Branch points of homotopies, Part II: Enumeration and general theory

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Abstract

Branch points arise from singularities along solution paths of a homotopy. This paper is the second in a systematic study of branch points of homotopies and elucidates the number of branch points and provides general theory regarding the set of branch points over the same ramification point. The general approach utilized in this paper is to view homotopies as lines in the parameter spaces of families of polynomial systems on a projective manifold. With this approach, the number of singularities of systems parameterized by pencils is computed under broad conditions. General conditions are also given for when the singularities have multiplicity two with at most one singularity in the solution set of any system parameterized by the line. Several examples are included to demonstrate the theoretical results.
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1 Introduction

The basic approach of homotopy continuation is to deform from the known solutions of system of equations $g = 0$, called the start system, to compute the solutions to a related system of equations $f = 0$, called the target system. To make this more concrete, consider the affine setting where $g = \{g_1, \ldots, g_N\}$ and $f = \{f_1, \ldots, f_N\}$ with g_j and f_j general polynomials on $\mathbb{C}^N$ of degree $d_j \in \mathbb{Z}_{>0}$. Thus, both $g = 0$ and $f = 0$ have $d_1 \cdots d_N$ nonsingular isolated solutions, i.e., the Bézout count is sharp. A linear homotopy $H : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ between g and f is

$$H(z, t) = tg(z) + (1 - t)f(z). \tag{1}$$

By genericity, there are no $t^* \in [0, 1]$ such that $H(z, t^*) = 0$ has a singular solution and thus the homotopy H is said to be trackable. That is, $H = 0$ defines $d_1 \cdots d_N$ smooth solution paths $z(t) : [0, 1] \to \mathbb{C}^N$ so that $H(z(t), t) \equiv 0$ which smoothly connect the solutions of $g = 0$ at $t = 1$ to the solutions of $f = 0$ at $t = 0$. However, if one instead considers $\mathbb{C}$ rather than $[0, 1]$, then there is always at least one $t^* \in \mathbb{C}$ such that $H(z, t^*) = 0$ has a singular solution whenever $d_1 \cdots d_N > 1$. With this formulation, it is natural to ask the following questions:

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1. For how many $t^* \in \mathbb{C}$ does $H(z, t^*) = 0$ have a singular solution?
2. What kinds of singularities actually occur as singular solutions to $H(z, t^*) = 0$?
3. For such $t^* \in \mathbb{C}$, what is the set of singular solutions for $H(z, t^*) = 0$?

The main results of this article show that these three questions can be answered in great generality. For example, for H in (1), the answers to the above questions are:

1. The number of t^* for which $H(z, t^*) = 0$ has at least one singular solution is (see (15))
$$d_1 \cdots d_N \left((d_1 + \cdots + d_N - N - 1) \left(\frac{1}{d_1} + \cdots + \frac{1}{d_N} \right) + N \right).$$
2. For such $t^* \in \mathbb{C}$, each singular solution of $H(z, t^*)$ has multiplicity two (see Remark 24).
3. For such $t^* \in \mathbb{C}$, there is exactly one singular solution of $H(z, t^*) = 0$ (see Corollary 27).

The last two are reminiscent of the classical situation for Lefschetz pencils, e.g., [18, Chap 2.1.1], and indeed the answer to these two questions follow from the existence of Lefschetz pencils and some standard vanishing theorems [15].

Example 1 (Cubic pencil on $\mathbb{C}$)**.** Let $f(z)$ and $g(z)$ be cubics in one variable. The results above imply that for $H(z, t) = tg(z) + (1 - t)f(z)$, there are 4 values of t^* where $H(z, t^*) = 0$ has a singular solution, and each singularity is a double root. Figure 1 plots $(\Re(t), \Im(t), \Re(z))$, the real and imaginary parts of t and the real part of z, respectively, for the three sheets of $H^{-1}(0)$. The color is based on the distance to the next nearest sheet and there is a contour plot of the minimum distance between sheets. (Distance is computed in $\mathbb{C}$, so zero distance implies equality of the real parts but the converse is not necessarily true.) One sees that the sheets meet in pairs at four points. The double roots are branch points and the values of t^* where they occur are ramification points.

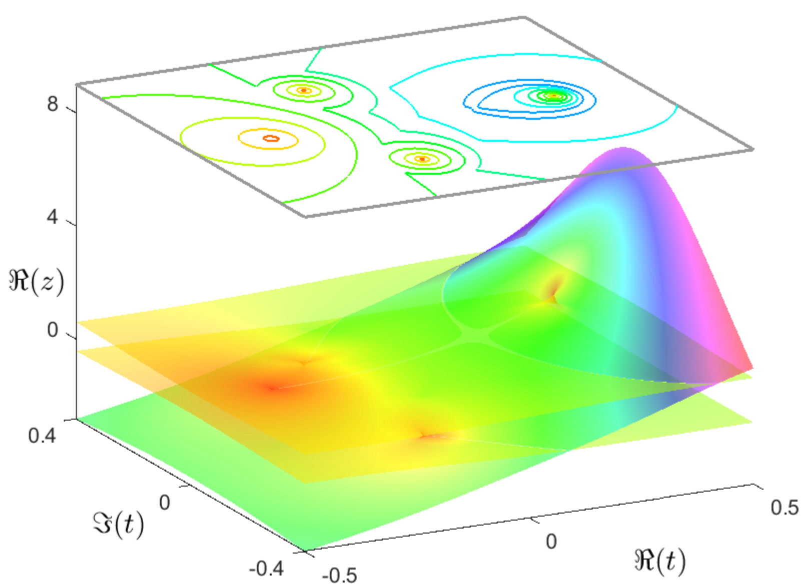

Figure 1: A visualization of the solution set of $H(z, t) = tg(z) + (1 - t)f(z) = 0$ for cubics $f(z)$ and $g(z)$ as t varies. This solution set is a three-sheeted cover of $\mathbb{C}$ with 4 branch points.

