A HYBRID SYMBOLIC-NUMERICAL APPROACH TO THE CENTER-FOCUS PROBLEM

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Abstract. We propose a new hybrid symbolic-numerical approach to the center-focus problem. The method allowed us to obtain center conditions for a three-dimensional system of differential equations, which was previously not possible using traditional, purely symbolic computational techniques.

1. Introduction

1.1. **Background.** Determination of the local stability of an isolated singular point for a system of ordinary differential equations (ODEs) is one of the fundamental problems encountered across various branches of applied sciences and engineering. For a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

where $\mathbf{f} : \mathbb{R}^n \supset \Delta \to \mathbb{R}^n$ is smooth, and x_0 is a singularity, i.e. $\mathbf{f}(\mathbf{x}_0) = 0$, the celebrated Hartman-Grobman theorem [12] states that the linearization of (1) or equivalently the set of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the Jacobian matrix $D\mathbf{f}(\mathbf{x}_0)$ characterizes the local qualitative behavior of the trajectories provided that the eigenvalues have non-zero real part, i.e. $Re(\lambda_j) \neq 0$. In this case, we say that x_0 is hyperbolic, otherwise we say that it is nonhyperbolic otherwise. To establish the local stability of nonhyperbolic singular points, higher order terms have to be taken into account.

One of the simplest and well-known stability questions is the *center-focus* (or *center*) problem, originally defined for planar polynomial differential systems, i.e., system (1) when $n = 2$ and $\mathbf{f}$ is a system of 2 polynomials in $\mathbb{R}[\mathbf{x}]$ of some degree m. It consists of obtaining conditions on the coefficients of $\mathbf{f}(\mathbf{x})$ to distinguish between a local focus (see Fig. 1(a)) or a center (see Fig. 1(b)), which has been the subject of intensive research (e.g., [59, 68, 72, 73, 15, 66, 10, 9, 56, 14, 23]). Although the problem is open in its full generality, it has been solved for some important subclasses of planar polynomial vector fields. As an example, consider the quadratic system defined by

$$\begin{aligned} \dot{u} &= v + a_1u^2 + a_2uv + a_3v^2 \\ \dot{v} &= -u + a_4u^2 + a_5uv + a_6v^2, \end{aligned} \tag{2}$$

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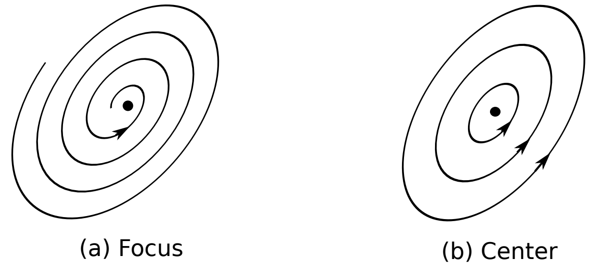

FIGURE 1. An example of a stable focus (a) and a center (b).

where $a_1, \ldots, a_6 \in \mathbb{R}$. The center conditions were established by Dulac [19] and Kapteyn [42]. It is well-known (see e.g. [73, 56]) that, for system (2), the Bautin ideal $\mathscr{B}$ is generated by the first three focus quantities of this system [7]. Moreover, the center variety $\mathbf{V}(\mathscr{B}) \subset \mathbb{R}^6$ of the ideal $\mathscr{B}$ has four irreducible components, namely

$$\mathbf{V}(\mathscr{B}) = \mathbf{V}(I_{Ham}) \cup \mathbf{V}(I_{sym}) \cup \mathbf{V}(I_{\triangle}) \cup \mathbf{V}(I_{con}),$$

corresponding to Hamiltonian systems, reversible systems, the Zariski closure of those systems having three invariant lines, and the Zariski closure of systems having an invariant conic and an invariant cubic, respectively.

The center-focus problem can also be defined for higher dimensional systems [8] and have recently been studied for a number of three-dimensional families [21, 11, 27, 49, 50, 51]. We continue this study here by applying our new symbolic-numerical approach to a three-dimensional system presented in Sec. 1.3 with results presented in Theorems 1 and 3.

1.2. **Computational challenges and the new approach.** The process of solving the center-focus problem for a specific system of differential equations can be divided into three steps [13]. First, the computation of certain number, say $p \in \mathbb{N}$, of *focus quantities* (also called Lyapunov quantities), which are polynomials in the parameters of the system. Second, finding the common zeros of the polynomial system formed by the focus quantities, or more precisely the determination of the irreducible component of the variety of the ideal generated by the first p focus quantities. Third, for the system restricted to each component, one checks if the necessary conditions for the existence of a center can be applied. This typically involves the application of the Darboux theory of integrability or reduction to the center manifold.

Techniques for efficient computation of Lyapunov quantities has been motivated both by mathematical and engineering problems. Over the years, a number of algorithms have been developed [57, 53, 29, 30, 48, 43, 71]. In this work, we used a method (described in [21]) for computing the focus quantities for a system in dimension three, which is based on the equivalence of the existence of a center and a local analytic first integral in the neighborhood of a singular point (more details are provided in Sec. 2). The advantage of this approach is that it allows to avoid center manifold approximation, which is especially important since its power series approximation of analytic or even polynomial systems need not converge (e.g., see [1, 60, 51]).

