MULTIPROJECTIVE WITNESS SETS AND A TRACE TEST

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ABSTRACT. In the field of numerical algebraic geometry, positive-dimensional solution sets of systems of polynomial equations are described by witness sets. In this paper, we define multiprojective witness sets which encode the multidegree information of an irreducible multiprojective variety. Our main results generalize the regeneration solving procedure, a trace test, and numerical irreducible decomposition to the multiprojective case. Examples are included to demonstrate this new approach.

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1. INTRODUCTION

Numerical algebraic geometry contains algorithms for computing and studying solution sets, called varieties, of systems of polynomial equations. Depending on the structure of the equations, we can view the variety as either affine or projective. Witness sets are the numerical algebraic geometric description of affine and projective varieties. If a variety is irreducible, its dimension and degree can be recovered directly from a witness set. When the variety is reducible, one can compute witness sets for each of the irreducible components thereby producing a numerical irreducible decomposition of the variety.

We will consider multiprojective varieties, which are defined by a polynomial system consisting of multihomogeneous polynomials. Multiprojective varieties naturally arise in many applications including kinematics [41], likelihood geometry [15, 23], and identifiability in tensor decomposition [14]. In fact, multihomogeneous homotopies [29] in numerical algebraic geometry developed from observing bihomogeneous structure of the inverse kinematics problem for 6R robots [41].

The paper is structured as follows. In Section 1.1, we define multiprojective witness sets and their collections. Section 2 summarizes homotopy continuation. Section 3 contains our first main contribution: a membership test using multiprojective witness sets. Section 4 contains our second main contribution: a generalization of the regeneration algorithm to compute multiprojective witness sets with examples presented in Section 5. Section 6 contains our third main contribution: a trace test for multiprojective varieties with examples presented in Section 7. In particular, this trace test provided the motivation to recompute the BKK bound for Alt’s problem (see Section 7.2) in Theorem 7.1.

1.1. Multiprojective witness sets. A projective variety intersected with a general linear space has an expected number of solutions. When the dimension of the linear space is complementary to the variety, the expected number of solutions is finite and is called the degree of the variety. A witness point set for a variety is such a finite set of points.

In the multiprojective setting, there are different types of linear spaces that may be taken which are related to the Chow ring (see [28 Chap. 8]). In particular, a collection of witness sets with all possible linear slicing can be used to describe a multiprojective variety.
Definition 1.1. Let $\mathcal{V}$ be a $c$-dimensional irreducible multiprojective variety in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and let $e = (e_1, e_2, \ldots, e_k) \in \mathbb{N}_{\geq 0}^k$ such that $c = |e| = e_1 + \cdots + e_k$. An $e^{th}$ witness set for $\mathcal{V}$ is a triple:

$$W^e(\mathcal{V}) := \{V, L^e, w^e(\mathcal{V})\}$$

where

1. $V$ is a set of polynomials that forms a witness system for $\mathcal{V}$, i.e., $\mathcal{V}$ is an irreducible component of the solution set $\bar{V} = \emptyset$.
2. $L^e = \bigcup_{i=1}^{k} \{e_i^{(1)}, \ldots, e_i^{(e_i)}\}$ is a set of $|e|$ linear polynomials with each $e_i^{(j)}$ being a general linear polynomial in the unknowns associated with $\mathbb{P}^{n_i}$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. The solution set $L^e = \emptyset$ defines a codimension $|e|$ linear space denoted by $L^e$.
3. $w^e(\mathcal{V}) = V \cap L^e$ is the witness point set for $\mathcal{V}$ with respect to $L^e$.

The number of points in $w^e(\mathcal{V})$, namely $|w^e(\mathcal{V})|$, is the degree of $\mathcal{V}$ with respect to $e$, which we will formally denote by $\deg W^e(\mathcal{V}) = |w^e(\mathcal{V})|\omega^e$.

Definition 1.2. A (complete) witness set collection for a $c$-dimensional irreducible multiprojective variety $\mathcal{V} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a formal union of witness sets:

$$\mathcal{W}(\mathcal{V}) = \bigcup_{e \in \mathbb{N}_{\geq 0}^k, c = |e|} W^e(\mathcal{V})$$

with degree defined to be a formal sum:

$$\deg \mathcal{W}(\mathcal{V}) = \sum_{e \in \mathbb{N}_{\geq 0}^k, c = |e|} |w^e(\mathcal{V})|\omega^e.$$ 

When the context is clear, we will write $\mathcal{W}$, $W^e$, $w^e$ for $\mathcal{W}(\mathcal{V})$, $W^e(\mathcal{V})$, $w^e(\mathcal{V})$, respectively.

Remark 1.3. One may disregard the terms where $|w^e(\mathcal{V})| = 0$ in both the formal union of witness sets and the formal sum of degrees.

Example 1.4. As an illustrative example, let $\mathcal{V}$ be the irreducible biprojective curve in $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $([x_0, x_1], [y_0, y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ defined by $V = \{x_1^2y_0 - x_0^2y_1\}$. The degree of $\mathcal{V}$ and $\mathcal{W}$ is $1\omega^{(1,0)} + 2\omega^{(0,1)}$, with the geometric meaning of this observed in Figure 1. In particular, since $\mathcal{V}$ is an irreducible hypersurface, $\deg \mathcal{V}$ can be observed directly from $\mathcal{V}$ based on the degree in each set of variables, i.e., $\deg \mathcal{V} = \deg_g(V)\omega^{(1,0)} + \deg_s(V)\omega^{(0,1)}$.

Notation 1.5. Our convention arises from a geometric interpretation based on slicing. This is the “reciprocal” of the algebraic convention, which is followed by the multidegree function in Macaulay2 [11]. For brevity, the degree of a hypersurface $G$ computed by Macaulay2 is

$$\deg G = \left(|w^{(n_1-1, n_2, \ldots, n_k)}|, |w^{(n_1, n_2-1, \ldots, n_k)}|, \ldots, |w^{(n_1, n_2, \ldots, n_k-1)}|\right).$$

Thus, the degree of the hypersurface $V$ in Ex. 1.4 may be written as $(2, 1)$. 


Remark 1.6. According to Definition 1.2, we denote $\mathbf{w}(V)$ as a set of points. This set contains no information about multiplicity that is necessary for describing generically nonreduced components, i.e., components with multiplicity greater than one. One can attach the local multiplicity structure in the form of a Macaulay dual basis, e.g., [9, 12, 13], to the points. Additionally, one can employ deflation methods, e.g., [20, 27], to perform computations on generically nonreduced components.

Just as for classical witness sets, e.g., see [37, Chap. 13], we extend the above definitions to reducible varieties by taking formal unions over the irreducible components.

2. Using Homotopy Continuation for Witness Sets

A witness set provides information needed to perform geometric computations on varieties. The tool that permits such computations is homotopy continuation, which we briefly summarize in this section.

2.1. Homotopies. Homotopy continuation is a fundamental tool in numerical algebraic geometry discussed in detail in [5, 37] and implemented in several software packages, e.g., [4, 24, 25, 38]. In this manuscript, we employ straight-line homotopies, e.g., [32, § 51]. One typical use of homotopy continuation below is to deform the linear space $L^e$ as in the witness set $\mathbf{w}(V)$ to another linear space of the same type, say $M^e$, along $V$. Let $V$ be a witness system for $V$ and suppose that $L^e$ and $M^e$ are defined by linear equations $L^e$ and $M^e$, respectively. We denote this homotopy by

$$H(V, L^e \rightarrow M^e) := \begin{cases} V \\ tL^e + (1-t)M^e. \end{cases}$$

The set of start points, at $t = 1$, for this homotopy is the witness point set $\mathbf{w}(V)$.

For membership testing (Section 3), one deforms to a general linear space of type $e$ passing through a given point $a^*$. The following specifies notation for such a linear space.

Notation 2.1. Given a point $a^* \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $e$, let $L^e_{a^*}$ be a general linear space of type $e$ that passes through $a^*$.

Example 2.2. Let $V \subset \mathbb{P}^1 \times \mathbb{P}^1$ and $V$ as in Ex. 1.4. Let $e = (0,1)$ and $L^e$ be the general linear space defined by $L^e = \{ y_0 - 5y_1 \}$. Then, $\mathbf{w}(V)$ consists of two points, namely

$$\mathbf{w}(V) = \left\{ \left[ \sqrt{5} : 1 \right], \left[ -\sqrt{5} : 1 \right] \right\}.$$
Consider \( \alpha^* = ([1 : 1], [1 : 2]) \) and let \( \mathcal{M} = \mathcal{L}_{\alpha}^e \), which is defined by \( \mathcal{M} = L_{\alpha}^e = \{2y_0 - y_1\} \). The homotopy \( \mathbf{H}(\mathcal{V}, \mathcal{L}^e \rightarrow \mathcal{M}^e) = \mathbf{H}(\mathcal{V}, \mathcal{L}_\alpha^e \rightarrow \mathcal{L}_{\alpha^*}^e) \) is thus

\[
\begin{bmatrix}
x_1^2y_0 - x_0^2y_1 \\
t(y_0 - 5y_1) + (1-t)(2y_0 - y_1)
\end{bmatrix} = 0.
\]

Starting, at \( t = 1 \), with the two points \( \mathbf{w}^e(\mathcal{V}) \), the set of endpoints for this homotopy is

\[
\left\{ \left( \left[ 1 : \sqrt{2} \right], [1 : 2] \right), \left( \left[ 1 : -\sqrt{2} \right], [1 : 2] \right) \right\}.
\]

For properly constructed homotopies, called complete homotopies in [18], each solution path defined by the homotopy, say \( P_i(t) \), is smooth on \( (0, 1] \) and thus can be tracked using numerical path tracking methods, e.g., a predictor-corrector based approach. Endgames, e.g., see [37] Ch. 10 and [22], are used to accurately compute the endpoints of the path, i.e., compute \( P_i(0) \).

Computationally, we perform path tracking in projective and multiprojective spaces by restricting to affine charts. That is, one imposes corresponding affine conditions on the coordinates thereby fixing a representation of the (multi)projective points. For example, in \( \mathbb{P}^1 \times \mathbb{P}^1 \), we can perform computations in \( \mathbb{C}^2 \times \mathbb{C}^2 \) by taking affine charts of the form

\[
\Box x_0 + \Box x_1 + \Box = 0 \quad \text{and} \quad \Box y_0 + \Box y_1 + \Box = 0.
\]

Here, we follow the convention of [5] where \( \Box \) represents a random or unspecified complex number.