The curve $C \subset \mathbb{C}^{N+1}$ consisting of all solutions to $H = 0$ in (1) is smooth and connected. Smoothness is an easy consequence of Bertini's Theorem, but connectedness requires vanishing theorems. The genus g of the smooth projective curve $\overline{C}$ that C is Zariski open in, satisfies

$$2g - 2 = d_1 \cdots d_N \left((d_1 + \cdots + d_N - N - 1) \left(\frac{1}{d_1} + \cdots + \frac{1}{d_N} \right) + N - 2 \right).$$

Consider the projection map $\pi : \mathbb{C}^{N+1} \to \mathbb{C}$ with $\pi(z, t) = t$. The image of C under π is a ramified branch covering with $d_1 \cdots d_N$ sheets. The ramification points are those $t^* \in \mathbb{C}$ such that $H(z, t^*) = 0$ has a singular solution, say (z^*, t^*), and (z^*, t^*) is a branch point of order two.

In fact, all of the aforementioned results are all true even if f is not necessarily general but still has exactly $d_1 \cdots d_N$ nonsingular solutions. That is, the genericity of the start system g and sharp Bézout count for f is enough. Moreover, for an arbitrary system f with the same degrees as g, many of the qualitative results still hold true. For example, if $Z_0 = \{(z, 0) \mid f(z) = 0\}$, then $C \setminus Z_0$ is still smooth with a unique branch point, which has order two, over each ramification point $t^* \neq 0$.

Although H in (1) was formulated in affine space, the generality assumption ensures that the same statements hold over projective space, i.e., no additional structure at infinity. This paper shows that all of the above results hold in much greater generality, namely for fairly general systems on slightly singular algebraic subsets of products of projective spaces. Besides allowing a reduction of the generality of the linear homotopy for the main results to hold, they also hold for weighted projective spaces [5] and the class of normal toric varieties [10] with only slightly more care.

The rest of the paper is organized as follows. Section 2 considers general theory using the natural representation of a polynomial system in algebraic geometry as an algebraic section of an algebraic vector bundle. Section 3 addresses the three questions above in greater generality. Some examples are provided in Section 4. Finally, Section 5 concludes with a question for further study. Specific examples used in this article were computed using `Bertini` [2, 3]. The files used for these runs plus two Maple worksheets are available at `https://doi.org/10.7274/c.7376080`.

2 Some general theory about polynomials

The following collects some general theory about polynomials.

2.1 Sections and bundles

In this article, a polynomial system is viewed an algebraic section of an algebraic vector bundle over an irreducible and reduced projective algebraic set X. This approach to polynomial systems matches up well with the usual notion of a polynomial system and is the approach used in [14]. For example, suppose that $f = \{f_1, \ldots, f_k\}$ is a system of k homogeneous polynomials in $N + 1$ variables, i.e., defined on $\mathbb{P}^N$, of degrees $d_1, \ldots, d_k$, respectively. On $\mathbb{P}^N$, f_j is not a function but is an algebraic section of the algebraic line bundle $[d_j]_{\mathbb{P}^N}$. Note that $[0]_{\mathbb{P}^N}$ is the trivial bundle, and that for all integers a, b,

$$[a]_{\mathbb{P}^N} \otimes_{\mathbb{C}} [b]_{\mathbb{P}^N} = [a + b]_{\mathbb{P}^N}.$$

Note also, that for any integer m,

$$[-m]_{\mathbb{P}^N} = [m]^*_{\mathbb{P}^N}$$

where $[m]^*_{\mathbb{P}^N}$ is the dual of $[m]_{\mathbb{P}^N}$. The system f is an algebraic section of

$$\mathcal{E} := [d_1]_{\mathbb{P}^N} \oplus \cdots \oplus [d_k]_{\mathbb{P}^N},$$

which is a rank k vector bundle. Systems of multihomogeneous polynomials are handled similarly.

Remark 2. It should be noted that any algebraic vector bundle on $\mathbb{C}^N$ is algebraically equivalent to the trivial bundle on $\mathbb{C}^N$ and that algebraic functions defined on $\mathbb{C}^N$ are polynomials. Therefore, an algebraic section of an algebraic vector bundle over $\mathbb{P}^N$ reduces to a polynomial system when restricted to $\mathbb{C}^N$. Though most systems that arise as start systems in homotopy continuation (including those arising from polyhedral methods) come from direct sums of line bundles, there are many systems that do not correspond to such direct sums yet nevertheless fit into our approach. We work out the details of one significant example in Section 4.1.

A *variety* is defined as an irreducible and reduced algebraic set and a *projective variety* is an irreducible and reduced projective algebraic set. Of the several different definitions of variety, we are following the usage in [11] and [9, Chap. 1.1]. An algebraic set M is said to be *smooth* if it is smooth in the scheme-theoretic sense, i.e., M is a manifold (and in particular, reduced). Given an algebraic set X, we denote its singular set by $\mathrm{Sing}\,(X)$ and its set of smooth points by $X_{\mathrm{reg}} = X \setminus \mathrm{Sing}\,(X)$. Over the complex numbers, the condition that an algebraic set X is irreducible is equivalent to the closure of X_{reg} in the usual topology (or in the Zariski topology) being equal to X. Hence, an algebraic manifold is irreducible if and only if it is connected.

For convenience, we follow the usual convention that $\dim \emptyset = -1$ and an empty subset of a variety X has codimension $\dim X + 1$.

Since all the bundles, manifolds, and sections we consider in this article are algebraic, outside of theorems, corollaries, and lemmas, we will usually drop the word algebraic in referring to them.

2.2 Projectivization of a vector bundle

Let $\mathcal{E}$ denote a rank k vector bundle on a projective variety X, i.e., an irreducible and reduced projective algebraic set. Let $\mathbb{P}\,(\mathcal{E})$ denote the space of lines in each fiber of $\mathcal{E}^*$ through the origin of the fiber. This is a $\mathbb{P}^{k-1}$-bundle over X with natural bundle projection $\pi_{\mathbb{P}(\mathcal{E})} : \mathbb{P}\,(\mathcal{E}) \to X$. This projective bundle over X may also be regarded as the quotient $(\mathcal{E}^* \setminus X)\,/\mathbb{C}^x$, where the nonzero complex numbers, $\mathbb{C}^x$, act by the natural multiplication on each fiber of $\mathcal{E}^*$.

There is a natural line bundle $\xi^*_{\mathcal{E}}$ on $\mathbb{P}\,(\mathcal{E})$ which, at each point $w \in \mathbb{P}\,(\mathcal{E})$, is the line in $\pi^*_{\mathbb{P}(\mathcal{E})}\mathcal{E}^*$ that w corresponds to in $\mathcal{E}^*$. The dual of $\xi^*_{\mathcal{E}}$, i.e., $\xi_{\mathcal{E}}$, is called the *tautological line bundle* on $\mathbb{P}\,(\mathcal{E})$.