From the computational point of view, the biggest obstacle in solving the center-focus problem for a specific system is the determination of the irreducible components of the variety (i.e., solution set) defined by a certain number of focus quantities. The most common approach [2, 32, 24] is the application of computer algebra algorithms for computing the primary decomposition of the ideal generated by the focus quantities such as Gianni-Trager-Zacharias (GTZ) [31] or Shimoyama-Yokoyama (SY) [58], which have been implemented in various symbolic packages (e.g. Singular [35], or Macaulay [34]). The computational difficulty related with Gröbner basis calculation over the field of characteristic zero was eased by implementation of modular arithmetics [70, 20, 55], and successfully used in numerous problems [22, 36, 24, 67]. Unfortunately, in practice, the application of algorithms that use Gröbner bases (also with modular arithmetics) is computationally very heavy and the center conditions can only be obtained for specific systems with few parameters. In this paper, we replace this particular step and find the common zeros of the polynomial systems formed by focus quantities using numerical algebraic geometry techniques (for more details, see Sec. 3 and the books [6, 65]). The parallelizablity of numerical algebraic geometry together with a regeneration based approach [38, 41] and exactness recovery [4] provides a natural alternative to Gröbner basis methods. In particular, for the first time, we are able to solve the center-focus problem for a quadratic, three-dimensional system described next.

1.3. **An application.** Consider a third-order differential equation of the form

$$\dddot{u} = \ddot{u} + \dot{u} + u + f(u, \dot{u}, \ddot{u}), \tag{3}$$

where $f = f(u, \dot{u}, \ddot{u}) \in \mathbb{R}[u, \dot{u}, \ddot{u}]$ is a polynomial of degree m. Following [49], we can equivalently write

$$\dot{u} = -v + h(u, v, w), \qquad \dot{v} = u + h(u, v, w), \qquad \dot{w} = -w + h(u, v, w), \tag{4}$$

where $h(u, v, w) = f(-u+w, v-w, u+w)/2$, which we call the *standard form* of system (3). Note that the origin of (4) is a nonhyperbolic singularity at which the associated Jacobian has two purely imaginary eigenvalues $\lambda_{1,2} = \pm i$ and $\lambda_3 = -1$. Various dynamic aspects of systems of the form (4) have recently been considered, including the center conditions [11, 18, 21, 50], limit cycle bifurcations [69, 51], Lie symmetries [27], and isochronicity [54]. In particular, the center conditions on the local center manifold for system (4), where

$$h(u, v, w) = a_1 u^2 + a_2 v^2 + a_3 w^2 + a_4 uv + a_5 uw + a_6 vw, \tag{5}$$

were studied in [49]. Although it was possible to compute the first eight focus quantities, standard symbolic algorithms (e.g. GTZ and SY) were not able to provide the decomposition of the Bautin ideal into primes for a general six-parameter system, even over the field of non-zero characteristics. On the other hand, the application of our hybrid approach using numerical algebraic geometry to decompose described in this paper, allowed us to obtain the center conditions for a general six-parameter system (4).

Theorem 1. *The system* (4) *with* $h(u, v, w)$ *as in* (5) *admits a center on the local center manifold if and only if one of the following holds:*

(1) $a_1 = a_2 = a_4 = 0$
(2) $a_1 - a_2 = a_3 = a_5 = a_6 = 0$
(3) $a_1 + a_2 = a_3 = a_5 = a_6 = 0$
(4) $a_1 + a_2 = 2a_2 - a_3 + a_6 = a_3 - a_4 - 2a_5 = 2a_4 + 3a_5 + a_6 = 0$
(5) $2a_1 - a_6 = 2a_2 + a_5 = 2a_3 - a_5 + a_6 = a_4 + a_5 + a_6 = 0$
(6) $a_1 - a_2 = 2a_2 + a_6 = a_4 = a_5 + a_6 = 0$
(7) $2a_1 + a_2 = 2a_2 + a_6 = 4a_3 + 5a_6 = a_4 = 2a_5 - a_6 = 0.$

As an easy conclusion, note that the irreducible components of the center variety (i.e. the variety of the Bautin ideal generated by the focus quantities) of system (4) for quadratic h (5) are vector subspaces of its six-dimensional parameter space, which was conjectured in [49].

1.4. **Outline.** The rest of the paper is organized as follows. Section 2 summarizes focus quantities and their computation. Section 3 summarizes the numerical algebraic geometric solving approach along with exactness recovery method used to prove Theorem 3 in Section 4. Appendix A presents the Dulac-Kapteyn criterion of quadratic planar systems with Appendix B summarizing Darboux theory of integrability.

2. Focus quantities computation in $\mathbb{R}^3$

This section is a review of the method described in [21] (see also [49, 51]) for studying the center problem on a center manifold for vector fields in dimension three. Let $X : U \to \mathbb{R}^3$ be a real analytic vector field, such that $DX(0)$ has one non-zero and two purely imaginary eigenvalues. By an invertible linear change of coordinates and a possible rescaling of time, the system of differential equations $\dot{\mathbf{u}} = X(\mathbf{u})$ can be written in the form

$$\begin{aligned} \dot{u} &= -v + P(u, v, w) \\ \dot{v} &= u + Q(u, v, w) \\ \dot{w} &= \beta w + R(u, v, w), \end{aligned} \tag{6}$$

where β is a non-zero real number. Let $X = (-v+P)\partial/\partial u + (u+Q)\partial/\partial v + (\beta w + R)\partial/\partial w$ denote the corresponding vector field. A *local first integral* of system (6) is a nonconstant differentiable function H defined in a neighborhood of the origin in $\mathbb{R}^3$ mapping into $\mathbb{R}$ that is constant on trajectories of (6), equivalently, H satisfies

$$XH := (-v + P)\frac{\partial H}{\partial u} + (u + Q)\frac{\partial H}{\partial v} + (\beta w + R)\frac{\partial H}{\partial w} \equiv 0 \tag{7}$$

sufficiently close to the origin. A *formal first integral* for system (6) is a non-constant formal power series H in u, v and w such that when P, Q, and R are expanded in power series at the origin, every coefficient in the formal power series in (7) is zero.

