methods. *Numer. Algorithms*, 58(4), 451–459, 2011.

[3] C. Beltrán and A. Leykin. Robust certified numerical homotopy tracking. *Found. Comput. Math.*, 13(2), 253–295, 2013.

[4] C. Beltrán and A. Leykin. Certified numerical homotopy tracking. *Exp. Math.*, 21, 69–83, 2012.

[5] C. Beltrán and L.M. Pardo. On Smale's 17th problem: a probabilistic positive solution. *Found. Comput. Math.*, 8, 1–43, 2008.

[6] L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and Real Computation*. Springer-Verlag, New York, 1998.

[7] J.D. Hauenstein, I. Haywood, and A.C. Liddell, Jr. An *a posteriori* certification algorithm for Newton homotopies. *ISSAC '14*, ACM, New York, 2014, pp. 248–255.

[8] J.D. Hauenstein and F. Sottile. Algorithm 921: alphaCertified: certifying solutions to polynomial systems. *ACM TOMS*, 38(4), 28, 2012.

[9] J.D. Hauenstein and F. Sottile. alphaCertified: software for certifying solutions to polynomial systems. Available at `www.math.tamu.edu/~sottile/research/stories/alphaCertified`.

[10] K.L. Judd. *Numerical Methods in Economics*. MIT Press, Cambride, MA, 1998.

[11] T.-L. Lee, T.-Y. Li, and C.-H. Tsai. *HOM4PS-2.0*: A software package for solving polynomial systems by the polyhedral homotopy continuation method. *Computing*, 83, 109–133, 2008. Software available at `www.math.msu.edu/~li/Software.htm`.

[12] A. Leykin. Numerical algebraic geometry for Macaulay2. *J. Softw. Algebr. Geom.* 3, 5-10, 2011.

[13] A. Leykin, and F. Sottile. Galois groups of Schubert problems via homotopy computation. *Math. Comp.*, 78, 1749–1765, 2009.

[14] D. Metha, T. Chen, J.D. Hauenstein, and D.J. Wales. Communication: Newton homotopies for sampling stationary points of potential energy landscapes. J. Chem. Phys., 141, 121104, 2014.

[15] M. Shub. Complexity of Bezout's theorem. VI. Geodesics in the condition (number) metric. *Found. Comput. Math.*, 9(2), 171–178, 2009.

[16] M. Shub and S. Smale. Complexity of Bézout's theorem. I. Geometric aspects. *J. Amer. Math. Soc.*, 6(2), 459–501, 1993.

[17] M. Shub and S. Smale. Complexity of Bezout's theorem. V. Polynomial time. *Theoret. Comput. Sci.*, 133, 141–164, 1994.

[18] S. Smale. The fundamental theorem of algebra and complexity theory. *Bull. Amer. Math. Soc. (N.S.)*, 4(1), 1–36, 1981.

[19] S. Smale. Newton's method estimates from data at one point. *The merging of disciplines: new directions in pure, applied, and computational mathematics (Laramie, Wyo., 1985)*, Springer, New York, 1986, pp. 185–196.

[20] A.J. Sommese, J. Verschelde, and C.W. Wampler. Using monodromy to decompose solution sets of polynomial systems into irreducible components. In *Applications of algebraic geometry to coding theory, physics and computation*, Vol. 36 of *NATO Sci. Ser. II Math. Phys. Chem.*, Kluwer Acad. Publ., Dordrecht, 2001, pp. 297–315.

[21] J. Verschelde. Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation. *ACM Trans. Math. Softw.*, 25(2), 251–276, 1999. Software available at `www.math.uic.edu/~jan`.

A Tables

	Number of steps (Total number of Newton and Euler iterations)				
m	`Newton`	`Euler`	`EulerNewton`	`ModifiedNewton`	`Robust` [3]
10	5 (8)	5 (5)	5 (5)	13 (12)	184 (183)
100	10 (18)	10 (10)	10 (10)	25 (24)	292 (291)
1000	15 (28)	15 (15)	15 (15)	38 (37)	395 (394)
10000	19 (36)	19 (19)	19 (19)	50 (49)	499 (498)
20000	21 (40)	21 (21)	21 (21)	54 (53)	530 (529)
30000	22 (42)	22 (22)	22 (22)	56 (55)	547 (546)

Table 1: Number of predictor-corrector steps and total number of Newton and Euler iterations using various tracking methods for selected m

	Steps/$\mathcal{L}(m)$	
m	`Newton`, `Euler` & `EulerNewton`	`ModifiedNewton`
10	8.34	21.69
100	8.67	21.67
1000	8.68	22.00
10000	8.25	21.71
20000	8.48	21.81
30000	8.54	21.73

Table 2: Number of predictor-corrector steps per unit length of the path in the γ-metric

m	Number of steps (Total number of Newton and Euler iterations)			
	`Newton`	`Euler`	`EulerNewton`	`ModifiedNewton`
-0.999	8 (14)	9 (9)	9 (9)	32 (31)
-0.99999	12 (22)	15 (15)	15 (15)	52 (51)
-0.9999999	17 (32)	21 (21)	21 (21)	73 (72)
-0.999999999	22 (42)	27 (27)	27 (27)	94 (93)
-0.99999999999	26 (50)	33 (33)	33 (33)	114 (113)
-0.9999999999999	31 (60)	39 (39)	39 (39)	135 (134)

Table 3: Number of predictor-corrector steps and total number of Newton and Euler iterations using various tracking methods for selected m

m	Steps/$\mathcal{L}(m)$		
	`Newton`	`Euler` & `EulerNewton`	`ModifiedNewton`
-0.999	4.63	5.21	18.53
-0.99999	4.17	5.21	18.07
-0.9999999	4.22	5.21	18.17
-0.999999999	4.25	5.21	18.14
-0.99999999999	4.11	5.21	18.00
-0.9999999999999	4.14	5.21	18.04