Given a vector bundle $\mathcal{E}$, let $\mathcal{O}_X(\mathcal{E})$ denote the *sheaf of germs of algebraic sections* of $\mathcal{E}$.

Remark 3. The convention of denoting this space of lines on $\mathcal{E}^*$ by $\mathbb{P}\,(\mathcal{E})$ and *not* by (the at first sight more natural) $\mathbb{P}\,(\mathcal{E}^*)$ is the standard convention in algebraic geometry, e.g., [11, pg. 162]. The original convention was to use $\mathbb{P}\,(\mathcal{E}^*)$ but this turned out to usually be inconvenient notationally. Nonetheless, one must remain aware of this since the older convention exists in the literature, e.g., [9], which is cited several times herein.

Let $\mathcal{E}$ be a vector bundle on X and let $V \subset H^0(X, \mathcal{E})$ denote a vector subspace of the vector space $H^0(X, \mathcal{E})$ consisting of all sections of $\mathcal{E}$ on X. We say V *spans* $\mathcal{E}$ if, for each $x \in X$, the evaluation of the sections of V at x gives all points in the fiber, $\mathcal{E}_x$, of $\mathcal{E}$ over x. When V spans $\mathcal{E}$, the evaluation map gives a surjective bundle map of $X \times V \to \mathcal{E}$. This case yields the exact sequence

$$0 \to \mathcal{F}^* \to X \times V \to \mathcal{E} \to 0, \tag{2}$$

with a vector bundle $\mathcal{F}^*$ as the kernel. Taking duals, we have the exact sequence

$$0 \to \mathcal{E}^* \to X \times V^* \to \mathcal{F} \to 0. \tag{3}$$

If we have a line bundle $\mathcal{L}$ on a projective variety X spanned by a vector space of sections V, then dualizing the evaluation map gives the embedding

$$0 \to \mathcal{L}^* \to X \times V^*.$$

This yields a map $\phi_V : X \to \mathbb{P}(V)$. Note that, by construction, $\mathcal{L} = \phi_V^*\left([1]_{\mathbb{P}(V)}\right)$.

The *Chern classes* of $\mathcal{E}$ are cohomology classes $c_j(\mathcal{E}) \in H^{2j}(X, \mathbb{Z})$ for $j = 1, \ldots, \dim X$. Some references for Chern classes are [9, § 3.1-3.2] and [17, § 11.2]. The Chern classes in [9] are more refined and defined to act on algebraic sets. Given a rank k vector bundle $\mathcal{E}$, the total Chern class is

$$c_t(\mathcal{E}) = 1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \cdots$$

where $c_\ell(\mathcal{E}) = 0$ for $\ell > \dim X$.

Lemma 4. *Let $\mathcal{E}$ be a rank k algebraic vector bundle on a projective variety X spanned by a vector space V of sections. Let $\mathcal{F}$ be as in* (3) *with rank $e+1 = \dim V - k$. Then, given an h-dimensional projective variety $Y \subset X$, we have*

$$c_1\left(\xi_{\mathcal{F}}\right)^{e+h} \cap \pi^{-1}_{\mathbb{P}(\mathcal{F})}(Y) = c_h(\mathcal{E}) \cap Y,$$

where $\cap$ denotes the cap-product pairing of cohomology and homology.

Proof. Let $s_h(\mathcal{E})$ be the h^{th} Segre class of $\mathcal{F}^*$, e.g., see [9, Chap. 3.1]. By definition,

$$c_1\left(\xi_{\mathcal{F}}\right)^{e+h} \cap \pi^{-1}_{\mathbb{P}(\mathcal{E})}(Y) = s_h(\mathcal{F}^*) \cap Y. \tag{4}$$

Denoting the total Segre class [9, Chap. 3.2] by

$$s_t(\mathcal{F}^*) = 1 + s_1(\mathcal{F}^*) + s_2(\mathcal{F}^*)t^2 + \cdots,$$

we have

$$c_t(\mathcal{F}^*) = s_t(\mathcal{F}^*)^{-1}.$$

It follows from (2) that

$$s_t(\mathcal{F}^*)^{-1} = c_t(\mathcal{F}^*) = c_t(\mathcal{E})^{-1} \qquad \text{yielding} \qquad s_t(\mathcal{F}^*) = c_t(\mathcal{E}).$$

The result now follows from (4). □

Remark 5. Let s denote a generic section of a spanned rank N vector bundle on a projective variety X. The number of smooth isolated zeroes of s is $c_N(\mathcal{E})$ evaluated on X. This number is familiar to people computing (multihomogeneous) Bézout numbers of start systems made up of (multihomogeneous) polynomials. We will for this reason often use the term Bézout number to refer to the number of smooth isolated zeroes of a general section of a rank N spanned vector bundle on an N-dimensional projective variety.

The other Chern classes also give useful structural information about start systems of homotopies. We will see both $c_1(\mathcal{E})$ and $c_{N-1}(\mathcal{E})$ used in (12) as part of Theorem 18. The number computed by this equation gives a useful measure of the quality of a homotopy.

Let T^*_X denote the *cotangent bundle* of a projective manifold X. Then, the *canonical bundle* is $K_X = \det(T^*_X)$. Though standard, we do not know a reference where the proof of the following is provided so we include one here.