It is well-known that system (6) admits a local center manifold W^c_{loc} at the origin, e.g., see [44, Thm. 5.1]. One of the main tools for detecting a center on a center manifold is the following theorem (see, e.g., [8, 21]).

Theorem 2. *The following statements are equivalent.*

(a) *The origin is a center for $X\,|_{W^c_{loc}}$.*
(b) *System (6) admits a local analytic first integral at the origin.*
(c) *System (6) admits a formal first integral at the origin.*

In fact, a real analytic local first integral from statement (b) (as well as a formal first integral from statement (c)) can always be chosen to be of the form $H(u,v,w) = u^2+v^2+\cdots$ where the dots mean higher order terms in a neighborhood of the origin in $\mathbb{R}^3$.

The equivalence of statements (a) and (b) is called the *Lyapunov Center Theorem* with a proof presented in, e.g., [8]. By this theorem, we can restrict our efforts to investigate the conditions for the existence of a first integral H which is equivalent to determine necessary and sufficient conditions for the existence of a center or a focus on the local center manifold.

From now on, we assume that P, Q and R in (6) are polynomials. We begin by introducing the complex variable $x = u + iv$. The first two equations in (6) are equivalent to a single equation $\dot{x} = ix + \cdots$, where the dots represent a sum of homogeneous polynomials of degrees between 2 and n. Let $\bar{x}$ denote the complex conjugate of x. We add to this equation its complex conjugate, replacing $\bar{x}$ everywhere by y which is regarded as an independent complex variable and replacing w by z simply as a notational convenience. This yields the following complexification of (6):

$$\begin{aligned}
\dot{x} &= ix + \sum_{p+q+r=2}^{n} a_{pqr} x^p y^q z^r,\\
\dot{y} &= -iy + \sum_{p+q+r=2}^{n} b_{pqr} x^p y^q z^r,\\
\dot{z} &= \beta z + \sum_{p+q+r=2}^{n} c_{pqr} x^p y^q z^r,
\end{aligned} \tag{8}$$

where $b_{qpr} = \bar{a}_{pqr}$ and c_{pqr} are such that $\sum_{p+q+r=2}^{n} c_{pqr} x^p \bar{x}^q w^r$ is real for all $x \in \mathbb{C}$ and $w \in \mathbb{R}$. Let X be the corresponding vector field of system (8) on $\mathbb{C}^3$. Existence of a first integral $H(u,v,w) = u^2 + v^2 + \cdots$ for system (6) is equivalent to the existence of a first integral for system (8), denoted again by H, of the form

$$H(x,y,z) = xy + \sum_{j+k+\ell=3} v_{jkl} x^j y^k z^\ell. \tag{9}$$

We now investigate the existence of a first integral H for system (8) by computing the coefficients of XH and equating them to zero. When H has the form (9), the coefficient $g_{jk\ell}$ of $x^j y^k z^\ell$ in XH can be calculated explicitly (see [21]). Except when $j = k$ and $\ell = 0$, the equation $g_{jk\ell} = 0$ can be solved uniquely for $\nu_{jk\ell}$ in terms of the known quantities $\nu_{\alpha\beta\gamma}$ with $\alpha+\beta+\gamma < j+k+\ell$. A formal first integral H thus exists if $g_{KK0} = 0$ for all $K \in \mathbb{N}$. An obstruction to the existence of the formal series H occurs when the coefficient g_{KK0} is non-zero. This coefficient is the K^{th} *focus quantity* and it can be expressed as

$$g_{KK0} = \sum_{\substack{j+k=2 \\ j\geq 0, k\geq 0}}^{2K-1} (j\, a_{K-j+1,K-k,0} + k\, b_{K-j,K-k+1,0})\, v_{j,k,0} \;+\; \sum_{\substack{j+k=2 \\ j\geq 0, k\geq 0}}^{2K-2} c_{K-j,K-k,0}\, v_{j,k,1}, \tag{10}$$

where we have made the natural assignments $v_{110} = 1$ and $v_{\alpha\beta\gamma} = 0$ for $\alpha+\beta+\gamma = 2$ but $(\alpha,\beta,\gamma) \neq (1,1,0)$. We know $g_{110} = 0$ and g_{220} is uniquely determined, but the remaining ones depend on the choices made for v_{KK0}, $K \in \mathbb{N}$, $K \geq 2$. Once such an assignment is made, H is determined and satisfies

$$XH(x,y,z) = g_{220}(xy)^2 + g_{330}(xy)^3 + \cdots .$$

It is known that if at least one focus quantity is non-zero for a choice of v_{KK0}, then the same is true for every other choice of the v_{KK0}. The vanishing of all focus quantities, i.e.,

$$g_{KK0} = 0 \qquad \text{for} \qquad K \geq 2 \tag{11}$$

is both a necessary and sufficient condition for the existence of a center on the center manifold, otherwise there is a focus (see [21]).

By Hilbert's basis theorem, there exists $K_0 \geq 2$ such that the set of solutions of $g_{KK0} = 0$ for all $2 \leq K \leq K_0$ is equivalent set defined by an infinite system (11). Since such a K_0 is not known *a priori*, we will apply an iterative approach that solves $g_{KK0} = 0$ for $2 \leq K \leq M+1$ given the solution set of $g_{KK0} = 0$ for $2 \leq K \leq M$. Without knowing K_0, solving using any $M \geq 2$ does always yield necessary conditions.

3. Numerical algebraic geometry

Symbolic methods, such as Gröbner basis techniques, take an algebraic viewpoint for solving systems of polynomial equations. In broad terms, they manipulate equations to obtain new relations describing the solution set. An alternative approach is to use a geometric viewpoint which manipulates solution sets. Following a numerical algebraic geometry approach, solution sets are represented by *witness sets* that we discuss below. A more detailed comparison of symbolic and numerical approaches is provided in [3].