2.2. Randomization. In a witness set for an irreducible variety \( \mathcal{V} \), the only condition on the polynomial system \( V \) is that it is a witness system (Def. 1.1). As in Remark 1.6, deflation techniques can be used to produce a system of polynomial equations \( V \) such that \( \mathcal{V} \) is a generically reduced irreducible component of \( \mathcal{V} \). That is, the dimension of the null space of the Jacobian matrix of \( V \) evaluated at a general point of \( \mathcal{V} \) is equal to the dimension of \( \mathcal{V} \). In particular, the number of polynomials in \( V \) is greater than or equal to the codimension of \( \mathcal{V} \). When the number of polynomials in \( V \) is equal to the codimension of \( \mathcal{V} \), homotopies \( \mathbf{H}(\mathcal{V}, \mathcal{L}^e \rightarrow \mathcal{M}^e) \) as constructed above are well-constrained systems, i.e., square.

If the number of polynomials in \( V \) is strictly greater than the codimension of \( \mathcal{V} \), the homotopies are over-determined. As discussed in [5] § 9.2, we will employ randomization for over-determined systems to improve numerical stability. That is, one forms numerical computations by replacing \( V \) with a generic randomization of \( V \), say \( \text{Rand}(\mathcal{V}) \). Hence, \( \mathcal{V} \) is a generically reduced irreducible component of \( \text{Rand}(\mathcal{V}) \) in which the number of polynomials in \( \text{Rand}(\mathcal{V}) \) is equal to the codimension of \( \mathcal{V} \).

In the multiprojective setting, we will maintain multihomogeneity by randomizing with respect to specified affine charts by fixing hyperplanes at infinity. To that end, in each \( \mathbb{P}^n_i \), suppose that \( \mathcal{H}_i \) is a general hyperplane defined by the linear polynomial \( \mathcal{H}_i \). We construct the affine charts by having \( \mathcal{H}_i = 1 \) and maintain multihomogeneity in the randomization via \( \mathcal{H}_i \). This is demonstrated in the following.

Example 2.3. Consider the following on \( \mathbb{P}^2 \times \mathbb{P}^2 \) with variables \([x_0, x_1, x_2]\) and \([y_0, y_1, y_2]\), respectively:

\[
\mathcal{V} = \begin{bmatrix}
x_1^2y_1 - x_2^3y_2 \\
x_1x_2y_0^2 - x_0^2y_1y_2 \\
x_1^2y_0 - x_0x_2y_2 \\
x_2^2y_0 - x_0x_1y_1
\end{bmatrix}.
\]
Let $\mathcal{V}$ be the solution set of $V = 0$, which is irreducible and has codimension 2. Hence, we aim to construct $\text{Rand}(V)$ consisting of 2 polynomials by using the following:

$$H_1 = x_0 + 2x_1 - 3x_2 \quad \text{and} \quad H_2 = 2y_0 - 5y_1 + 3y_2.$$ 

For example, we can take $\text{Rand}(V)$ to have the form

$$\text{Rand}(V) = \begin{bmatrix} x_1^2y_1 - x_2^3y_2 + \Box H_1(x_1^2y_0 - x_0x_2y_2) + \Box H_1(x_2^2y_0 - x_0x_1y_1) \\
 x_1x_2y_0^2 - x_0^3y_1y_2 + \Box H_2(x_1^2y_0 - x_0x_2y_2) + \Box H_2(x_2^2y_0 - x_0x_1y_1) \end{bmatrix}$$

so that both polynomials in $\text{Rand}(V)$ are homogeneous of degree $(3,1)$ and $(2,2)$, respectively.

One downside of utilizing randomization is the destruction of sparsity structure. For example, in Ex. 2.3 $V$ consists of binomials while $\text{Rand}(V)$ does not. Another downside of randomization is the increase of degrees. In $\mathbb{C}^N$ or $\mathbb{P}^N$, one can order the polynomials based on degrees to minimize the degrees of the polynomials in the randomization, i.e., adding random linear combinations of smaller degree polynomials to polynomials of higher degree. However, in the multiprojective case, the degrees of the polynomials, each of which is a vector of integers, need not have a well-ordering.

### 3. Membership Test

One application of witness sets is to use homotopy continuation to decide membership in the corresponding variety [34] which was extended to images of algebraic sets using pseudowitness sets in [16]. In this section, we describe our first main contribution which is using a multiprojective witness set collection to test membership in the corresponding multiprojective variety.

Suppose that $\mathcal{V} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is an irreducible multiprojective variety and we want decide if a given point $\alpha^*$ is a member of $\mathcal{V}$. The following highlights the difficulty of producing a membership test for multiprojective varieties, namely the loss of transversality of slices passing through $\alpha^*$.

**Example 3.1.** Let $\mathcal{V} \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the irreducible surface defined by

$$V = \begin{bmatrix} x_1y_1 - x_2y_2 \\
x_0y_1y_2 - x_1y_0^2 \\
x_0y_1^2 - x_2y_0^2 \end{bmatrix}$$

and $\alpha^* = ([1 : 0 : 0], [1 : 0 : 3])$. Since $\mathcal{V}$ is the only irreducible component defined by $V$, we verify that $\alpha^* \in \mathcal{V}$ by observing $V(\alpha^*) = 0$.

The multiprojective witness set collection for $\mathcal{V}$ consists of three witness sets

$$\mathfrak{W}(\mathcal{V}) = \mathcal{W}^{(2,0)} \sqcup \mathcal{W}^{(1,1)} \sqcup \mathcal{W}^{(0,2)} \quad \text{with} \quad \deg \mathfrak{W}(\mathcal{V}) = 2\omega^{(2,0)} + 2\omega^{(1,1)} + 1\omega^{(0,2)}.$$ 

In particular, for $e = (2,0)$, the witness set $\mathcal{W}^e$ has two witness points. For simplicity, let $\mathcal{L}^e$ be the general codimension 2 linear space defined by $\mathcal{L}^e = \{ -x_0 + 3x_1 - 2x_2, x_0 + x_1 + 3x_2 \}$ and $\mathcal{L}^e_{\alpha^*}$ be defined by $\mathcal{L}^e_{\alpha^*} = \{ x_1, x_2 \}$. The homotopy $H(\mathcal{V}, \mathcal{L}^e \rightarrow \mathcal{L}^e_{\alpha^*})$ defines two paths, both of which end at $([1 : 0 : 0], [1 : 0 : 0])$. Since $\alpha^*$ is not an endpoint of this homotopy, a natural conclusion based on previous membership tests [34, 16] is that $\alpha^* \notin \mathcal{V}$. However, this perceived failure is obtained since these membership tests are based on the intersection of the variety and the linear space passing through the test point $\alpha^*$ to be transverse at $\alpha^*$ if $\alpha^*$ is indeed contained in the variety. Here, $\mathcal{V} \cap \mathcal{L}^e_{\alpha^*}$ is actually
a positive-dimensional set that contains $\alpha^*$, i.e., the intersection of the dimension 2 variety $V$ and the codimension 2 linear space $L_{e}^{\alpha^*}$ is not transverse at $\alpha^*$.

The following algorithm takes into account transversality for a membership test.

**Algorithm 3.2.** [Membership test using multiprojective witness sets] Given a multiprojective witness set collection $\mathfrak{W}(V)$ for an irreducible variety $V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, determine if a given point $\alpha^*$ is contained in $V$.

1. For each $W^{e}(V)$ in $\mathfrak{W}(V)$ with $|w^{e}(V)| > 0$
   - (a) Construct a general linear space $L_{e}$ of type $e$ that passes through $\alpha^*$.
   - (b) With start points $w^{e}(V)$, compute the endpoints $E^{e}$ defined by $H(V, L^{e} \to L_{e}^{\alpha^*})$.
   - (c) If $\alpha^* \in E^{e}$, return "$\alpha^*$ is a member of $V$.",
   - (d) If every point in $E^{e}$ is isolated in $V \cap L_{e}^{\alpha^*}$, and $\alpha^* \notin E^{e}$, return "$\alpha^*$ is not a member of $V$.".
2. Return "$\alpha^*$ is not a member of $V$.".

Before proving correctness of Algorithm 3.2, we first show that, for each $\alpha^* \in V$, there exists an element $W^{e}(V) \in \mathfrak{W}(V)$ with $|w^{e}(V)| > 0$ such that a general slice $L_{e}^{\alpha^*}$ is transverse to $V$ at $\alpha^*$.

**Proposition 3.3** (Existence of transversal slices). If $V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is an irreducible variety and $\alpha^* \in V$, there exists $W^{e}(V) \in \mathfrak{W}(V)$ such that $|w^{e}(V)| > 0$ in which a general linear space $L_{e}^{\alpha^*}$ of type $e$ passing through $\alpha^*$ is transverse to $V$ at $\alpha^*$, i.e., $\alpha^*$ is an isolated point in $V \cap L_{e}^{\alpha^*}$.

**Proof.** For simplicity in our constructive proof, we write $\alpha^* = (\alpha_{1}^{*}, \ldots, \alpha_{k}^{*}) \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and set

$$M_i := \mathbb{P}^{n_i} \times \cdots \times \mathbb{P}^{n_{i-1}} \times \{\alpha_{i}^{*}\} \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_k}.$$ 

Define $e_1 := \dim V - \dim(V \cap M_1)$ and let $K_1 \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_1}$ be a general linear space passing through $\alpha^*$ that imposes $e_1$ conditions on $\mathbb{P}^{n_1}$, i.e., $K_1$ is defined by $e_1$ general linear equations in the unknowns of $\mathbb{P}^{n_1}$ that vanish on $\alpha_{1}^{*}$. We construct $e_2, \ldots, e_k$ recursively. For $i = 2, \ldots, k$, define

$$e_i := \dim (V \cap M_1 \cap \cdots \cap M_{i-1}) - \dim (V \cap M_1 \cap \cdots \cap M_{i-1} \cap M_i),$$

and let $K_i$ be a general linear space passing through $\alpha^*$ that imposes $e_i$ conditions on $\mathbb{P}^{n_i}$.