Lemma 6. *Given a rank k algebraic vector bundle $\mathcal{E}$ on a connected algebraic manifold X, we have*

$$K_{\mathbb{P}(\mathcal{E})} = \pi^*_{\mathbb{P}(\mathcal{E})}\left(K_X \otimes \det\left(\mathcal{E}\right)\right) \otimes \xi_{\mathcal{E}}^{-k}.$$

Proof. Since the Jacobian of the map $\pi_{\mathbb{P}(\mathcal{E})} : \mathbb{P}\left(\mathcal{E}\right) \to X$ is surjective, we have a short exact sequence

$$0 \to T_{\mathbb{P}(\mathcal{E})/X} \to T_{\mathbb{P}(\mathcal{E})} \to \pi^*_{\mathbb{P}(\mathcal{E})}T_X \to 0,$$

where $T_{\mathbb{P}(\mathcal{E})/X}$ is the bundle of tangents to the fibers of $\pi_{\mathbb{P}(\mathcal{E})}$. Taking determinants, we have

$$K^*_{\mathbb{P}(\mathcal{E})} = \pi^*_{\mathbb{P}(\mathcal{E})}K^*_X \otimes \det\left(T_{\mathbb{P}(\mathcal{E})/X}\right). \tag{5}$$

The relative version of the Euler sequence [9, App. B5.8] (which we have adjusted for the different convention for $\mathbb{P}\left(\mathcal{E}\right)$ in [9]) is

$$0 \to \mathbb{C} \times \mathbb{P}\left(\mathcal{E}\right) \to \pi^*_{\mathbb{P}(\mathcal{E})}\mathcal{E}^* \otimes \xi_{\mathcal{E}} \to T_{\mathbb{P}(\mathcal{E})/X} \to 0 \tag{6}$$

which yields

$$\det\left(T_{\mathbb{P}(\mathcal{E})/X}\right) = \det\left(\pi^*_{\mathbb{P}(\mathcal{E})}\mathcal{E}^* \otimes \xi_{\mathcal{E}}\right) = \det\left(\pi^*_{\mathbb{P}(\mathcal{E})}\mathcal{E}^*\right) \otimes \xi^k_{\mathcal{E}} = \pi^*_{\mathbb{P}(\mathcal{E})}\det\left(\mathcal{E}\right)^* \otimes \xi^k_{\mathcal{E}}.$$

The result now follows from (5). □

Remark 7. If X is a point and $V = \mathcal{E} = \mathbb{C}^{N+1}$, then $\mathbb{P}\left(\mathcal{E}\right) = \mathbb{P}^N$, $\xi_{\mathcal{E}} = [1]_{\mathbb{P}^N}$, and (6) becomes

$$0 \to [0]_{\mathbb{P}^N} \to \bigoplus_{j=1}^{N+1}[1]_{\mathbb{P}^N} \to T_{\mathbb{P}^N} \to 0. \tag{7}$$

From this sequence, we conclude that $K_{\mathbb{P}^N} = [-(N+1)]_{\mathbb{P}^N}$ and that the total Chern class of $T_{\mathbb{P}^N}$ is

$$\left(1 + tH_1\right)^{N+1}$$

where H_1 is the first Chern class of $[1]_{\mathbb{P}^N}$. In particular, $c_j(T_{\mathbb{P}^N}) = \binom{N+1}{j}H_1^j$.

2.3 Theorem of Bertini

There are many results in algebraic geometry loosely connected under the name *Bertini's Theorem*. Some references for the results used here are [11, § III.10], [6, § 1.7], and [16, § A.9]. Among the simplest Bertini's Theorem is the algebraic version of Sard's Theorem.

Theorem 8 (Algebraic Sard's Theorem)**.** *Let $f : X \to Y$ be an algebraic map from a connected algebraic manifold to an algebraic variety Y. Assume that there is at least one point $y \in Y$ such that the irreducible component Z of the fiber $f^{-1}(y)$ satisfies $\dim Z = \dim X - \dim Y$. Then, there is a nonempty Zariski open set $U \subset Y$ such that $f : f^{-1}(U) \to U$ is onto and of maximal rank. In particular, all the fibers of f over points of U are smooth with all connected components of each fiber having dimension $\dim Z = \dim X - \dim Y$.*

Proof. This is just [11, Cor. 10.7] combined with the upper semi-continuity of dimension. □

Corollary 9. *Given an algebraic vector bundle $\mathcal{E}$ on an algebraic manifold X with $\mathcal{E}$ spanned by a finite dimensional space of algebraic sections V, there is a nonempty Zariski open set $U \subset V$ such that, for $s \in U$, $Z = s^{-1}(0)$ is smooth and either Z is empty or $\mathrm{codim}_X\ Z = \mathrm{rank}\left(\mathcal{E}\right)$.*

There are many generalizations dealing with the situations when there are singularities. Letting $\mathrm{Sing}\,(X)$ denote the singular set of a algebraic set X, we will use the following result.

Theorem 10. *Let X denote a reduced algebraic set with all components of X of the same dimension, say N, and let ξ denote an algebraic line bundle on X spanned by a finite dimensional vector space of algebraic sections V. Then, there exists a nonempty Zariski open set $U \subset V$ such that, for $s \in U$, $D = s^{-1}(0)$ is either empty or*

1. *D is reduced and with all components of D having dimension $N-1$;*
2. *$\mathrm{Sing}\,(D) \subset \mathrm{Sing}\,(X)$ and $D \cap X_{\mathrm{reg}}$ is smooth; and*
3. *each irreducible component Z of $\mathrm{Sing}\,(D)$ is a proper algebraic subset of any irreducible component of $\mathrm{Sing}\,(X)$ that Z belongs to. Hence, $\dim \mathrm{Sing}\,(D) < \dim \mathrm{Sing}\,(X)$ if $\mathrm{Sing}\,(D) \neq \emptyset$.*

Proof. This follows from the stronger result stated in [6, Thm. 1.7.1]. □

In Theorem 10, since the restriction of V to D spans the restriction ξ_D of ξ to D, we can repeat the above result to obtain a sequence of algebraic sets

$$D_1 \subset D_2 \subset \cdots \subset D_{N-1} \subset D_N = X \tag{8}$$

such that, if $D_j \neq \emptyset$, then the following hold:

1. all components of D_j have dimension j;
2. $\mathrm{Sing}\,(D_j) \subset \mathrm{Sing}\,(D_{j+1})$; and
3. $\dim \mathrm{Sing}\,(D_j) < \dim \mathrm{Sing}\,(D_{j+1})$ if $\mathrm{Sing}\,(D_j) \neq \emptyset$.

Thus, if $\mathrm{Sing}\,(D_j) \neq \emptyset$, then

$$\dim \mathrm{Sing}\,(D_j) \leq \dim \mathrm{Sing}\,(X) - (N - j).$$

This gives the important fact that we state explicitly.

Corollary 11. *If $\dim \mathrm{Sing}\,(X) \leq k$, then D_j is smooth for $j < N - k$. In particular, if $\mathrm{Sing}\,(X)$ has codimension at least two, then D_1 is smooth.*

Homotopies are typically constructed with enough conditions for vanishing theorems to guarantee that the D_j are irreducible. The following provides sufficient conditions to ensure irreducibility.