The field of numerical algebraic geometry grew out of the use of homotopy continuation for computing isolated solutions to a system of polynomial equations. We will first briefly explain using basic homotopy continuation on a polynomial system $F : \mathbb{C}^N \to \mathbb{C}^N$, that is, $F(x) = 0$ defines a system of N polynomial equations in N variables. The idea is to select another polynomial system $G : \mathbb{C}^N \to \mathbb{C}^N$ related to F such that $G(x) = 0$ is "easy" is to solve. The simplest example is $G_i(x) = x_i^{d_i} - 1$ where $d_i = \deg F_i$, but there is a wide range of constructing the so-called *start systems* which exploit structure in F (see [6, 65] for a broad overview). Let $S \subset \mathbb{C}^N$ denote the set of isolated nonsingular solutions of $G = 0$.

The next step is to construct a homotopy $H : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ connecting G and F, say

$$H(x,t) = F(x) \cdot (1-t) + \gamma \cdot t \cdot G(x)$$

for a randomly selected $\gamma \in \mathbb{C}$. For each $s \in S$, the homotopy H defines a solution path $x_s(t)$ such that $x_s(1) = s$ and $H(x_s(t), t) \equiv 0$ with the goal of computing the endpoint $x_s(0) = \lim_{t \to 0^+} x_s(t)$. In fact, this limit is either a point in $\mathbb{C}^N$ which must be a solution of $F = 0$ or the path is said to be diverging to infinity. By differentiating $H(x_s(t), t)$ with respect to t, one obtains the Davidenko differential equation

$$J_x H(x_s(t), t) \cdot \dot{x}_s(t) = -J_t H(x_s(t), t).$$

By including the randomly selected γ, called the "gamma trick," the Jacobian matrix $J_x H$ is invertible along the path for $t \in (0, 1]$ with probability one and thus one can use predictor-correct techniques to track the solution path $x_s(t)$ starting at $x_s(1) = s$ in order to approximate $x_s(0)$. We refer the interested reader to [6, 65] for more details about path tracking and using endgames to estimate $x_s(0)$. In the end, the set $E \subset \mathbb{C}^N$ of convergent endpoints of all the paths $x_s(t)$ for $s \in S$ is a superset of the isolated nonsingular solutions of $F = 0$.

We now turn our attention to computing the solution set of $F = 0$, denoted $\mathcal{V}(F) \subset \mathbb{C}^N$, for a polynomial system $F : \mathbb{C}^N \to \mathbb{C}^n$. Geometrically, $\mathcal{V}(F)$ can be decomposed into a union of irreducible components $\mathcal{V}(F) = \cup_{i=1}^r V_i$. This corresponds algebraically to a prime decomposition of the radical ideal generated by F, namely $\sqrt{I(F)} = \cap_{i=1}^r I(V_i)$. Numerical algebraic geometry describes an irreducible decomposition of $\mathcal{V}(F)$ by computing a witness set for each V_i, called a numerical irreducible decomposition.

Suppose that V is an irreducible component of $\mathcal{V}(F)$ for some polynomial system F. A witness set for V is the triple $\{F, \mathcal{L}, W\}$ where $\mathcal{L} \subset \mathbb{C}^N$ is general linear subspace of codimension $d = \dim V$ and $W = V \cap \mathcal{V}(\mathcal{L})$ so that $|W| = \deg V$. Here, the definition of general means that $\mathcal{L}$ intersects V transversely, which is a Zariski open condition on the Grassmannian of codimension d linear subspaces in $\mathbb{C}^N$. The books [6, 65] provide for more information about witness sets including performing computations on irreducible components which have multiplicity > 1 with respect to F.

A witness set for an irreducible component $V \subset \mathbb{C}^N$ facilitates additional computations that can be performed on V. Of particular interest to the problems discussed in this article include the recovery of exact polynomials that vanish on V, determining the existence of real points in V, and intersecting V with another solution set.

With an input polynomial system with exact coefficients, e.g., in $\mathbb{Q}$, one often would like exact output. Although the internal computations and witness sets rely upon numerical approximations, there exist techniques for recovering exact answers which can then be verified using exact symbolic methods, which is typically computationally inexpensive. For the problems at hand here, we use the exactness recovery technique described in [4] which uses a sufficiently accurate numerical approximation of a sufficiently general point on V to compute polynomials with integer coefficients that vanish on V. This method is based on using a lattice-base reduction technique such as LLL [45] or PSLQ [25].

In many applications, only real solutions or components which contain real points are of interest, which is the case here. The approach of [37] uses critical points conditions of the distance function to determine if V, represented by a witness set, contains real points. If $V \cap \mathbb{R}^N = \emptyset$, then we can disregard this component from further computations.

We conclude this section by describing the intersection approach built from witness sets which is used in the subsequent section. For this situation, we consider a sequence of polynomial systems of interest, namely $F_k = \{f_1, \ldots, f_k\}$ for $k \geq 1$. Given witness sets for the irreducible components of $\mathcal{V}(F_k)$, our goal is to compute witness sets for the irreducible components of $\mathcal{V}(F_{k+1}) = \mathcal{V}(F_k) \cap \mathcal{V}(f_{k+1})$. For consistency, we assume $\mathcal{V}(F_k) \subset \mathbb{C}^N$ since one can easily adjust the methods to work on projective space which will arise below since each g_{KK0} is homogeneous in $a_1, \ldots, a_6$, i.e., $\mathcal{V}(g_{KK0})$ is naturally a hypersurface in $\mathbb{P}^5$.

For the base case, we need to decompose the hypersurface $\mathcal{V}(F_1) = \mathcal{V}(f_1)$, which can be readily performed, e.g., via [64].