By construction, the intersection $\cap_{i=1}^{k} K_i$ is a general linear space of type $e$ that passes through $\alpha^*$. Thus, we can take $L_{e}^{\alpha^*}$ to be $\cap_{i=1}^{k} K_i$. It remains to show that for the constructed $e$, the linear space $\cap_{i=1}^{k} K_i$ is transverse to $V$ at $\alpha^*$. Since $e_i$ is the dimension of the fiber over $\alpha_{i}^{*}$ with respect to $V \cap K_1 \cap \cdots \cap K_{i-1}$, which is nonempty and has dimension at most $n_i$, we know $0 \leq e_i \leq n_i$. If $c = \dim V$, then it follows that $c = |e|$ because $V$ is irreducible and $\alpha^* \in V$. Since $\cap_{i=1}^{k} K_i$ is sequentially imposing $e_i$ conditions on the corresponding fibers which have dimension $e_i$, we know that $\alpha^*$ is an isolated point in $V \cap K_1 \cap \cdots \cap K_k$. Upper semicontinuity, e.g., [37, Thm A.4.5], provides $|w^{e}(V)| \geq 1$. \hfill $\square$

The following illustrates this construction.

**Example 3.4.** With the setup from Ex. 3.1 we follow the proof of Prop. 3.3. First,

$$e_1 = \dim V - \dim (V \cap (\{[1 : 0 : 0]\} \times \mathbb{P}^2)) = 2 - 1 = 1.$$ 

Hence, only one condition on the first $\mathbb{P}^2$ is needed to be imposed, say by $K_1$ as in Prop. 3.3. Next,

$$e_2 = \dim (V \cap K_1) - \dim (V \cap K_1 \cap (\mathbb{P}^2 \times \{[1 : 0 : 3]\})) = 1 - 0 = 1.$$
showing that we also need to impose one condition on the second \( \mathbb{P}^2 \). Hence, \( e = (1, 1) \) will yield slices that are transversal to \( \mathcal{V} \) at \( \alpha^* \).

To illustrate, we consider the linear spaces \( \mathcal{L}^e \) and \( \mathcal{L}^{\alpha^*}_e \) defined by

\[ L^e = \{ x_0 + x_1 + 3x_2, y_0 - 2\sqrt{-1}y_1 - y_2 \} \quad \text{and} \quad L^{\alpha^*}_e = \{ x_1 + 3x_2, y_0 - 2\sqrt{-1}y_1 - y_2/3 \}, \]

respectively. The endpoints of the two solution paths defined by the homotopy \( H(\mathcal{V}, \mathcal{L}^e \to \mathcal{L}^{\alpha^*}_e) \) are \( \alpha^* \) and \( ([3 + 4\sqrt{-1} : 3 : -1], [1 - 2\sqrt{-1} : -1 : 3]) \) with \( \alpha^* \) indeed being an isolated point of \( \mathcal{V} \cap \mathcal{L}^{\alpha^*}_e \).

We note that the constructive proof of Prop. 3.3 implicitly used an ordering of the spaces in the multiprojective space \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \). If the spaces were ordered differently, one could yield a different \( e \).

Example 3.5. With the setup from Ex. 3.1, we follow the proof of Prop. 3.3 but take the \( \mathbb{P}^2 \)'s is reverse order, i.e., constructing \( e = (e_1, e_2) \) by first computing \( e_2 \) and then computing \( e_1 \). In this case,

\[ e_2 = \dim V - \dim (V \cap (\mathbb{P}^2 \times \{ [1 : 0 : 3] \})) = 2 - 0 = 2 \]

and thus \( e_1 = 0 \). Hence, \( e = (0, 2) \) will also yield transversality at \( \alpha^* \) and therefore can also be used in a membership test. In fact, since \( w^e(\mathcal{V}) \) only consists of one point in this case, only one path is needed to be tracked to determine that \( \alpha^* \in \mathcal{V} \).

We use Prop. 3.3 to show that Algorithm 3.2 provides a membership test for multiprojective varieties.

**Theorem 3.6 (Correctness of Algorithm 3.2).** Algorithm 3.2 is a valid membership test based on multiprojective witness sets.

**Proof.** If \( e \) is chosen such that \( \alpha^* \in \mathcal{E}^e \), then we know \( \alpha^* \in \mathcal{V} \) because \( \mathcal{E}^e \subset \mathcal{V} \cap \mathcal{L}^{\alpha^*}_e \) which justifies Item 1 of Algorithm 3.2. If every point in \( \mathcal{E}^e \) is isolated in \( \mathcal{V} \cap \mathcal{L}^{\alpha^*}_e \), then coefficient-parameter homotopy theory [30] provides that \( \mathcal{E}^e = \mathcal{V} \cap \mathcal{L}^{\alpha^*}_e \). Hence, in this case, \( \alpha^* \in \mathcal{V} \) if and only if \( \alpha^* \in \mathcal{E}^e \) which justifies Item 1d. By Prop. 3.3, there exists \( e \) such that \( |w^e(\mathcal{V})| \geq 1 \) and we have the property that \( \alpha^* \in \mathcal{V} \) if and only if \( \alpha^* \in \mathcal{E}^e \) which justifies Item 2.

\[ \square \]

Since Algorithm 3.2 requires the input of multihomogeneous witness set collection \( \mathfrak{W}(\mathcal{V}) \), the rest of the paper is devoted to computing such collections.

### 4. Multipiregeneration

In this section, we will compute a witness set collection of a variety \( \mathcal{V} \). Thus, the aim is to compute isolated points of \( \mathcal{V} \) \( \mathcal{L}^e \) for all possible \( e \). When \( \mathcal{V} \) is equidimensional, by all possible \( e \) we mean \( e \in \mathbb{N}^k_{\geq 0} \) such that \( |e| = \dim V \); otherwise, we mean \( 0 \leq e \leq (n_1, \ldots, n_k) \). We take an equation-by-equation approach called regeneration [18] [19]. The key idea is the following: given a witness set collection for a variety \( \mathcal{V} \) and a hypersurface \( \mathcal{G} \), compute a witness set collection for \( \mathcal{V} \cap \mathcal{G} \). Iterating this process we will compute \( \mathfrak{W}(\mathcal{V}) \) with \( \mathcal{V} \) defined as the intersection of hypersurfaces. From \( \mathfrak{W}(\mathcal{V}) \), Section 6.2 describes computing witness sets for the irreducible components of \( \mathcal{V} \).

By utilizing an approach based on regeneration, one is actually constructing a sequence of witness point sets for subproblems as part of the computation. When the subproblems have physical meaning, the intermediate steps of regeneration provide useful information. For example, Alt’s problem [17] counts the number of four-bar linkages whose coupler curve passes through nine general points in
the plane (see Section 7.2). By using a regeneration-based approach, one actually solves the two-point, three-point, . . . , and eight-point problems in the process of solving Alt’s nine-point problem.

The key step of multiregeneration is presented in Section 4.1. We summarize the complete algorithm in Section 4.2. Examples are presented in Section 5.

4.1. Intersection with a hypersurface. Given a witness set collection for \( Y \) and a hypersurface \( G \), we compute a witness set collection for \( Y \cap G \). For the ease of exposition, we assume, in this subsection, that no irreducible component of \( Y \) is contained in \( G \). The general case is provided in Section 4.2.

This computation has two steps: regenerate to union of hyperplanes and then deform to the hypersurface \( G \). To simplify the presentation, we use the following.

**Notation 4.1.** We denote \( \delta_i \) to be the vector with 1 in the \( i \)th position and 0 elsewhere.

4.1.1. Regenerating to a union of hyperplanes. Following the notation above, let \( Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) and \( \deg G = (g_1, \ldots, g_k) \). For \( i = 1, \ldots, k \) and \( j = 1, \ldots, g_i \), let \( s_i^{(j)} \) be a general linear polynomial in the unknowns associated with \( \mathbb{P}^{n_i} \). Suppose that \( S \) is the union of \( g_1 + \cdots + g_k \) hyperplanes defined by

\[
S = \bigcap_{i=1}^{k} \bigcap_{j=1}^{g_i} s_i^{(j)}. \tag{1}
\]

Let \( S_i^{(j)} \) be the hyperplane defined by \( s_i^{(j)} \) so that

\[
S = \bigcup_{i=1}^{k} \bigcup_{j=1}^{g_i} S_i^{(j)}. \]

With this setup, both \( G \) and \( S \) are hypersurfaces of degree \( (g_1, \ldots, g_k) \) and no irreducible component of \( Y \) is contained in either \( G \) or \( S \). From \( \mathcal{W}(Y) \), the following computes the isolated points in \( Y \cap S \cap L^d \) for all possible \( d \), thereby producing \( \mathcal{W}(Y \cap S) \).

**Algorithm 4.2.** [Regenerating to a union] Given a witness set collection \( \mathcal{W}(Y) \) and \( \{s_i^{(j)}\} \) whose product \( S \) defines the union of hyperplanes \( S \), compute a witness set collection \( \mathcal{W}(Y \cap S) \).

1. Initialize \( \mathcal{W}(Y \cap S) \) to be the empty set.
2. For \( d \in \mathbb{N}_{\geq 0} \) such that there exist \( i \in \{1, 2, \ldots, k\} \) and \( W^e(Y) \in \mathcal{W}(Y) \) with \( d + \delta_i = e \):
   a. Initialize \( P^d = \emptyset \) and append to \( \mathcal{W}(Y \cap S) \) the set
      \[
      W^d(Y \cap S) := \{Y \cup \{S\}, L^d, P^d\}.
      \]
3. For \( i = 1, \ldots, k \) such that \( g_i > 0 \):
   a. For each \( W^e(Y) \in \mathcal{W}(Y) \) such that \( e \geq \delta_i \):
      i. Define \( d := e - \delta_i \) and hyperplane \( M \) such that \( L^d \cap M = L^e \).
      ii. For \( j = 1, \ldots, g_i \), append to \( P^d \) the endpoints of the homotopy
         \[
         H \left(V \cap L^d, M \rightarrow S_i^{(j)}\right) = \begin{cases} V \\ L^d \\ tM + (1-t)s_i^{(j)} \end{cases}.
         \]
with start points $w^e(V)$, which are the isolated points of $V \cap L^e$.