Theorem 12. *Let X be a projective variety and let ξ be an algebraic line bundle on X spanned by a vector space V of algebraic sections. If $c_1(\xi)^N \neq 0$, then, for a general $s \in V$, each D_j is irreducible for $j \geq 1$.*

Proof. First, assume that X is smooth and let $D = s^{-1}(0)$ be a smooth zero set for a general $s \in V$. If $D = \emptyset$, then ξ would be the trivial bundle and $c_1(\xi) = 0$. Hence, $D \neq \emptyset$ with exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0. \tag{9}$$

By the Kawamata-Viehweg vanishing theorem [15, Cor 7.50], we know $H^0(X, \mathcal{O}_X(-D)) = 0$ and $H^1(X, \mathcal{O}_X(-D)) = 0$. Therefore, using the long exact cohomology sequence associated to (9), $\dim H^0(D, \mathcal{O}_D) = 1$. Since D is smooth, this implies it is connected. For manifolds, irreducibility is equivalent to being connected.

If X is not smooth, then let $\pi : \widehat{X} \to X$ be a desingularization. Here, $\widehat{X}$ is a connected projective manifold and π gives a one-to-one and onto map from $\widehat{X} \setminus \pi^{-1}(\mathrm{Sing}\,(X)) \to X_{\mathrm{reg}}$. Since $c_1(\pi^*\xi)^N \neq 0$ and π^*V spans $\pi^*\xi$, a general choice of $s \in V$ gives rise to the set

$$\overline{D} = \{x \in X \mid s(x) = 0\} \subset X$$

which π maps to D such that

1. $\overline{D}$ is connected and smooth; and
2. π maps $\widehat{D} \setminus \pi_D^{-1}(\mathrm{Sing}\,(D))$ one-to-one and onto D_{reg}.

Since the image of an irreducible set is irreducible, this shows that D is irreducible.

Noting that $c_1(\xi_D)^{N-1} = c_1(\xi)^N \neq 0$, the argument may be repeated until we reach D_1. □

When studying the number of branch points in a fiber in Section 3.3, we will need the following when $k = N-1$. Recall that a line bundle $\mathcal{L}$ on a projective variety is called *ample* if some power $\mathcal{L}^n$ of $\mathcal{L}$ is spanned and the map of X to projective space associated to $H^0(\mathcal{L}^n)$ is an embedding. A vector bundle $\mathcal{E}$ is said to be *ample* if $\xi_{\mathcal{E}}$ on $\mathbb{P}(\mathcal{E})$ is ample. On projective space, the line bundles $[d]_{\mathbb{P}^N}$ that are ample are those associated to homogeneous polynomials of positive degrees d.

Lemma 13. *Let X be an N-dimensional projective variety with* $\dim \mathrm{Sing}\,(X) \leq k-1$. *Let $\mathcal{G}$ denote an algebraic vector bundle on X of rank $k < N$. Assume that $\mathcal{G}$ is a direct sum of algebraic line bundles $\mathcal{L}_j$ with each $\mathcal{L}_j$ spanned by a vector space V_j of algebraic sections. Let H_X be an ample line bundle on X. Assume that, for each $j \geq k$,*

$$\left(\prod_{i=1}^{j-1} c_1(\mathcal{L}_i)\right) c_1(\mathcal{L}_j)^2 c_1(H_X)^{N-j-1} \neq 0. \tag{10}$$

Then, a general section of $\mathcal{G}$ from $V_1 \oplus \cdots \oplus V_k$ has a smooth, nonempty, connected solution set of dimension $N-k$.

Proof. From the methods used earlier, we may reduce to the situation where X is smooth. Let $D_N = X$. For each $j = N-k, \ldots, N-1$, let D_j denote the solution set of a general element of V_{N-j} restricted to D_{j+1}. We have

$$X = D_N \supseteq D_{N-1} \supseteq \cdots \supseteq D_{N-k}.$$

The condition $c_1(\mathcal{L}_1)^2 \cdot H_X^{N-2} \neq 0$ implies the solution set D_{N-1} is nonempty and thus is smooth of dimension $N-1$ by Bertini's Theorem.

By [15, Cor 7.50], $H^r(\mathcal{L}_1^*) = 0$ for $r \leq 1$. Thus, from the exact sequence

$$0 \to \mathcal{O}(\mathcal{L}_1^*) \to \mathcal{O}_{D_N} \to \mathcal{O}_{D_{N-1}} \to 0,$$

the long cohomology sequence yields $H^0(\mathcal{O}_{D_N}) = H^0(\mathcal{O}_{D_{N-1}})$. Hence, D_{N-1} is connected.

Denote the restriction of H_X and $\mathcal{L}_r$ to D_j by H_{D_j} and $\mathcal{L}_{j,D_r}$, respectively. Noting that

$$c_1(\mathcal{L}_{j,D_{N-j+1}})^2 \cdot H_{D_{N-j+1}}^{\dim D_{N-j+1}-2} = \left(\prod_{i=1}^{j-1} c_1(\mathcal{L}_i)\right) c_1(\mathcal{L}_j)^2 c_1(H_X)^{N-j-1} \neq 0,$$

we may continually repeat the above argument to prove the result by downward induction. □

The key to Lemma 13 is (10). Consider what it means for (10) to fail. To that end, assume for simplicity that X in Lemma 13 has no singularities and (10) fails for $j = 1$, i.e.,

$$c_1(\mathcal{L}_1)^2 \cdot c_1(H_X)^{N-2} = 0.$$

Failing at other values of j is similar but the mass of indices obscures a conceptual understanding.

Using Bertini's Theorem, this implies that the solution sets of two general elements $a_1, a_2 \in V_1$ have empty intersection. Let A denote the span of a_1 and a_2. Thus, the two dimensional vector subspace A of V_1 spans $\mathcal{L}_1$ under the evaluation map giving

$$X \times A \to \mathcal{L}_1 \to 0.$$

The map ϕ associated to this surjection maps X to $\mathbb{P}^1$ with $\mathcal{L}_1 = \phi^*[1]_{\mathbb{P}^1}$. The solution sets of elements of A are fibers of the map. Depending on the situation, the fibers may be connected or disconnected. For example, if $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{L}_1$ was the line bundle corresponding to a bihomogeneous polynomial of bidegree $(1, 0)$, the condition would fail and the map ϕ would be a product projection. Here the fibers and hence the solution sets of elements of V_1 are connected. However, if $\mathcal{L}_1$ was the line bundle associated to bihomogeneous polynomials of bidegree $(2, 0)$, then ϕ would be a product projection composed with a degree two map of $\mathbb{P}^1$ to $\mathbb{P}^1$. In this case, the solution sets of general elements of V_1 would be disconnected (with two components). The condition given by (10) does not see which powers of the $\mathcal{L}_j$ are used and the condition is nonzero if the connectedness is true for all the powers of the $\mathcal{L}_j$.