Now, suppose that we are given witness sets for the irreducible components $V_{k,1}, \ldots, V_{k,n_k}$ of $\mathcal{V}(F_k)$. For each $j \in \{1, \ldots, n_k\}$, we need to compute $V_{k,j} \cap \mathcal{V}(f_{k+1})$ using the provided witness set for $V_{k,j}$, say $\{F_k, \mathcal{L}_{k,j}, W_{k,j}\}$. Clearly, if f_{k+1} vanishes identically on $V_{k,j}$, we know that $V_{k,j}$ is an irreducible component of $\mathcal{V}(F_{k+1})$. Thus, we shall assume that $V_{k,j}$ is not contained in the hypersurface $\mathcal{V}(f_{k+1})$ so that $V_{k,j} \cap \mathcal{V}(f_{k+1})$ is either empty or consists of irreducible components of dimension one less than $V_{k,j}$. With this assumption, if $d := \dim V_{k,j}$ is zero, we know $V_{k,j} \cap \mathcal{V}(f_{k+1}) = \emptyset$. Thus, we also assume that $d > 0$.

We compute $V_{k,j} \cap \mathcal{V}(f_{k+1})$ using a regenerative intersection approach developed in [40, 41] which builds on the diagonal intersection [63] and the regenerative cascade [39, 38]. It can be performed using Bertini [5]. To perform this computation, we select a general hyperplane $\mathcal{H}$ and codimension $d-1$ linear space $\mathcal{K}$ so that $\mathcal{L}_{k,j} = \mathcal{H} \cap \mathcal{K}$. Our first goal is to compute the finite set of points $V_{k,j} \cap \mathcal{K} \cap \mathcal{V}(f_{k+1})$ given $W_{k,j} = V_{k,j} \cap \mathcal{K} \cap \mathcal{H}$. If $g = \deg f_{k+1}$, we select general hyperplanes $\mathcal{H}_1, \ldots, \mathcal{H}_g$ and compute $V_{k,j} \cap \mathcal{K} \cap \mathcal{H}_\ell$ for $\ell = 1, \ldots, g$ by standard homotopy continuation from $V_{k,j} \cap \mathcal{K} \cap \mathcal{H}$. Thus, we have computed

$$V_{k,j} \cap \mathcal{K} \cap (\cup_{\ell=1}^{g} \mathcal{H}_\ell)$$

which, again by standard homotopy continuation, can be used to compute $V_{k,j} \cap \mathcal{K} \cap \mathcal{V}(f_{k+1})$.

Now, to compute witness sets for the irreducible components of $V_{k,j} \cap \mathcal{V}(f_{k+1})$, which will have the form $\{F_{k+1}, \mathcal{K}, \bullet\}$, we simply need to partition the set of points $V_{k,j} \cap \mathcal{K} \cap \mathcal{V}(f_{k+1})$

into subsets corresponding to distinct irreducible components. This is accomplished by using random monodromy loops [61] with the decomposition certified by the trace test [62]. In short, the idea is to move the linear space $\mathcal{K}$ in a random loop in the Grassmannian and observe which points are connected by such paths: the points in $V_{k,j} \cap \mathcal{K} \cap \mathcal{V}(f_{k+1})$ which are path connected by such random loops must lie on the same irreducible component. Thus, monodromy loops yield necessary conditions. The (linear) trace test yields a sufficient condition which follows from the fact that, as the linear space $\mathcal{K}$ is moved in parallel, the centroid of points arising from a union of irreducible components must move linearly. One can read about further details of computing such decompositions in [6, 65].

4. Center conditions for a three dimensional quadratic system

The following provides a proof of Theorem 1. We note that, without loss of generality, we can always assume that either $a_6 = 0$ or $a_6 = 1$. The latter follows immediately by the change of variables $(u, v, w) \mapsto (x/a_6, y/a_6, z/a_6)$ and rescalling of time $dt = a_6 d\tau$. Thus, the seven cases in Theorem 1 can be split into ten cases, five each for $a_6 = 0$ and $a_6 = 1$. After showing these ten cases, we then describe how the seven cases of Theorem 1 follow.

Theorem 3. *Consider system* (4) *with $h(u, v, w)$ as in* (5).

The system (4) *with $a_6 = 0$ admits a center on the local center manifold if and only if one of the following holds:*

(a) $a_1 - a_2 = a_3 = a_5 = 0$
(b) $a_1 + a_2 = a_3 = a_5 = 0$
(c) $a_1 = a_2 = a_4 = 0$;
(d) $a_1 + a_2 = 2a_1 + a_3 = 6a_1 - a_4 = 4a_1 + a_5 = 0$
(e) $a_1 = a_2 + a_3 = 2a_2 - a_4 = 2a_2 + a_5 = 0$.

The system (4) *with $a_6 = 1$ admits a center on the local center manifold if and only if one of the following holds:*

(f) $a_1 = a_2 = a_4 = 0$
(g) $2a_1 - 1 = a_4 + a_5 + 1 = 2a_2 + a_5 = 2a_3 - a_5 + 1 = 0$
(h) $2a_1 + 1 = 2a_2 + 1 = a_4 = a_5 + 1 = 0$
(i) $a_1 + a_2 = 4a_2 - a_5 + 3 = 6a_2 + a_4 + 5 = 2a_2 - a_3 + 1 = 0$
(j) $4a_1 - 1 = 2a_2 + 1 = 4a_3 + 5 = a_4 = 2a_5 - 1 = 0$.

Necessary conditions.