(4) Return $\mathcal{W}(V \cap S)$.

**Remark 4.3.** In Item 2 of Algorithm 4.2 if $V$ is equidimensional then $|d| = \dim V - 1$ and $|e| = \dim V$.

**Proposition 4.4.** Algorithm 4.2 correctly returns the multiprojective witness set collection $\mathcal{W}(V \cap S)$.

Proof. By construction,

$$V \cap S \cap L^d = \bigcup_{i=1}^{k} \bigcup_{j=1}^{g} V \cap S_i^{(j)} \cap L^d.$$ 

In Item 3(a)ii, the hyperplanes $M$ and $S_i^{(j)}$ are both defined by general linear polynomials in the unknowns associated with $\mathbb{P}^{n_i}$. Hence, the isolated points in $V \cap S_i^{(j)} \cap L^d$ are obtained by the homotopy $H \left(V \cap L^d, M \rightarrow S_i^{(j)} \right)$ starting at the isolated points of $V \cap L^e = V \cap M \cap L^d$. $\square$

4.1.2. **Deforming hypersurfaces.** The next step is to deform the hypersurface $S$ to $G$ along $V$ to compute $\mathcal{W}(V \cap G)$ from $\mathcal{W}(V \cap S)$.

**Algorithm 4.5.** [Deforming hypersurface] Given $\mathcal{W}(V \cap S)$ and a hypersurface $G$ such that $S$ and $G$ have the same degree and no irreducible component of $V$ is contained in $G$, compute a witness set collection $\mathcal{W}(V \cap G)$.

(1) Initialize $\mathcal{W}(V \cap G)$ to be the empty set.

(2) For $W^d(V \cap S) \in \mathcal{W}(V \cap S)$ do

(a) Set $Q^d$ to the set of endpoints of the homotopy $H \left(V \cap L^d, S \rightarrow G \right)$ starting at the isolated points of $V \cap S \cap L^d$.

(b) Remove from $Q^d$ the nonisolated points of $V \cap G \cap L^d$, e.g., via [3].

(c) Append to $\mathcal{W}(V \cap G)$ the multiprojective witness set

$$W^d(V \cap G) := \left\{ V \cup \{G\}, L^d, Q^d \right\}.$$ 

(3) Return $\mathcal{W}(V \cap G)$.

**Proposition 4.6.** Algorithm 4.5 correctly returns the multiprojective witness set collection $\mathcal{W}(V \cap G)$.

Proof. By [30], the set of endpoints of $H \left(V \cap L^d, S \rightarrow G \right)$ starting at the isolated points of $V \cap S \cap L^d$ consists of a superset of the isolated points of $V \cap G \cap L^d$. Hence, removing the nonisolated points, e.g., using [3], yields the isolated points of $V \cap G \cap L^d$. $\square$

**Example 4.7.** Consider $V$ to be as in Ex. 1,4 with $L^{(1,0)} = \{x_0 - 2x_1\}$ and $L^{(0,1)} = \{y_0 - y_1\}$. In particular, $\mathcal{W}(V) = \{W^{(1,0)}(V), W^{(0,1)}(V)\}$ where $|w^{(1,0)}(V)| = 1$ and $|w^{(0,1)}(V)| = 2$. Consider computing $V \cap G$ where $G$ is defined by $27x_0y_0 - 50y_0x_1 - 25x_0y_1 + 50x_1y_1 = 0$ having degree $(g_1, g_2) = (1, 1)$.

We can simplify Algorithm 4.2 by taking $s_1^{(1)} = x_0 - 2x_1$ and $s_2^{(1)} = y_0 - y_1$. Hence, we have that $\mathcal{W}(V \cap S) = \{W^{(0,1)}(V \cap S)\}$ where $w^{(0,1)}(V \cap S) = w^{(1,0)}(V) \cup w^{(0,1)}(V)$ consisting of three points.

For Algorithm 4.5, we only need to consider $d = (0, 0)$ for which Item 2a deforms from the three paths defined by deforming $S$ to $G$ along $V$ as shown in Figure 2. The set of three endpoints is $V \cap G$. 

MULTIPROJECTIVE WITNESS SETS AND A TRACE TEST 9
Algorithm 4.8. [Multiregeneration]
Given \( V = \{ G_1, \ldots, G_\ell \} \), compute a witness set collection \( \mathcal{W}(V) \).

1. Initialize a multiprojective witness set collection \( \mathcal{W}(P^{n_1} \times \cdots \times P^{n_\ell}) \) by solving a linear system of equations \( L^{(n_1, n_2, \ldots, n_\ell)} = 0 \)

2. For \( i = 0, \ldots, \ell - 1 \)
   a. Define \( X := \cap_{i=1}^{i} G_i \) and \( G := G_{i+1} \).
   b. Define \( Y \) to be the union of irreducible components of \( X \) not contained in \( G \).
   c. Define \( Z \) to be the union of irreducible components of \( X \) contained in \( G \).
   d. Initialize the witness set collections \( \mathcal{W}(Y) \) and \( \mathcal{W}(Z) \) to each be the empty set.
   e. For each \( W^e(X) \in \mathcal{W}(V) \), do:
      i. Partition \( W^e(X) = N^e \cup U^e \) where \( N^e = W^e(X) \setminus G \) and \( U^e = W^e(X) \cap G \).
      ii. Append to \( \mathcal{W}(Y) \) the witness set \( W^e(Y) := \{ X, L^e, N^e \} \).
      iii. Append to \( \mathcal{W}(Z) \) the witness set \( W^e(Z) := \{ X \cup \{ G \}, L^e, U^e \} \).
   f. For \( \deg G = (g_1, \ldots, g_\ell) \), construct general linear forms \( \{ s_i^{(j)} \} \) and let \( S \) be their product as in (1) which defines the hypersurface \( S \).
   g. Input \( \mathcal{W}(Y) \) and \( \{ s_i^{(j)} \} \) into Algorithm 4.2 to compute \( \mathcal{W}(Y \cap S) \).
   h. Input \( \mathcal{W}(Y \cap S) \) and \( G \) into Algorithm 4.5 to compute \( \mathcal{W}(Y \cap G) \).
   i. Merge \( \mathcal{W}(Z) \) and \( \mathcal{W}(Y \cap G) \) to form \( \mathcal{W}(\cap_{i=1}^{i} G_i) \).

3. Return \( \mathcal{W}(V) = \mathcal{W}(\cap_{i=1}^{i} G_i) \).

Proposition 4.9. Algorithm 4.8 correctly returns the multiprojective witness set collection \( \mathcal{W}(V) \).

Proof. For each \( e \), we have \( N^e = W^e(Y) \) so that Algorithms 4.2 and 4.5 compute \( \mathcal{W}(Y \cap G) \).

The set \( Z = X \setminus Y \) is the union of irreducible components of \( X \) which are contained in \( G \). For each \( e \), \( U^e = W^e(Z) \). Since each irreducible component of \( Z \) is an irreducible component of \( X \cap G \), we know that each point in \( U^e \) is an isolated point in \( X \cap G \cap L^e \). The result now follows immediately from

\[
X \cap G \cap L^e = (Y \cup Z) \cap G \cap L^e = (Y \cap G \cap L^e) \cup (Z \cap L^e).
\]

\( \square \)
Remark 4.10. The performance of Algorithm 4.8 is based on the ordering of the polynomials and the structure of the polynomials. Example systems which can be viewed using different multihomogenizations are considered in Section 5.

Remark 4.11. As formulated, some of the paths which need to be tracked in Algorithm 4.8 may be singular. Following the notation in the proof of Prop. 4.9, this occurs when an irreducible component of \( \mathcal{Y} \) for \( \ell < \ell' \) has multiplicity greater than one with respect to \( \{G_1, \ldots, G_i\} \). One can use deflation (see Remark 1.6) to produce paths with are nonsingular on \( (0, 1] \).

Another option is to apply Algorithm 4.8 to a randomization of \( V \) (see Section 2.2). In this case, all paths which need to be tracked are nonsingular on \( (0, 1] \) by Bertini’s Theorem so that one does not need to deflate paths. Two drawbacks of randomization, as discussed in Section 2.2, are the typical destruction of structure and the increase of degrees resulting in more paths which need to be tracked.

The following is an illustrative example using Algorithm 4.8.

Example 4.12. Consider the following polynomial system defined on \( \mathbb{P}^2 \times \mathbb{P}^2 \):

\[
V := \{ G_1, G_2, G_3 \} := \{ x_0 y_2 - x_2 y_1, \ x_1 y_2 - x_2 y_1, \ x_0 y_1 y_2 - x_1 y_0 y_2 \}. 
\]

It is easy to verify that \( V \) has four irreducible components:

- \( S_1 = \{ [[x_0, x_1, x_2], [1, 0, 0]] \} \) having dimension 2 and degree \( 1 \omega^{(2, 0)} \);
- \( S_2 = \{ [[x_0, x_1, 0], [y_0, y_1, 0]] \} \) having dimension 2 and degree \( 1 \omega^{(1, 1)} \);
- \( C_1 = \{ [[x_0, x_0, x_2], [y_0, y_0, y_2]] \mid x_2 y_0 = x_0 y_2 \} \) having dimension 1 and degree \( 1 \omega^{(1, 0)} + 1 \omega^{(0, 1)} \);
- \( C_2 = \{ [[0, 0, 1], [y_0, 0, y_2]] \} \) having dimension 1 and degree \( 1 \omega^{(0, 1)} \).

We demonstrate using Algorithm 4.8 to compute witness sets for the pure-dimensional components, i.e., \( S_1 \cup S_2 \) and \( C_1 \cup C_2 \), with a decomposition computed in Ex. 6.5.