2.4 Spaces of systems and spaces of solutions

Fix a projective variety X and an algebraic vector bundle $\mathcal{E}$ on X spanned by a vector space of algebraic sections V. Let $\mathbb{P}(V)$ be the projective space of lines through the origin in V^*. From (2), we have an embedding

$$\mathbb{P}(\mathcal{F}) \to X \times \mathbb{P}(V^*).$$

Let p and q denote the projections from $X \times \mathbb{P}(V^*)$ to X and $\mathbb{P}(V^*)$ respectively. Let $p_{\mathbb{P}(\mathcal{F})}$ and $q_{\mathbb{P}(\mathcal{F})}$ denote their restrictions to $\mathbb{P}(\mathcal{F})$. Note that $p_{\mathbb{P}(\mathcal{F})}$ is simply $\pi_{\mathbb{P}(\mathcal{F})}$. Each $v \in V$ is a section of $\mathcal{E}$ which, for us, is a polynomial system. Thus, $\mathbb{P}(V^*)$ is the space of polynomial systems associated to V. For $v \in V$, let $Z(v) \subset X$ denote the solution set of v. This yields the following.

Theorem 14. *Given $v \in \mathbb{P}(V^*)$,*

$$\pi_{\mathbb{P}(\mathcal{F})}\left(q^{-1}_{\mathbb{P}(\mathcal{F})}(v)\right) = Z(v).$$

In particular, this identifies the projective variety $\mathbb{P}(\mathcal{F})$ with the total space of all solutions corresponding to the nonzero elements of the vector space of sections V.

Proof. Fix a point $v \in \mathbb{P}(V^*)$. Choose a $\hat{v}$ in V over $v \in \mathbb{P}(V^*)$. Note that the evaluation map takes $\{x\} \times \hat{v}$ to $\hat{v}(x)$, which is 0 if and only if $x \in Z(v)$. Thus, over each of the points $x \in X$, $\{x\} \times \hat{v}$ comes from a point w_x in the fiber $\mathcal{F}^*_x$ of $\mathcal{F}^*$ over x if and only if $\hat{v}(x) = 0$. □

Note that V^* spanning $\mathcal{F}$ implies that V^* yields a space of sections, namely $\pi^*_{\mathbb{P}(\mathcal{F})}V^*$, which span $\pi^*_{\mathbb{P}(\mathcal{F})}\mathcal{F}$. By the surjection $\pi^*_{\mathbb{P}(\mathcal{F})}\mathcal{F} \to \xi_{\mathcal{F}}$, $\pi^*_{\mathbb{P}(\mathcal{F})}V^*$ spans $\xi_{\mathcal{F}}$. This is summarized in the following.

Lemma 15. *The map $q_{\mathbb{P}(\mathcal{F})} : \mathbb{P}(\mathcal{F}) \to \mathbb{P}(V^*)$ is the map associated to $\pi^*_{\mathbb{P}(\mathcal{F})}V^*$ spanning $\xi_{\mathcal{F}}$ and therefore $\xi_{\mathcal{F}} = q^*_{\mathbb{P}(\mathcal{F})}[1]_{\mathbb{P}(V^*)}$.*

Given these identifications, we have the strong conclusion involving a monodromy action.

Theorem 16. *Let X be a N-dimensional projective variety with $\dim \mathrm{Sing}\,(X) \leq N-2$. Let $\mathcal{E}$ be a rank N algebraic vector bundle spanned by a vector space V of algebraic sections. Assume that at least one section $s \in V$ has at least one isolated solution on X. Then, the map*

$$q_{\mathbb{P}(\mathcal{F})} : \mathbb{P}\,(\mathcal{F}) \to \mathbb{P}\,(V^*)$$

is generically finite to one. Moreover, given a line $\ell \subset \mathbb{P}\,(V^*)$, $C = q^{-1}_{\mathbb{P}(\mathcal{F})}(\ell)$ is a smooth, connected algebraic curve. In fact, there is a dense Zariski open set $U \subset \mathbb{P}\,(V^*)$ with $q^{-1}_{\mathbb{P}(\mathcal{F})}(u)$ being finite and smooth for $u \in U$. Given a solution $\hat{u}$ over any $u \in U \cap \ell$ having smooth isolated zeroes, continuation using loops in $U \cap L$ starting and ending at u will give all solutions over u.

Proof. By Theorem 8, there is a nonempty Zariski open set $U \subset \mathbb{P}\,(V^*)$ such that $\dim q^{-1}_{\mathbb{P}(\mathcal{F})}(u) = 0$ for smooth for $u \in U$. Hence,

$$c_1(\xi_{\mathcal{F}})^{\dim \mathbb{P}(\mathcal{F})} = c_1\left(q^*_{\mathbb{P}(\mathcal{F})}[1]_{\mathbb{P}(V^*)}\right)^{\dim \mathbb{P}(V^*)} = q^*_{\mathbb{P}(\mathcal{F})} c_1\left([1]_{\mathbb{P}(V^*)}\right)^{\dim \mathbb{P}(V^*)} \neq 0.$$

Theorem 10 yields smoothness of C and Theorem 12 yields connectedness of C. The monodromy statement is an immediate consequence of connectedness. □

3 Singular points of the systems in a general pencil

The setting under consideration is the space of systems on a projective manifold X such that at least one of the systems in the space has isolated solutions. There are three questions to ask about the singular points of systems parameterized by a general line in such a space of systems. We shall refer to the set of systems parameterized by a line in the space of systems as a *pencil of systems.* Moreover, those parameterized by a general line with be called a *general pencil of systems.* With this setup, this section considers the following questions:

1. How many singular points are there?
2. What are their multiplicities?
3. How many can there be in a fiber?

The answers to these questions constitute the main results of this paper. The first is primarily topological and answered in Section 3.1 counting with respect to ramification index. The second question is answered in Section 3.2. under very general hypotheses covering practically all cases of interest for systems arising by restriction from products of projective spaces. The last question is considered in Section 3.3.