We first consider $a_6 = 0$ and take $(a_1, \ldots, a_5) \in \mathbb{P}^4$. Using the notation from Section 3, $\mathcal{V}(F_2)$ and $\mathcal{V}(F_3)$ are irreducible of codimension 1 and 2 of degree 2 and 8, respectively. Now, $\mathcal{V}(F_4)$ has codimension 3 and decomposes into the following components: 5 linear spaces, 3 of multiplicity 1 and 2 of multiplicity 3, and an irreducible algebraic set of degree 39. The three linear spaces of multiplicity 1 are (a), (b), and (c). The other two linear spaces are complex conjugates of each other with their union is defined in $\mathbb{P}^4$ by

$$a_1 + a_2 = 4a_2^2 + a_4^2 = a_5 = 0.$$

Since the real points on this union are contained in (c), we only need to further investigate the degree 39 component, denoted $X_{4,6}$, which is not contained in $\mathcal{V}(g_{550})$. Regenerating from $X_{4,6}$ to compute $X_{4,6} \cap \mathcal{V}(g_{550})$ yields 189 distinct points in $\mathbb{P}^4$, of which 19 correspond to real points. There are 14 real points that do not lie on (a), (b), or (c) of which only 2 satisfy $g_{660} = 0$, namely (d) and (e). We note that (e) has multiplicity 2 with respect to F_5.

We next consider $a_6 = 1$ and take $(a_1, \dots, a_5) \in \mathbb{C}^5$. Similar to the case above, $\mathcal{V}(F_2)$ and $\mathcal{V}(F_3)$ are irreducible of codimension 1 and 2 of degree 2 and 8, respectively. Also, $\mathcal{V}(F_4)$ has codimension 3 and decomposes into the following components: 3 linear spaces, one having multiplicity 1, namely (f), with the other 2 having multiplicity 3, and an irreducible algebraic set of degree 41. As above, the two linear spaces of multiplicity 3 are complex conjugates of each other with their union defined in $\mathbb{C}^5$ by

$$a_1 + a_2 = 4a_2^2 + a_4^2 = 2a_2 + a_4 a_5 = 2a_2 a_5 - a_4 = a_5^2 + 1 = 0.$$

Since there are no real points on this union, we only need to further investigate the degree 41 components, denoted $X_{4,4}$, which is not contained in $\mathcal{V}(g_{550})$. Regenerating $X_{4,4}$ yields 4 irreducible components of $X_{4,4} \cap \mathcal{V}(g_{550})$ not contained in (f) or the hyperplane $a_5^2 + 1 = 0$. Three of these are the lines (g), (h), and (i) with the fourth being an irreducible curve of degree 244, denoted $X_{5,4}$, not contained in $\mathcal{V}(g_{660})$. Regenerating $X_{5,4}$ yields 71 distinct real points not contained in the hyperplane $a_5^2 + 1 = 0$ nor satisfying (f), (g), (h), or (i). Of these, only one satisfies $g_{770} = 0$, namely (j).

Sufficient conditions.

Cases (a) and (b). If the condition (a) (resp. (b)) holds, system (4) reduces to

$$\begin{aligned}
\dot{u} &= -v + a_1 u^2 + a_2 v^2 + a_4 uv,\\
\dot{v} &= u + a_1 u^2 + a_2 v^2 + a_4 uv,\\
\dot{w} &= -w + a_1 u^2 + a_2 v^2 + a_4 uv,
\end{aligned}$$

with $a_2 = a_1$ (resp. $a_2 = -a_1$). Note that by Theorem 2, it is enough to show that this system admits a local analytic first integral at the origin. Since the first two equations are decoupled from the third we only need to show that

$$\dot{u} = -v + a_1 u^2 + a_2 v^2 + a_4 uv, \quad \dot{v} = u + a_1 u^2 + a_2 v^2 + a_4 uv, \tag{12}$$

admits a local analytic first integral. In fact, if $a_4 \neq 0$ and $a_2 = a_1$, system (12) has the inverse integrating factor

$$V(u,v) = -a_4 + a_4 (a_4 + 2a_1)(x - y) + a_1 (a_4 + 2a_1)^2 \left(x^2 + yx + y^2\right).$$

As $V(0,0) = -a_4$, it follows that system (12) has a first integral defined at the origin. If $a_4 = 0$ and $a_2 = a_1$, applying Theorem 4(ii) with $a = c = a_1$, $b = d = -a_1$, $A = 2a_1$ and

$B = -2a_1$, we have that (12) has a center at the origin and so it is integrable. The case $a_2 = -a_1$ (i.e. case (b)) is analogous, since

$$V(u,v) = 1 + (2a_1 - a_4)\,x + (2a_1 + a_4)\,y - a_1a_4x^2 - {a_4}^2xy + a_1a_4y^2,$$

is an inverse integrating factor for system (12), which is also nonzero at the origin.

Case (c). In this case system (4) becomes

$$\begin{array}{rcl} \dot{u} & = & -v + a_3w^2 + a_5uw, \\ \dot{v} & = & u + a_3w^2 + a_5uw, \\ \dot{w} & = & -w + a_3w^2 + a_5uw. \end{array}$$

Note that w $= 0$ is invariant and is a center manifold for this system. Moreover, the restriction of the associated vector field to w $= 0$ gives rise to a linear center.