\( \ell = 0 \). Since \( X = \mathbb{P}^2 \times \mathbb{P}^2 \) and \( \mathcal{G} = \mathcal{G}_1 \) a hypersurface of degree \( (1, 1) \), the witness point for \( \mathbb{P}^2 \times \mathbb{P}^2 \) is regenerated into two points as summarized in the following chart.

\[
\begin{array}{c|ccc}
\ell & (2, 1) & (1, 2) \\
\hline
w^e(\mathcal{G}_1) & 1 & 1 \\
\end{array}
\]

\( \ell = 1 \). For both \( d = (2, 1) \) and \( d = (1, 2) \), \( U^d = \emptyset \) so that all points are “nonsolutions” and must be regenerated. Since \( \mathcal{G} = \mathcal{G}_1 \) is a hypersurface of degree \( (1, 1) \), two start points are regenerated into four points as summarized in the following chart.

\[
\begin{array}{c|ccc}
\ell & (2, 0) & (1, 1) & (0, 2) \\
\hline
w^e(\mathcal{G}_1 \cap \mathcal{G}_2) & 1 & 2 & 1 \\
\end{array}
\]

\( \ell = 2 \). For \( d = (2, 0) \), \( N^d = \emptyset \) so that no points are regenerated. The point in \( U^d \) is the witness point for \( S_1 \). For \( d = (1, 1) \), both \( U^d \) and \( N^d \) consist of one point. The point in \( U^d \) is a witness point for \( S_2 \). Since \( \mathcal{G}_3 \) is a hypersurface of degree \( (1, 2) \), regenerating this point yields two nonisolated endpoints, one each on \( S_1 \) and \( S_2 \), and one isolated endpoint which is a witness point for \( C_1 \). For \( d = (0, 2) \), \( N^d \) consists of one point which must be regenerated. This yields two isolated endpoints, one each on \( C_1 \)
and $C_2$. Since $V = G_1 \cap G_2 \cap G_3$, the results for $V$ are summarized in the following chart.

<table>
<thead>
<tr>
<th>$e$</th>
<th>$(2,0)$</th>
<th>$(1,1)$</th>
<th>$(1,0)$</th>
<th>$(0,1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w^e(V)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

5. Multiregeneration Examples

The following provides some examples demonstrating multiregeneration described in Algorithm 4.8.

5.1. Comparison on a zero-dimensional variety. As mentioned in the introduction, multihomogeneous homotopies [29] were developed by observing a bihomogeneous structure when solving the inverse kinematics problem for 6R robots. Following [5] Ex. 5.2 with $\times$ denoting the cross product and $\circ$ denoting the dot product, the polynomial system $f(z_2, z_3, z_4, z_5)$ defined on $(\mathbb{C}^3)^4$ is

$$f_1 := z_1 \circ z_2 - c_1, \quad f_2 := z_5 \circ z_6 - c_5$$

$$(f_3, f_4, f_5) := (a_1 z_1 \times z_2 + d_2 z_2 + a_2 z_2 \times z_3 + d_3 z_3 + a_3 z_3 \times z_4 + d_4 z_4 + a_4 z_4 \times z_5 + d_5 z_5 + a_5 z_5 \times z_6 - p)$$

$$(f_6, f_7, f_8) := (z_2 \circ z_3 - c_2 z_3 z_4 - c_3 z_2 z_5 - c_4).$$

Each setup corresponds with homogenizing $f$ to yield a polynomial system $\bar{V}$ defined on a product of 1, 2, and 4 projective spaces based on natural groupings of the variables, namely $\{z_2, z_3, z_4, z_5\}$, $\{z_2, z_4\} \times \{z_3, z_5\}$, and $\{z_2\} \times \{z_3\} \times \{z_4\} \times \{z_5\}$, respectively. The corresponding multihomogeneous Bezout counts are 1024, 320, and 576, while the BKK root count is 288. These four counts are the total number of paths one needs to track using a 1-, 2-, and 4-homogeneous homotopy and a polyhedral homotopy, respectively.

We consider solving $f = 0$ using various multihomogeneous homotopies, a polyhedral homotopy, and various multiregressions. In particular, we consider using a 1-, 2-, and 4-homogeneous setup. Each setup corresponds with homogenizing $f$ to yield a polynomial system $V$ defined on a product of 1, 2, and 4 projective spaces based on natural groupings of the variables, namely $\{z_2, z_3, z_4, z_5\}$, $\{z_2, z_4\} \times \{z_3, z_5\}$, and $\{z_2\} \times \{z_3\} \times \{z_4\} \times \{z_5\}$, respectively. The corresponding multihomogeneous Bezout counts are 1024, 320, and 576, while the BKK root count is 288. These four counts are the total number of paths one needs to track using a 1-, 2-, and 4-homogeneous homotopy and a polyhedral homotopy, respectively.

In the multiregeneration, we only use slices which could lead to isolated solutions yielding partial information about the multidegrees in the intermediate stages. Tables 1 and 2 summarize the multiregeneration computations by listing the total number of start points and the total number of points in the resulting witness sets for each codimension.

5.2. Comparison on a system with singular solutions. Our next example arises from computing the Lagrange points of a three-body system where the small body is assumed to exert a negligible gravitational force on the other two bodies, e.g., Earth, Moon, and man-made satellite. The nondimensionalized system $f$ defined on $\mathbb{C}^6$ derived in [5] § 5.5.1, with variables $\rho_1, w, \delta_{13}, \delta_{23}, x, y$ and parameter $\mu$, the ratio of the masses of the two large bodies, is

$$\left\{ \begin{array}{l}
wp_1 - 1, \quad wp_2 - \mu, \quad (\rho_1 - x)^2 + y^2 - \delta_{13}^2, \\
(\omega_1^3 \delta_{23}^3 - \mu \delta_{23}^2 - \delta_{13}^4) x + \rho_1 \mu \delta_{23}^2 - \rho_2 \delta_{13}^2, \\
(\omega_1^3 \delta_{23}^3 - \mu \delta_{23}^2 - \delta_{13}^4) y \end{array} \right\}$$

where $p_2 = 1 - \rho_1$. For generic $\mu$, this system has 64 solutions counting multiplicity: 32 of multiplicity 1, 4 of multiplicity 2, 2 of multiplicity 3, and 2 of multiplicity 9.

We first consider solving $f = 0$ using a multiregeneration applied to $V$, a homogenization of $f$ based on using a 1-homogeneous and a 5-homogeneous structure defined by $\{\rho_1\} \times \{w\} \times \{\delta_{13}\} \times \{\delta_{23}\} \times \{x, y\}$. 
Table 1. Summary of multiregeneration solving the inverse kinematics of 6R robot using a 1- and 2-homogeneous setup.

<table>
<thead>
<tr>
<th>codim</th>
<th>1-homogeneous</th>
<th>2-homogeneous</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>start points</td>
<td>witness points</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>7</td>
<td>32</td>
<td>20</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>20</td>
</tr>
<tr>
<td>9</td>
<td>124</td>
<td>40</td>
</tr>
<tr>
<td>10</td>
<td>180</td>
<td>68</td>
</tr>
<tr>
<td>11</td>
<td>124</td>
<td>56</td>
</tr>
<tr>
<td>12</td>
<td>88</td>
<td>80</td>
</tr>
<tr>
<td>total</td>
<td>644</td>
<td>314</td>
</tr>
</tbody>
</table>

Table 2. Summary of multiregeneration solving the inverse kinematics of 6R robot using a 4-homogeneous setup.

<table>
<thead>
<tr>
<th>codim</th>
<th>start points</th>
<th>witness points</th>
<th>4-homogeneous</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>multidegree</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\omega^{(2,3,3,3)}$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>$\omega^{(2,3,3,2)}$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
<td>$\omega^{(2,3,3,2)} + \omega^{(2,2,3,2)} + \omega^{(2,2,2,2)} + \omega^{(2,2,2,1)} + \omega^{(2,2,1,2)} + \omega^{(2,1,2,2)} + \omega^{(2,1,1,2)} + \omega^{(2,1,1,1)} + \omega^{(2,1,1,1)}$</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>11</td>
<td>$\omega^{(2,3,3,2)} + \omega^{(2,2,3,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,1,2)} + \omega^{(2,1,2,2)} + \omega^{(2,1,2,1)} + \omega^{(2,1,1,2)} + \omega^{(2,1,1,1)} + \omega^{(2,1,1,1)}$</td>
</tr>
<tr>
<td>5</td>
<td>28</td>
<td>18</td>
<td>$\omega^{(2,3,3,2)} + \omega^{(2,2,3,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,1,2)} + \omega^{(2,1,2,2)} + \omega^{(2,1,2,1)} + \omega^{(2,1,1,2)} + \omega^{(2,1,1,1)} + \omega^{(2,1,1,1)}$</td>
</tr>
<tr>
<td>6</td>
<td>18</td>
<td>18</td>
<td>$\omega^{(2,3,3,2)} + \omega^{(2,2,3,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,1,2)} + \omega^{(2,1,2,2)} + \omega^{(2,1,2,1)} + \omega^{(2,1,1,2)} + \omega^{(2,1,1,1)} + \omega^{(2,1,1,1)}$</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>16</td>
<td>$\omega^{(2,3,3,2)} + \omega^{(2,2,3,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,1,2)} + \omega^{(2,1,2,2)} + \omega^{(2,1,2,1)} + \omega^{(2,1,1,2)} + \omega^{(2,1,1,1)} + \omega^{(2,1,1,1)}$</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
<td>14</td>
<td>$\omega^{(2,3,3,2)} + \omega^{(2,2,3,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,1,2)} + \omega^{(2,1,2,2)} + \omega^{(2,1,2,1)} + \omega^{(2,1,1,2)} + \omega^{(2,1,1,1)} + \omega^{(2,1,1,1)}$</td>
</tr>
<tr>
<td>9</td>
<td>28</td>
<td>24</td>
<td>$\omega^{(2,3,3,2)} + \omega^{(2,2,3,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,1,2)} + \omega^{(2,1,2,2)} + \omega^{(2,1,2,1)} + \omega^{(2,1,1,2)} + \omega^{(2,1,1,1)} + \omega^{(2,1,1,1)}$</td>
</tr>
<tr>
<td>10</td>
<td>48</td>
<td>24</td>
<td>$\omega^{(2,3,3,2)} + \omega^{(2,2,3,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,1,2)} + \omega^{(2,1,2,2)} + \omega^{(2,1,2,1)} + \omega^{(2,1,1,2)} + \omega^{(2,1,1,1)} + \omega^{(2,1,1,1)}$</td>
</tr>
<tr>
<td>11</td>
<td>48</td>
<td>20</td>
<td>$\omega^{(2,3,3,2)} + \omega^{(2,2,3,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,1,2)} + \omega^{(2,1,2,2)} + \omega^{(2,1,2,1)} + \omega^{(2,1,1,2)} + \omega^{(2,1,1,1)} + \omega^{(2,1,1,1)}$</td>
</tr>
<tr>
<td>12</td>
<td>40</td>
<td>16</td>
<td>$\omega^{(2,3,3,2)} + \omega^{(2,2,3,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,2,1)} + \omega^{(2,2,1,2)} + \omega^{(2,1,2,2)} + \omega^{(2,1,2,1)} + \omega^{(2,1,1,2)} + \omega^{(2,1,1,1)} + \omega^{(2,1,1,1)}$</td>
</tr>
<tr>
<td>total</td>
<td>264</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This 5-homogeneous structure was selected since it minimizes the multihomogeneous Bézout count amongst all possible partitions, namely 248 compared with the classical Bézout count of 1024. Even though the system has singular solutions, they only arise at the final stage of the multiregeneration.
with this setup so that all paths which need to be tracked are nonsingular on \((0,1]\). The results are summarized in Table 3 with the number of witness points counted with multiplicity.