3.1 How many singular points are there?

Let $q : \mathbb{P}\,(\mathcal{F}) \to \mathbb{P}\,(V^*)$ be the generically finite-to-one map in Theorem 16. Let $\mathcal{A} \subset \mathbb{P}\,(V^*)$ denote the union of $\mathrm{Sing}\,(\mathbb{P}\,(\mathcal{F}))$ and the algebraic set of points $y \in \mathbb{P}\,(V^*)$ with $\dim q^{-1}(y) > 0$. Let $\mathcal{B}$ denote $q^{-1}(\mathcal{A})$, $\mathcal{V} = \mathbb{P}\,(V^*) \setminus \mathcal{A}$, and $\mathcal{U} = \mathbb{P}\,(\mathcal{F}) \setminus \mathcal{B}$. This gives rise to the following.

Lemma 17. *With this setup, $\mathcal{U}$ is smooth and the map $q_{\mathcal{U}} : \mathcal{U} \to \mathcal{V}$ is finite-to-one. Since $\dim \mathcal{A} \leq \dim \mathbb{P}\,(V^*) - 2$, a general linear $\ell \subset \mathbb{P}\,(V^*)$ lies in $\mathcal{V}$. Moreover, given an arbitrary point $y \in \mathbb{P}\,(V^*)$ and general point $x \in \mathbb{P}\,(V^*)$, the line containing x and y meets $\mathcal{A}$ in at most y.*

Proof. The statements about smoothness and $q_{\mathcal{U}}$ being finite-to-one are true by the construction of $\mathcal{V}$ and $\mathcal{U}$. The fact that a general line containing y meets $\mathcal{V}$ in at most y follows by dimension counting. Consider the closure Q of the union of all lines on $\mathbb{P}(V^*)$ containing y meeting $\mathcal{V}$ in at least one point distinct from y. Thus, one has $\dim Q \leq \dim \mathcal{V} + 1 \leq \dim X - 1$. □

Let ℓ be a general line in $\mathbb{P}(V^*)$, which lies in $\mathcal{V}$. By Bertini's Theorem, $C = q^{-1}(\ell)$ is smooth and, by Theorem 16, C is connected.

The line bundle associated to the ramification locus R of $q_{\mathcal{U}}$ is identified [9, Ex. 3.2.20] with $K_{\mathcal{U}} \otimes q^* K^*_{\mathcal{V}}$. Note that $R \cap \ell$ are the ramification points of q_C. A straightforward check shows that, for $x \in C \cap R$, the multiplicity of the component of R at x is the ramification index of q_C at x. In particular, over $\mathcal{U}$, R is the zero set of a section of the line bundle

$$\mathcal{R} := K_{\mathbb{P}(\mathcal{F})} \otimes q^*_{\mathbb{P}(\mathcal{F})} \left(K^*_{\mathbb{P}(V^*)} \right).$$

It immediately follows that $\deg c_1(\mathcal{R})_C = C \cap R$. Note that the singular points of a system $y \in \ell$ are the points in $q_C^{-1}(y) \cap R$. Additionally, since $K_{\mathbb{P}(V^*)} = [-\dim V]_{\mathbb{P}(V^*)}$ and $\xi_{\mathcal{F}} = q^*_{\mathbb{P}(\mathcal{F})}[1]_{\mathbb{P}(V)^*}$, we conclude from Lemma 6 that

$$\mathcal{R} = \pi^*_{\mathbb{P}(\mathcal{F})} \left(K_X \otimes \det(\mathcal{F}) \right) \otimes \xi^N_{\mathcal{F}}. \tag{11}$$

The following counts singular points with respect to the ramification index. Thus, for example, a multiplicity two singular point of a system contributes one to the count.

Theorem 18. *Let X be an N-dimensional projective variety with singular set of codimension at least two. Let $\mathcal{E}$ be a rank N vector bundle spanned by a vector space V of sections. Assume that at least one section $s \in V$ has at least one isolated solution on X. Then, the number of singular points of solutions of the systems parameterized by a general line $\ell \subset \mathbb{P}(V^*)$ is*

$$\left(c_1(K_X) + c_1(\mathcal{E}) \right) c_{N-1}(\mathcal{E}) + N c_N(\mathcal{E}). \tag{12}$$

In particular, when $\mathcal{E}$ is a direct sum of line bundles $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_N$, this number equals

$$\left(c_1(K_X) + \sum_{j=1}^{N} c_1(\mathcal{L}_j) \right) \sum_{i=1}^{N} \left(\prod_{j \neq i} c_1(\mathcal{L}_j) \right) + N \prod_{j=1}^{N} c_1(\mathcal{L}_j). \tag{13}$$

Proof. Letting $C = q^{-1}(\ell)$, the number of singular points is

$$\mathcal{R} \cap C = \left(\pi^*_{\mathbb{P}(\mathcal{F})} \left(c_1(K_X) + c_1(\det(\mathcal{F})) \right) + N c_1(\xi_{\mathcal{F}}) \right) \cdot c_1(\xi_{\mathcal{F}})^{\dim \mathbb{P}(\mathcal{F}) - 1}.$$

Using Lemma 4, we have

$$\mathcal{R} \cap C = \left(\pi^*_{\mathbb{P}(\mathcal{F})} \left(c_1(K_X) + c_1(\det(\mathcal{F})) \right) \right) \cdot c_{N-1}(\mathcal{E}) + N c_N(\mathcal{E}).$$

This immediately implies (12) with (13) trivially following from (12). □

The following specializes Theorem 18 to a product of projective spaces.