Case (d). For $a_2 = -a_1$, $a_3 = -2a_1$, $a_4 = 6a_1$ and $a_5 = -4a_1$ the vector field associate to system (4) has the invariant algebraic surface $F(u,v,w) = w + a_1(u-v)^2 - 2a_1(v-w)^2 = 0$ with cofactor $K(u,v,w) = -1$. Since $F = 0$ is tangent to $w = 0$ at the origin, it is a center manifold for this system. To determine the dynamics on it first we use the change of coordinates $(u,v,w) \mapsto (x+z, y+z, z)$ that transforms the system into

$$\begin{array}{rcl} \dot{x} & = & -y, \\ \dot{y} & = & x + 2z, \\ \dot{z} & = & -z + a_1x^2 + 6a_1xy + 4a_1xz - a_1y^2 + 4a_1yz. \end{array} \tag{13}$$

The center manifold $F = 0$ in the new variables is given by $F(x,y,z) = z + a_1(x-y)^2 - 2a_1y^2 = 0$. The restriction of system (13) to $F = 0$ is given by

$$\dot{x} = -y, \quad \dot{v} = x - 2a_1x^2 + 4a_1xy + 2a_1y^2.$$

Since this system has the following inverse integrating factor (nonzero at the origin)

$$V(u,v) = 1 - 4a_1\,(x-y) + 4a_1^2\left(x^2 - 2xy - y^2\right)$$

thus in this case system (4) has a center on the center manifold.

Case (e). For $a_1 = 0$, $a_3 = -a_2$, $a_4 = 2a_2$ and $a_5 = -2a_2$ system (4) has the invariant algebraic surface $F(u,v,w) = w - a_2(y-z)^2 = 0$ with cofactor $K(u,v,w) = -1$. Since $F = 0$ is tangent to $w = 0$ at the origin, it is a center manifold for this system. To determine the dynamics on it first we use the change of coordinates $(u,v,w) \mapsto (x, y+z, z)$ that transforms the system into

$$\begin{array}{rcl} \dot{x} & = & -y - z + a_2y^2 + 2a_2xy + 2a_2yz, \\ \dot{y} & = & x + z, \\ \dot{z} & = & -z + a_2y^2 + 2a_2yz + 2a_2xy. \end{array} \tag{14}$$

The center manifold $F = 0$ in the new variables writes as $F(x, y, z) = z - a_2 y^2 = 0$. The restriction of system (14) to $F = 0$ is

$$\dot{x} = -y + 2a_2 xy + 2a_2^2 y^3, \quad \dot{y} = x + a_2 y^2.$$

This system is invariant by the change of variables $(x, y, t) \mapsto (x, -y, -t)$ so that it has a center at the origin. Hence, system (4) restricted to (e) has a center on the center manifold.

Case (f). In this case system (4) becomes

$$\begin{aligned} \dot{u} &= -v + a_3 w^2 + a_5 uw + vw, \\ \dot{v} &= u + a_3 w^2 + a_5 uw + vw, \\ \dot{w} &= -w + a_3 w^2 + a_5 uw + vw. \end{aligned}$$

It is clear that the plane w $= 0$ is invariant and is a center manifold for this system. Moreover, the restriction of the associated vector field to w $= 0$ gives rise to a linear center.

Case (g). If $a_1 = 1/2$, $a_3 = -a_2 - 1/2$, $a_4 = 2a_2 - 1$ and $a_5 = -2a_2$, then system (4) has the invariant algebraic surface $F(u, v, w) = -2w + (u - w)^2 + 2a_2(v - w)^2 = 0$ with cofactor $K(u, v, w) = -1$. Since $F = 0$ is tangent to $w = 0$ at the origin, it is a center manifold for this system. To determine the dynamics on it first we use the change of coordinates $(u, v, w) \mapsto (x + z, y + z, z)$ that transforms system (4) with conditions (l1) into

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + 2z, \\ \dot{z} &= -z + x^2/2 + (2a_2 - 1)xy + a_2 y^2 + 4a_2 yz. \end{aligned} \tag{15}$$

The center manifold $F = 0$ in the new variables writes as $F(x, y, z) = -2z + x^2 + 2a_2 y^2 = 0$. The restriction of system (15) to $F = 0$ is

$$\dot{x} = -y, \quad \dot{v} = x + x^2 + 2a_2 y^2.$$

As this system is invariant under $(x, y, t) \mapsto (x, -y, -t)$, it follows that it has a center at the origin, i.e. system (4) under the conditions (g) has a center on the center manifold.

Case (h). If $a_1 = -1/2$, $a_2 = -1/2$, $a_4 = 0$ and $a_5 = -1$. Then the vector field associate to system (4) has the invariant algebraic surface $F(u, v, w) = w + \big[(u + w)^2 + (v - w)^2\big]/2 - w^2\,(1 + a_3) = 0$ with the cofactor $K(u,\, v, w) = -1 - 2u + 2a_3 w$. Since $F = 0$ is tangent to $w = 0$ at the origin, it is a center manifold for this system. To determine the dynamics on it first we use the change of coordinates $(u, v, w) \mapsto (x - z, y + z, z)$, that transforms system (4) with condition (l2) into

$$\begin{aligned} \dot{x} &= -y - 2z + 2(1 + a_3)z^2 - x^2 - y^2, \\ \dot{y} &= x, \\ \dot{z} &= -z + (1 + a_3)z^2 - x^2/2 - y^2/2. \end{aligned} \tag{16}$$

The center manifold $F = 0$ in the new variables is given by

$$F(x, y, z) = z - (1 + a_3)z^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 = 0.$$

The restriction of system (16) to $F = 0$ gives rise to a linear center.