<table>
<thead>
<tr>
<th>codim</th>
<th>1-homogeneous ([18])</th>
<th>5-homogeneous multidegree</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>start points</td>
<td>witness points</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
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</tr>
<tr>
<td>total</td>
<td>268</td>
<td>160</td>
</tr>
</tbody>
</table>

Table 3. Summary of using multiregeneration to solve the Lagrange points system

Due to the presence of singular solutions, we also applied these two multiregenerations to the homogenization of a perturbation of \(f\) as suggested in \([2]\), namely \(f + \epsilon\) where \(\epsilon \in \mathbb{C}^6\) is general. From the isolated roots of \(f + \epsilon\), the isolated roots of \(f\) are recovered using the parameter homotopy \(f + t \cdot \epsilon = 0\). The results matched those presented in Table 3 with the only change being the endpoints in the final stage are all nonsingular.

5.3. Comparison on a positive-dimensional variety. Our last example of multiregeneration arises from computing a rank-deficiency set of a skew-symmetric matrix. In particular, consider the system

\[
f(x, \lambda) = \begin{bmatrix} S(x) \cdot B \cdot \begin{bmatrix} 1 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}^T \end{bmatrix}
\]

where \(S(x) = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ -x_1 & 0 & x_6 & x_7 & x_8 & x_9 \\ -x_2 & -x_6 & 0 & x_{10} & x_{11} & x_{12} \\ -x_3 & -x_7 & -x_{10} & 0 & x_{13} & x_{14} \\ -x_4 & -x_8 & -x_{11} & -x_{13} & 0 & x_{15} \\ -x_5 & -x_9 & -x_{12} & -x_{14} & -x_{15} & 0 \end{bmatrix}\)

and \(B \in \mathbb{C}^{6 \times 6}\) is a fixed general matrix.

We focus on the skew-symmetric matrices \(S(x)\) of rank 4. That is, we aim to compute

\[
\{(x^*, \lambda^*) \in \mathbb{C}^{15} \times \mathbb{C}^8 \mid \text{rank } S(x^*) = 4 \text{ and } \lambda^* \text{ is the unique solution to } f(x^*, \lambda) = 0\}.
\]

Since \(f\) consists of 12 polynomials and \(\lambda \in \mathbb{C}^8\), the codimension of the projection onto \(x\) is at most 4. Therefore, from a witness set computational standpoint, we can intrinsically restrict to a 4-dimensional linear space in \(\mathbb{C}^{15}\), namely by fixing a general matrix \(A \in \mathbb{C}^{15 \times 4}\) and vector \(b \in \mathbb{C}^{15}\), and considering the elements of \(\mathbb{C}^{15}\) of the form \(Ay + b\) where \(y \in \mathbb{C}^4\). Thus, we consider the polynomial system

\[
F(y, \lambda) = f(Ay + b, \lambda).
\]

We consider polynomial systems \(V\) arising from homogenizing \(F\) using a 1-homogeneous \((\mathbb{C}^{12})\) and 2-homogeneous \((\mathbb{C}^4 \times \mathbb{C}^8)\) structure. Since each polynomial in \(V\), respectively, has degree 2 and multidegree \((1,1)\), we apply multiregeneration to a randomization of \(V\). In particular, the 1-homogeneous regeneration setup is equivalent to the regenerative cascade \([19]\). Table 4 summarizes the multiregenerations for each codimension by listing the total number of start points, the total number of endpoints.
MULTIPROJECTIVE WITNESS SETS AND A TRACE TEST

1-homogeneous [19]

<table>
<thead>
<tr>
<th>codim</th>
<th>start points</th>
<th>iso. solns</th>
<th>noniso. solns</th>
<th>nonsolns</th>
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</table>

Table 4. Summary of various regeneration procedures for positive-dimensional solving related to the rank-deficiency system

that satisfy the system and are isolated ($U^d$), the total number of nonisolated solutions (removed in multiregeneration algorithm), and the total number of “nonsolutions” ($N^d$). The only difference in this example between the 1- and 2-homogeneous regenerations are the number of start points of paths which need to be tracked yielding two additional costs for the 1-homogeneous setup. The first is the added cost in computing the additional start points themselves and the second is the typical added cost of tracking these extra paths which often become ill-conditioned near the end.

In this example, the solution set of $F = 0$ is actually an irreducible component of codimension 9 and degree 45 in $\mathbb{C}^{12}$ with multidegree $3\omega^{(3,0)} + 6\omega^{(2,1)} + 12\omega^{(1,2)} + 24\omega^{(0,3)}$ in $\mathbb{C}^4 \times \mathbb{C}^8$. The closure of the projection of this irreducible component onto the $y$ coordinates, namely

$$\{ y \in \mathbb{C}^4 \mid \text{rank } S(Ay + b) \leq 4 \},$$

is a hypersurface of degree 3 that is defined by the cubic polynomial $\sqrt{\det S(Ay + b)}$.

6. Decomposition and a trace test

After computing witness point sets for multihomogeneous varieties using multiregeneration described in Section 4, the last remaining piece is to decompose into multiprojective witness sets for each irreducible component. One key part of this decomposition is a trace test, first proposed in [36] for affine and projective varieties, which validates that a collection of witness points forms a witness point set for a union of irreducible components. We extend this to the multiprojective case in Section 6.1. The original motivation for such a test was to verify the generic number of solutions to a parameterized system with examples of this presented in Section 7.

Equipped with the membership test from Section 3, the trace test from Section 6.1 and monodromy [35], we complete the decomposition in Section 6.2.

6.1. Trace test. Given a collection of witness points lying on a pure-dimensional multiprojective variety $V$, the goal is to verify that they form a collection of witness point sets for some variety $Z \subset V$. 

We accomplish this by generalizing the trace test proposed in \cite{36} for affine and projective varieties to the multiprojective setting. To that end, for \( V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \), we first fix generic hyperplanes \( \mathcal{H}_i \) at infinity for each projective space \( \mathbb{P}^{n_i} \). Let \( H_i \) be the polynomial defining \( \mathcal{H}_i \),
\[
H = \prod_{i=1}^{n} H_i \quad \text{and} \quad \mathcal{H} = \bigcup_{i=1}^{n} \mathcal{H}_i
\]
so that \( \mathcal{H} \) is defined by the polynomial \( H \).

With this setup, we perform our trace using computations on the product space \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) with a general coordinate from the Segre product space \( \mathbb{P}^{(n_1+1) \cdots (n_k+1)-1} \) defined as follows.

**Definition 6.1.** A general coordinate in \( \mathbb{P}^{(n_1+1) \cdots (n_k+1)-1} \) derived from \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) has the form
\[
\rho := (\Box x_0 + \cdots + \Box x_{n_1}) \cdot (\Box y_0 + \cdots + \Box y_{n_2}) \cdots (\Box z_0 + \cdots + \Box z_{n_k})
\]
where \((x, y, \ldots, z)\) represents the coordinates in \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \).

Hyperplanes in the Segre product space \( \mathbb{P}^{(n_1+1) \cdots (n_k+1)-1} \) are defined by polynomials which are multilinear, i.e., linear in the variables for each \( \mathbb{P}^{n_i} \). We say that \( \mathcal{R} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) is a Segre linear slice defined by the polynomial \( R \) if \( R \) is multilinear. For example, \( \mathcal{H} \) in (2) is a Segre linear slice since the polynomial \( H \) in (2) is multilinear. With this, we are able to define the trace.

**Definition 6.2.** Let \( V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) be a pure \( c \)-dimensional multiprojective variety, \( 1 \leq c' \leq c \), \( \mathcal{R}_1, \ldots, \mathcal{R}_{c'} \) be general Segre linear slices defined by \( R_1, \ldots, R_{c'} \), respectively, and \( \mathcal{L}^e \) be a linear space of codimension \( |e| = c - c' \) defined by \( L^e \). Let \( \mathfrak{m} \subset V \cap \mathcal{L}^e \cap \mathcal{R}_1 \cap \cdots \cap \mathcal{R}_{c'} \) and consider the homotopy
\[
H^e (V \cap \mathcal{L}^e, \mathcal{R} \rightarrow \mathcal{R} + \mathcal{H}) = \begin{cases} 
V \\
L^e \\
R_1 + (1 - t) H \\
\vdots \\
R_{c'} + (1 - t) H
\end{cases}
\]
where \( H \) as in (2). For each \( m \in \mathfrak{m} \), let \( m(t) \) denote the path defined by this homotopy starting at \( m \). Then, the trace of \( m \) with respect to \( H^e \) and a general coordinate \( \rho \) is the average of the \( \rho \) coordinate of the paths \( m(t) \), i.e.,
\[
\text{Trace}^e_m (H^e (V \cap \mathcal{L}^e, \mathcal{R} \rightarrow \mathcal{R} + \mathcal{H}) (t) := \frac{1}{|\mathfrak{m}|} \sum_{m \in \mathfrak{m}} \rho(m(t)).
\]
The trace is said to be affine linear with respect to \( \mathcal{H}_1, \ldots, \mathcal{H}_k \) if it is a linear function in \( t \) when restricted to the affine chart where \( H_1 = \cdots = H_k = 1 \).