Table 4: Number of predictor-corrector steps per unit length of the path in the γ-metric

N	d	Average number of steps (Average number of Newton and Euler iterations)			
		`Newton`	`Euler` & `EulerNewton`	`ModifiedNewton`	`Robust` [3]
1	2	12.2 (22.4)	11.5 (11.5)	33.1 (32.1)	218.0 (217.0)
	3	35.4 (68.8)	33.8 (33.8)	100.7 (99.7)	661.6 (660.6)
	4	51.5 (101.0)	49.5 (49.5)	149.1 (148.1)	1082.3 (1081.3)
	5	93.8 (185.6)	90.5 (90.5)	274.4 (273.4)	1710.5 (1709.5)
2	2	69.2 (136.4)	66.8 (66.8)	201.2 (200.2)	932.5 (931.5)
	3	117.5 (233.0)	113.9 (113.9)	345.6 (344.6)	2167.8 (2166.8)
	4	237.4 (472.8)	230.8 (230.8)	704.3 (703.3)	2810.2 (2809.2)
	5	386.0 (770.0)	375.8 (375.8)	1149.8 (1148.8)	6583.9 (6582.9)
3	2	132.3 (262.6)	128.5 (128.5)	390.1 (389.1)	1553.4 (1552.4)
	3	311.5 (621.0)	303.2 (303.2)	925.7 (924.7)	3439.4 (3438.4)
	4	517.9 (1033.8)	504.8 (504.8)	1545.5 (1544.8)	10458.8 (10457.8)
	5	923.6 (1845.2)	901.0 (901.0)	2762.0 (2761.0)	12385.5 (12384.5)

Table 5: Average number of predictor-corrector steps and average number of Newton and Euler iterations using various tracking methods for 100 random systems for selected N and d.

N	d	Average number of steps (Average number of Newton and Euler iterations)		
		`Newton`	`Euler` & `EulerNewton`	`ModifiedNewton`
4	2	231.9 (461.8)	225.6 (225.6)	687.6 (686.6)
	3	655.2 (1308.4)	638.9 (638.9)	1955.4 (1954.4)
	4	1130.8 (2259.6)	1103.4 (1103.4)	3382.4 (3381.4)
	5	1985.1 (3968.2)	1937.8 (1937.8)	5944.9 (5943.9)
5	2	409.7 (817.4)	399.3 (399.3)	1220.1 (1219.1)
	3	872.4 (1742.8)	850.9 (850.9)	2606.8 (2605.8)
	4	1957.9 (3913.8)	1911.3 (1911.3)	5862.4 (5861.4)
	5	3361.7 (6721.4)	3282.6 (3282.6)	10074.4 (10073.4)
6	2	551.6 (1101.2)	538.0 (538.0)	1645.4 (1644.4)
	3	1472.3 (2942.6)	1436.9 (1436.9)	4405.1 (4404.1)
	4	2876.5 (5751.0)	2808.6 (2808.6)	8617.9 (8616.9)
7	2	707.1 (1400.2)	683.8 (683.8)	2092.9 (2091.9)
	3	2043.9 (4085.8)	1995.3 (1995.3)	6119.2 (6118.2)
	4	3899.5 (7797.0)	3808.0 (3808.0)	11686.8 (11685.8)
8	2	869.6 (1737.2)	848.4 (848.4)	2598.4 (2597.4)
	3	2853.6 (5705.2)	2786.3 (2786.3)	8547.9 (8546.9)
9	2	1110.4 (2218.8)	1083.7 (1083.7)	3319.9 (3318.9)
	3	4016.9 (8031.8)	3922.7 (3922.7)	12036.7 (12035.7)
10	2	1492.5 (2983.0)	1457.0 (1457.0)	4465.6 (4464.6)
	3	4482.0 (8692.0)	4377.1 (4377.1)	13432.5 (132431.5)

Table 6: Average number of predictor-corrector steps and average number of Newton and Euler iterations using various tracking methods for 100 random systems for selected N and d.

N	d	$\text{Log}_{\text{input size}}$(Average number of steps)		
		`Newton`	`Euler` & `EulerNewton`	`ModifiedNewton`
1	2	2.28	2.22	3.18
	3	2.57	2.54	3.33
	4	2.45	2.42	3.11
	5	2.53	2.51	3.13
2	2	1.71	1.69	2.13
	3	1.59	1.58	1.95
	4	1.61	1.60	1.93
	5	1.59	1.59	1.89
3	2	1.44	1.43	1.75
	3	1.40	1.40	1.67
	4	1.34	1.34	1.58
	5	1.33	1.33	1.55
4	2	1.33	1.32	1.60
	3	1.31	1.31	1.53
	4	1.25	1.24	1.44
	5	1.22	1.22	1.40
5	2	1.29	1.29	1.53
	3	1.20	1.20	1.40
	4	1.18	1.17	1.35
	5	1.14	1.13	1.29
6	2	1.23	1.23	1.45
	3	1.17	1.17	1.35
	4	1.12	1.11	1.27
7	2	1.19	1.18	1.38
	3	1.13	1.13	1.29
	4	1.07	1.06	1.21
8	2	1.15	1.15	1.34
	3	1.11	1.10	1.26
9	2	1.13	1.13	1.31
	3	1.09	1.09	1.24
10	2	1.13	1.12	1.29
	3	1.06	1.05	1.19

Table 7: Logarithm of the average number of predictor-corrector steps with the base being the input size $N \cdot \binom{N+d}{d}$.