Case (i). For $a_2 = -a_1$, $a_3 = -2a_1 + 1$, $a_4 = 6a_1 - 5$ and $a_5 = -4a_1 + 3$ system (4) admits an invariant algebraic surface $F(u, v, w) = w + (a_1 - 1)(u - w)^2 + (1 - 2a_1)(u - w)(v - w) + (1 - a_1)(v - w)^2 = 0$ with cofactor $K(u, v, w) = -1$. Since $F = 0$ is tangent to $w = 0$ at the origin, it is a center manifold for this system. The change of coordinates $(u, v, w) \mapsto (x + z, y + z, z)$ transforms system (4) under the conditions (i) into

$$\begin{array}{rcl} \dot{x} & = & -y, \\ \dot{y} & = & x + 2z, \\ \dot{z} & = & -z + a_1x^2 + (6a_1 - 5)xy + 2(2a_1 - 1)xz - a_1y^2 + 4(a_1 - 1)yz. \end{array} \tag{17}$$

Again, in the new variables the center manifold is given by $F(x, y, z) = z + (a_1 - 1)x^2 + (1 - 2a_1)xy + (1 - a_1)y^2 = 0$ and the restriction of (17) to $F = 0$ reduces to

$$\dot{x} = -y, \quad \dot{v} = x + 2(1 - a_1)x^2 + 2(2a_1 - 1)xy + 2(a_1 - 1)y^2.$$

This system has the following inverse integrating factor (nonzero at the origin)

$$\begin{array}{rcl} V(u, v) & = & 1 + 4(1 - a_1)x + 2\,(2a_1 - 1)\,y + 4(a_1 - 1)^2x^2 \\ & & -4(a_1 - 1)(2a_1 - 1)xy - 4(a_1 - 1)^2y^2. \end{array}$$

Hence system (4) has a center on the center manifold.

Case (j). For $a_1 = 1/4$, $a_2 = -1/2$, $a_3 = -5/4$, $a_4 = 0$ and $a_5 = 1/2$ the vector field associated to system (4) admits a polynomial first integral

$$\begin{array}{rcl} H(x, y, z) & = & x^2 + y^2 - \frac{1}{2}x^3 - \frac{1}{2}x^2y + 2x^2z - \frac{3}{2}xz^2 - y^2x \\ & & +y^2z - \frac{1}{2}yz^2 - \frac{1}{2}x^3z + \frac{5}{4}x^2z^2 + \frac{1}{2}y^2x^2 \\ & & -\frac{3}{2}xz^3 + \frac{1}{2}y^2z^2 - yz^3 + xyz + \frac{1}{8}x^4 \\ & & -x^2yz - y^2xz + 2yxz^2 + \frac{5}{8}z^4, \end{array}$$

and so it has a center on the center manifold.

□

Proof of Theorem 1.

Case (1). Follows from Cases (c) and (f) of Theorem 3.

Case (2). Follows from Case (a) of Theorem 3.

Case (3). Follows from Case (b) of Theorem 3.

Case (4). Follows from Cases (d) and (i) of Theorem 3.

Case (5). Follows from Cases (e) and (g) of Theorem 3.
Case (6). Follows from Cases (c) and (h) of Theorem 3.
Case (7). Follows from Cases (c) and (j) of Theorem 3.

□

Appendix A. Dulac-Kapteyn criterion

The following theorem provides a criterion in order to determine when a quadratic planar polynomial system has a center at the origin. It was first proven by Dulac [19] and Kapteyn [42], but we present the version given in [16].

Theorem 4 (Quadratic Center). *The system*

$$\begin{array}{rcl}\dot{u} &=& -v - bu^2 - (B+2c)uv - dv^2,\\ \dot{v} &=& u + au^2 + (A+2b)uv + cv^2,\end{array}$$

has a center at the origin if and only if at least one of the following three hold:

(i) $a + c = b + d$;
(ii) $A(a+c) = B(b+d)$ *and* $aA^3 - (3b+A)A^2B + (3c+B)AB^2 - dB^3 = 0$;
(iii) $A + 5b + 5d = B + 5a + 5c = ac + bd + 2a^2 + 2d^2 = 0$.

Appendix B. Basic Darboux theory of integrability

Since, by Poincaré theorem, the integrability is closely related to the existence of a center on a center manifold (also on the plane), we provide a short overview of the basic notions of the Darboux the ory of integrability used in Section 4; for more information see [46, 33] and some applications see [28, 47, 52].

We say that $F = F(x,y,z) \in \mathbb{C}[x,y,z]$ is a *Darboux polynomial* and $F = 0$ is an *invariant algebraic surface* of the vector field X if and only if there exists a polynomial $K(x,y,z) \in \mathbb{C}[x,y,z]$, the *cofactor* of F, such that $XF = KF$. A the heart of the Darboux theory of integrability is the following result [17]: if there exists some number n of pairs (F_j, K_j) for which there exists a nontrivial dependency relation $\sum \alpha_j K_j = 0$ then $F_1^{\alpha_1} \cdots F_n^{\alpha_n}$ is a first integral of X.

Consider now the planar system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{18}$$

where $P, Q \in \mathbb{R}[x,y]$, and the associate vector field $X = P\partial/\partial x + Q\partial/\partial y$. Let U be an open subset of $\mathbb{R}^2$, and let $R, V : U \to \mathbb{R}$ be two analytic functions which are not identically zero on U. We say that R is an *integrating factor* of this polynomial system on U if one of the following three equivalent conditions holds

$$\frac{\partial RP}{\partial x} = -\frac{\partial RQ}{\partial x}, \quad \mathrm{div}(RP, RQ) = 0, \quad XR = -R\,\mathrm{div}(P,Q),$$

where div denotes the divergence. The first integral H associated to the integrating factor R can be easily obtained by

$$H(x,y) = \int R(x,y)P(x,y)dy + h(x),$$

where $h(x)$ is chosen such that it satisfies $\partial H/\partial x = -RQ$. Note that $\partial H/\partial y = RP$, so that $XH \equiv 0$. The function V is an *inverse integrating* factor of the polynomial system (18) on U if

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V. \tag{19}$$

We note that $\{V = 0\}$ is formed by orbits of system (18) and $R = 1/V$ defines on $U\backslash\{V = 0\}$ an integrating factor of (18). We note that if P and Q are quadratic polynomials and the origin of system (18) is a center, then there always exits a polynomial function $V : \mathbb{R}^2 \to \mathbb{R}$ of degree 3 or 5 satisfying equation (19), see [26].

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