We illustrate with the following example.

**Example 6.3.** Consider the pure 1-dimensional variety \( V \subset \mathbb{P}^2 \times \mathbb{P}^1 \) defined by
\[
V = \{(y_1^2 - y_0^2)x_1 - y_1^2x_0, \quad (y_1^2 - y_0^2)x_2 - y_1^2x_0\}.
\]
In particular, \( V \) is the union of three irreducible varieties:
\begin{itemize}
\item \( \mathcal{L}_1 = \{([0, x_1, x_2], [1, 1])\} \),
\item \( \mathcal{L}_2 = \{([0, x_1, x_2], [1, -1])\} \), and
\item \( \mathcal{C} = \{(x_0, x_1, x_1), [y_0, y_1]) \mid (y_1^2 - y_0^2)x_1 = y_1^2x_0 \} \).
\end{itemize}

For simplicity, we take \( H_1 = x_2 \) and \( H_2 = y_1 \), and the Segre linear slice \( R_1 \) be defined by
\[
R_1 = (6x_0/7 + 3x_1/5 + 2x_2/7)(y_0 - y_1/2).
\]

The set \( \mathcal{V} \cap R_1 \) consists of five points \( m_1, \ldots, m_5 \), say where \( \mathcal{L}_1 \cap R_1 = \{m_1\} \), \( \mathcal{L}_2 \cap R_1 = \{m_2\} \), and \( \mathcal{C} \cap R_1 = \{m_3, m_4, m_5\} \). With the general coordinate
\[
\rho = (2x_0/7 - 5x_1/12 + 3x_2/17)(4y_0/13 - 3y_1/14),
\]
we consider the homotopy
\[
H^e (\mathcal{V}, R_1 \to R_1 + H) \begin{cases}
(x_1^2 - y_0^2)x_1 - y_1^2x_0 \\
(x_1^2 - y_0^2)x_2 - y_1^2x_0 \\
(6x_0/7 + 3x_1/5 + 2x_2/7)(y_0 - y_1/2) + (1 - t)x_2y_1.
\end{cases}
\]

Restricting to \( x_2 = y_1 = 1 \), Figure 3 plots the trace of this homotopy for two sets of start points, namely \( \{m_1, m_2, m_3, m_4\} \) and \( \{m_1, m_2, m_3, m_4, m_5\} \). This plot shows that the trace of \( \{m_1, m_2, m_3, m_4\} \) is not affine linear while the trace \( \{m_1, m_2, m_3, m_4, m_5\} \) is indeed affine linear.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Plot of the trace for \( \{m_1, \ldots, m_4\} \) in blue (nonlinear) and \( \{m_1, \ldots, m_5\} \) in red (linear).}
\end{figure}

As with the classical trace test [36], the trace with start points \( m \) is affine linear if and only if \( m \) is a collection of witness points for a multiprojective variety.

**Theorem 6.4 (Trace test).** Suppose that \( \mathcal{V} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) is a pure \( c \)-dimensional variety defined by \( \mathcal{V} \), \( \mathcal{H}_i \) is a general hyperplane at infinity defined by \( \mathcal{H}_i \) for \( \mathbb{P}^{n_i} \), \( \rho \) is a general coordinate, \( R_1, \ldots, R_c \) are general Segre linear slices defined by \( R_1, \ldots, R_c \) for \( 1 \leq c^' \leq c \), \( \mathcal{L}^e \) is a linear space of codimension \( |e| = c - c^' \) defined
by \( L^e \), and \( m \subset V \cap L^e \cap R_1 \cap \cdots \cap R_c \). Then, Trace\(^\rho\) \( H^e (V \cap L^e, R \to R + H) \) \((t)\) is affine linear in \( t \) if and only if there exists \( Z \subset V \cap L^e \), which is a union of irreducible components of dimension \( c' \), such that \( Z \cap R_1 \cap \cdots \cap R_c \cap m = m \).

**Proof.** By applying [36, Thm. 3.6] to the Segre product space \( \mathbb{P}^{(n_1+1)\cdots(n_k+1)-1} \), one has the result if the trace in every coordinate of \( \mathbb{P}^{(n_1+1)\cdots(n_k+1)-1} \) is affine linear. Clearly, if the trace is affine linear in every coordinate of \( \mathbb{P}^{(n_1+1)\cdots(n_k+1)-1} \), then it is affine linear in the general coordinate \( \rho \).

Conversely, we know, for all \( \alpha_{a_1}, \beta_{a_2}, \ldots, \gamma_{a_k} \in \mathbb{C} \),

\[
\sum_{(i_1, \ldots, i_k)} (\alpha_{i_1} \cdot \beta_{i_2} \cdots \gamma_{i_k}) q_{i_1, \ldots, i_k} = 0
\]

if and only if every \( q_{i_1, \ldots, i_k} = 0 \). Hence, if there exists \( q_{i_1, \ldots, i_k} \neq 0 \), then

\[
\sum_{(i_1, \ldots, i_k)} (\alpha_{i_1} \cdot \beta_{i_2} \cdots \gamma_{i_k}) q_{i_1, \ldots, i_k} \neq 0
\]

for general \( \alpha_{a_1}, \beta_{a_2}, \ldots, \gamma_{a_k} \in \mathbb{C} \). Therefore, if the trace of some coordinate in the Segre product space \( \mathbb{P}^{(n_1+1)\cdots(n_k+1)-1} \) is not affine linear in \( t \), then the trace for \( \rho \) will also not be affine linear in \( t \).  

In order to numerically test for affine linearity in \( t \), a classical approach is to evaluate the trace at 3 distinct values of \( t \) and decide if they lie on a line. Another approach is described in [6] which computes derivatives with respect to \( t \) of the trace. Two savings in this numerical computation are to work intrinsically on the hyperplanes \( H_i = 1 \) to reduce the number of variables, and to use slices which preserve structure and/or simplify the computation. These savings are utilized in Section 7.

### 6.2. Decomposition

Decomposition of a pure-dimensional variety corresponds with partitioning the collection of witness point sets into collections corresponding to each irreducible component. Similar to the classical trace test [36], one can use Theorem 6.4 to perform this partition by finding the smallest subsets for which the trace is affine linear. When the total number of points is small, such as in Ex. 6.3 which has only 5 points to consider, this is an effective approach. However, this quickly becomes impractical as the number of points increases. Thus, we propose using two methods to reduce the number of possible partitions one needs to consider. Both are based on the fact that smooth points of irreducible components are path connected.

The first approach is currently used in the classical affine or projective setting, namely to utilize random monodromy loops [35]. That is, one tracks the points along the variety as a linear slice is moved in a general loop in the corresponding Grassmannian. Each start point and corresponding endpoint defined by this loop must lie on the same irreducible component.

In the multiprojective setting, there is a second approach that one may utilize, namely to use one set of slices to perform membership testing (Section 3) on the points arising from a different set of slices. As with monodromy loops, all points which are connected by smooth paths must lie on the same irreducible component thereby reducing the possible number of partitions to consider.

**Example 6.5.** The multiregeneration computation in Ex. 4.12 yielded pure-dimensional witness sets for dimensions 1 and 2. We now illustrate how to decompose into the four irreducible components.
We first start with the pure 2-dimensional variety which has degree $1ω^{(2,0)} + 1ω^{(1,1)}$. Since the trace of $m_{(2,0)}$, which is the unique point in $V \cap L^{(2,0)}$, is affine linear, this shows that there are two irreducible components, one of degree $1ω^{(2,0)}$ (namely, $S_1$) and one of degree $1ω^{(1,1)}$ (namely, $S_2$).

We now turn to the pure 1-dimensional variety which has degree $1ω^{(1,0)} + 2ω^{(0,1)}$. For simplicity and concreteness, let $H_1 = x_2$ and $H_2 = y_2$ define the hyperplanes as infinity $H_1$ and $H_2$, respectively. Let $L^{(1,0)} = x_0 + x_1 - 2x_2$ and $L^{(0,1)} = y_0 - 2y_1 - y_2$ define the linear slices $L^{(1,0)}$ and $L^{(0,1)}$, respectively, with $R_1 = L^{(1,0)} \cdot L^{(0,1)}$ define the Segre linear slice $R_1$. Thus, $V \cap R_1$ consists of 3 isolated points

$$m_1 = ([1,1,1], [1,1,1]), \quad m_2 = ([1,1,1], [1,1,1]), \quad m_3 = ([0,0,1], [1,0,1])$$

where $\{m_1\}$ and $\{m_2, m_3\}$ are the sets of isolated points of $V \cap L^{(1,0)}$ and $V \cap L^{(0,1)}$, respectively. We now employ the membership test (see Section 3) applied to $m_1$ using $L^{(0,1)}$. For simplicity, we take $L^{(0,1)}_{m_1} = (1 + \sqrt{-1})(y_0 - 2y_1 + y_2)$ which defines $L^{(0,1)}_{m_1}$ and utilize the homotopy $H(V, L^{(0,1)} \to L^{(0,1)}_{m_1})$ with start points $m_2$ and $m_3$. The endpoints of the path starting at $m_2$ and $m_3$ are $m_1$ and $([0,0,1], [-1,0,1])$, respectively. Hence, $m_1$ and $m_2$ lie on the same irreducible component. The trace test shows that there are indeed two irreducible components, one from $\{m_1, m_2\}$ having degree $1ω^{(1,0)} + 1ω^{(0,1)}$ (namely $C_1$) and the other from $\{m_3\}$ having degree $1ω^{(0,1)}$ (namely $C_2$).

7. Trace test examples

We close with several examples using the multihomogeneous trace test from Section 6.

7.1. Tensor decomposition. In [8], the problem of computing the number $k$ of tensor decompositions of a general tensor of rank 8 in $C^3 \otimes C^6 \otimes C^6$ is formulated with [8] Thm. 3.5] proving $k \geq 6$. Computation 4.2 of [14] uses numerical algebraic geometry together with the fact the number of minimal decompositions of a general tensor of $\text{Sym}^3 C^3 \otimes C^2 \otimes C^2$ is also $k$ to conclude that $k = 6$. We use the trace test from Theorem 6.4 to confirm this result. To that end, we first consider a natural formulation as an affine variety. For $a, b \in C$, let $v_3(1,a,b) \in C^{10}$ be the degree 3 Veronese embedding, i.e.,

$$v_3(1,a,b) = (1, a, b, a, b, b, a, b, a, b, b, b).$$

For $x \in C^5$, consider the tensor $T(X) = v_3(1,x_1,x_2) \otimes (x_3,x_4) \otimes (1,x_5) \in C^{40}$. For general $P, Q \in C^{40}$, we consider the set

$$C = \left\{ \left( s, X^{(1)}, \ldots, X^{(8)} \right) \middle| P \cdot s + Q = \sum_{i=1}^{8} T \left( X^{(i)} \right) \right\} \subset C \times (C^5)^8.$$

It follows from [33] Thm. 3.42] that $C$ is an irreducible variety of dimension 1. Using the natural embedding of $C \times (C^5)^8 \hookrightarrow P^1 \times P^{40}$, we have the two natural hyperplanes at infinity and can restrict each being set to 1 resulting in computations back on $C \subset C \times (C^5)^8$. In order to maintain the invariance under the symmetric group on 8 elements, namely $S_8$, for $r_1, r_2 \in C$, we consider the Segre linear space $R_1$ defined by

$$R_1 = (s - r_1) \left( x_5^{(1)} + \cdots + x_5^{(8)} - r_2 \right).$$

After randomly selecting $P, Q \in (C^5)^8$ and $r_1, r_2 \in C$, we used Bertini [4] to compute $C \cap R_1$, which yielded a total of $528 \cdot 8! = 21,288,960$ points with precisely $6 \cdot 8!$ points satisfying $s - r_1 = 0$, i.e., $k = 6$. 
To verify that we have computed every point in \( C \cap R_1 \), we applied the trace test from Theorem 6.4 which showed that the trace was indeed affine linear confirming [14 Computation 4.2].

7.2. Alt’s problem. In 1923, Alt [11] formulated the problem of counting the number of coupler curves generated by four-bar linkages that pass through nine general points in the plane. The solution to this problem, namely 1442, was found using homotopy continuation in [40]. We use the trace test from Theorem 6.4 to confirm this result and study two related systems.

Following the formulation in [40], only the displacements between one selected point and the other eight points are of interest. These displacements are represented using isotropic coordinates \((\delta_j, \gamma_j) \in \mathbb{C}^2\) for \( j = 1, \ldots, 8 \). The polynomial system under consideration consists of 12 polynomials in 12 variables:

\[
a, b, m, n, x, y, \bar{a}, \bar{b}, \bar{m}, \bar{n}, \bar{x}, \bar{y}.
\]

The first four polynomials are:

\[
f_1 = n - ax, \quad f_2 = n - \bar{a}x, \quad f_3 = m - by, \quad f_4 = m - \bar{b}y.
\]

Next, for \( j = 1, \ldots, 8 \), we have \( f_{4+j} = \gamma_j \bar{r}_j + \gamma_j \gamma_j^0 + \bar{r}_j \gamma_j^0 \) where:

\[
\gamma_j := q_j r_j^y - q_j r_j^x, \quad \bar{r}_j := r_j p_j^y - r_j p_j^x, \quad \gamma_j^0 := p_j^x q_j - p_j^y q_j^x, \quad \text{and}
\]

\[
\bar{p}_j^y := \bar{p} - \delta_j x, \quad q_j^x := m - \delta_j x, \quad r_j^x := \delta_j (\bar{a} - x) + \delta_j (a - x) - \delta_j \bar{a},
\]

\[
\bar{p}_j^y := \bar{m} - \delta_j y, \quad q_j^y := m - \delta_j y, \quad r_j^y := \delta_j (\bar{b} - y) + \delta_j (b - y) - \delta_j \bar{b}.
\]

7.2.1. Original problem. To study Alt’s problem, we randomly selected \((\delta_j, \gamma_j) \in \mathbb{C}^2\) for \( j = 1, \ldots, 7 \) and \( u_1, v_\ell \in \mathbb{C} \) for \( \ell = 1, 2 \). For \( \delta_8 = u_1 s + v_1 \) and \( \bar{\delta}_8 = u_2 s + v_2 \), we consider the irreducible variety \( C \subset \mathbb{C} \times \mathbb{C}^{12} \) defined by \( f_1 = \cdots = f_{12} = 0 \) consisting of nondegenerate four-bar linkages whose coupler curve passes through \((\delta_j, \gamma_j)\) for \( j = 1, \ldots, 8 \). As in Section 7.1, we actually consider the natural embedding of \( \mathbb{C} \times \mathbb{C}^{12} \rightarrow \mathbb{P}^1 \times \mathbb{P}^{12} \) and restrict each natural hyperplane at infinity to be 1. To further simplify the computation, we actually model the computation related to computing the closure of the image of \( C \) under the map

\[
\pi(s, a, b, m, n, x, y, \bar{a}, \bar{b}, \bar{m}, \bar{n}, \bar{x}, \bar{y}) = (s, x).
\]

That is, for random \( r_1, r_2 \in \mathbb{C} \), we consider intersecting \( C \) with the Segre linear space \( R_1 \) defined by

\[
R_1 = (s - r_1)(x - r_2).
\]

We used Bertini to compute \( C \cap R_1 \) resulting in 32,358 points with precisely 8652 = 1442 · 2 · 3 of them satisfying \( s - r_1 = 0 \). In particular, 8652 corresponds to the number of distinct four-bar mechanisms while 1442 is the number of distinct coupler curves. The 6-fold reduction from mechanisms to coupler curves is due to a two-fold symmetry and Roberts cognates. Since the trace of the 32,358 points is indeed affine linear, this confirms the result of [40].
7.2.2. Product decomposition bound. A product decomposition bound is constructed in [31] by replacing 
\[ f_{4+j} = \gamma_j \gamma_j^0 + \gamma_j \gamma_j^0 = \gamma_j \gamma_j^0 + \gamma_j \gamma_j^0 \]
for \( j = 1, \ldots, 8 \) with 
\[ g_{4+j} = (a_j \gamma_j + \beta_j \gamma_j^0) \cdot (\mu_j \gamma_j + \nu_j \gamma_j^0) \]
where \( a_j, \beta_j, \mu_j, \nu_j \in C \). For generic choices, [31] showed that the resulting system has 18,700 isolated solutions, which is a product decomposition bound on the number of isolated solutions for Alt’s problem. We can repeat a similar computation for Alt’s original problem with this product decomposition system by letting \( C \) denote the union of the irreducible components of dimension 1 defined by randomly selecting all parameters except \( \delta_8 \). We then intersected \( C \) with the Segre linear slice \( R_1 \) defined by 
\[ R_1 = (\delta_8 - r_1)(x - r_2) \]
for random \( r_1, r_2 \in C \). Using Bertini, we find that \( C \cap R_1 \) consists of 37,177 points, precisely 18,700 satisfy \( \delta_8 - r_1 = 0 \). We confirm the result of [31] since the trace of the 37,177 points is affine linear.

7.2.3. Polyhedral bound. In our last example, we consider the family of systems \( F \) with the same monomial support as the polynomial system \( f_1, \ldots, f_{12} \) defining Alt’s problem as above. The number of solutions to a generic member of \( F \) is called the polyhedral bound or BKK bound [39]. We consider a line in \( F \) by randomly fixing all coefficients except the coefficient of \( ax \) in \( f_2 \), which we call \( p_2 \). Consider the set of isolated solutions over this line yields a variety of dimension 1 which we intersect with \( R_1 \) defined by 
\[ R_1 = (p_2 - r_1)(x - r_2) \]
for random \( r_1, r_2 \in C \). Using Bertini, we obtain 132,091 points, which was verified to be complete by the trace test, with exactly 79,135 satisfying \( p_2 - r_1 = 0 \). Therefore, our computation shows that the BKK (polyhedral) bound for this system is 79,135. Since this contradicts the bound of 83,977 reported in [39], we provide a proof based on exact computations in polymake [10] that 79,135 is indeed correct.

Theorem 7.1 (BKK bound for Alt’s problem). The BKK (polyhedral) bound for \( f_1, \ldots, f_{12} \) as above for Alt’s problem is equal to 79,135.

Proof. Following [39], we can use \( f_1, \ldots, f_4 \) to remove \( n, n, m, \) and \( m \) resulting in an unmixed system, i.e., all polynomials have the same monomial support, consisting of 8 polynomials of degree 7 in 8 variables with the same BKK (polyhedral) bound as the original system. Since \( \gamma_j, \gamma_j^0, \) and \( \gamma_j^0 \) are quartic polynomials, one naively would have expected the polynomials to have degree 8, which is not the case due to exact cancellation in the coefficients. Hence, to properly recover the monomial support, we used the following computation in Maple to find that the support of these 8 polynomials is 239 monomials.

```maple
#input
n := a*xHat: nHat := aHat*x: m := b*yHat: mHat := bHat*y:
pH := nHat - x*dHat: py := mHat - y*dHat:
qx := n - xHat*d: qy := m - yHat*d:
rx := d*(aHat - xHat) + dHat*(a - x) - d*dHat:
ry := d*(bHat - yHat) + dHat*(b - y) - d*dHat:
g := qx*ry - qy*rx:
```
\[
gHat := rx*py - ry*px:
gZero := px*qy - py*qx:
f := g*gHat + g*gZero + gHat*gZero:
\]
\[
nops([coeffs(expand(f),[a,b,x,y,aHat,bHat,xHat,yHat])])
\]
\[
239 \quad \#output
\]

We then used the software \texttt{polymake} \cite{10} to compute the vertices of the polytope which is the convex hull of these 239 monomials resulting in 150 vertices. We note that \cite{39} reported 259 monomials and 158 vertices with the lower values in our computation possibly resulting from using symbolic computations in \texttt{Maple} to remove monomials whose coefficients are identically zero. To complete the proof, we used \texttt{polymake} to compute the volume of this polytope, which was 2261/1152. Hence, the BKK (polyhedral) bound is equal to

\[
8! \cdot \frac{2261}{1152} = 79,135.
\]

\[\square\]

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