Corollary 19. *Let $X = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_k}$ be a product of projective spaces and $N = a_1 + \cdots + a_k$. For $j = 1, \ldots, N$, let $\mathcal{L}_j$ be a line bundle of multidegrees $d_{j,1}, \ldots, d_{j,k}$ and $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_N$. Let V be a vector space of sections of $\mathcal{E}$ that spans $\mathcal{E}$ and assume that at least one section $s \in V$ has at least one isolated solution on X. For $j = 1, \ldots, N$, let δ_j be the linear function $\sum_{i=1}^{k} d_{j,i} H_i$ in the variables H_i. Then, the number of singular points of solutions of the systems parameterized by a general line $\ell \subset \mathbb{P}(V^*)$ is the coefficient of $H_1^{a_1} \cdots H_k^{a_k}$ in the polynomial*

$$\left(\sum_{j=1}^{N} \delta_j - \sum_{i=1}^{k} (a_i + 1) H_i \right) \delta_1 \cdots \delta_N \left(\frac{1}{\delta_1} + \cdots + \frac{1}{\delta_N} \right) + N \delta_1 \cdots \delta_N. \tag{14}$$

In particular, when $X = \mathbb{P}^N$ and $\mathcal{E} = [d_1]_{\mathbb{P}^N} \oplus \cdots \oplus [d_N]_{\mathbb{P}^N}$, then this number equals

$$(d_1 + \cdots + d_N - N - 1)\, d_1 \cdots d_N \left(\frac{1}{d_1} + \cdots + \frac{1}{d_N} \right) + N d_1 \cdots d_N \tag{15}$$

Proof. Let $\pi_j : X \to \mathbb{P}^{a_j}$ be the projection of X onto its j^{th} factor and let $H_j = c_1(\pi^*([1]_{\mathbb{P}^{a_j}}))$. Note that for a vector of nonnegative integers $(b_1, \ldots, b_k)$ such that $b_1 + \cdots + b_k = N$, $H_1^{b_1} \cdots H_k^{b_k}$ evaluated on X equals 1 if $b_j = a_j$ for all $j = 1, \ldots, k$ and equals zero otherwise. With this, the result immediately follows from Theorem 18. □

Remark 20. One can easily use (14) to count the number of singular points for product of projective spaces using a computer algebra system. As mentioned at the end of the Introduction, we have developed a Maple worksheet for this. The following summarizes the steps used to perform this calculation following the notation in Corollary 19 and its proof. First, the total Chern class is computed via

$$c_t(\mathcal{E}) = \prod_{j=1}^{N} \left(1 + t c_1(\mathcal{L}_j)\right).$$

To compute the Bézout number, compute the coefficient of t^N in $c_t(\mathcal{E})$, which is a polynomial in the H_j's. For this polynomial, the coefficient of $H_1^{a_1} \cdots H_k^{a_k}$ is the Bézout number since this coefficient multiplied by $H_1^{a_1} \cdots H_N^{a_k}$ is precisely $c_N(\mathcal{E})$. In the Maple worksheet, this is performed in stages by first computing the coefficient, say A_1 of $H_1^{a_1}$, and then taking the coefficient, say A_2, of $H_2^{a_2}$ in A_1, and so on.

Next, $c_{N-1}(\mathcal{E})$ is equal to the coefficient of t^{N-1} in the polynomial $c_t(\mathcal{E})$. Moreover, from (14),

$$c_1(K_X) = -\sum_{j=1}^{N} (a_j + 1) H_j.$$

Thus, the Maple worksheet computes the product

$$(c_1(K_X) + c_1(\mathcal{E})) c_{N-1}(\mathcal{E})$$

and extracts the coefficient of $H_1^{a_1} \cdots H_N^{a_N}$.

Finally, the two coefficients are combined via (12) to yield the number of singular points.

The following defines the ratio of the number of singularities in a general pencil with the Bézout number, which is one measure of the quality of a homotopy.

Definition 21. Let B be the Bézout number and σ be the number of singularities in a general pencil. Then, the ratio $r = \sigma/B$ is called the *singularity ratio.*

Hence, for a homotopy, the singularity ratio is equal to the number of singular points per solution path that needs to be tracked. A high singularity ratio means that there are many branch points in relation to the number of solution paths to track. Similarly, a low singularity ratio means that there are few branch points in relation to the number of solution paths to track.

We close this subsection with two examples.

3.1.1 Cyclic example

Let $X = \left(\mathbb{P}^1\right)^N$ be a product of N copies of $\mathbb{P}^1$'s with $N \geq 2$. Fix multidegrees $d_1, \ldots, d_N$ with $d_j = (d_{j,1}, \ldots, d_{j,N})$ such that

$$d_{j,k} = \begin{cases} 1 & \text{if } k = j \text{ or } k = j+1 \mod N \\ 0 & \text{otherwise} \end{cases}$$

Consider the general pencil $\lambda f + \mu g$ where $f = (f_1, \ldots, f_N)$ and $g = (g_1, \ldots, g_N)$ are general systems of multihomogeneous polynomials on X such that both f_j and g_j have multidegree d_j. The corresponding line bundles $\mathcal{L}_j$ and the vector bundle $\mathcal{E}$ are clear. Moreover, it is straightforward to check that the coefficient of $H_1 \cdots H_N$ in

$$(H_1 + H_2) \cdots (H_N + H_1)$$

is 2. So the Bézout number of systems in this pencil is $B = 2$.

Since $\det \mathcal{E}$ is the line bundle of multidegrees $(2, \ldots, 2)$ and K_X is a line bundle of multidegrees $(-2, \ldots, -2)$, $K_X \otimes \det \mathcal{E}$ has multidegrees all 0. Hence, $c_1(K_X) + c_1(\det \mathcal{E}) = 0$ and so the number of singular points of solutions of the systems parameterized by the general pencil is $\sigma = 2N$. Thus, the singularity ratio is $r = \sigma/B = N$ and the genus g of the smooth curve of all the singularities of the pencil satisfies $2g - 2 = -4 + 2N$, i.e., $g = N - 1$.

3.1.2 Polynomials with the same multidegrees

A special case of this situation occurs in the companion paper [12] shown using multihomogeneous counts. The following considers a generalization with all polynomials having the same multidegree on the product of projective spaces

$$X = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_k}.$$

Let $N = a_1 + \cdots + a_k$ and $\mathcal{E}$ be a direct sum $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_N$ where all of the line bundles $\mathcal{L}_j$ have the same multidegree $(d_1, \ldots, d_k)$. Then, the number of singularities in a general pencil is

$$\binom{N}{a_1, \ldots, a_k} d_1^{a_1} \cdots d_k^{a_k} \left(N(N+1) - \sum_{j=1}^{k} \frac{a_j(a_j+1)}{d_j} \right) \tag{16}$$

where $\binom{N}{a_1, \ldots, a_k}$ is the usual multinomial coefficient, namely

$$\binom{N}{a_1, \ldots, a_k} = \frac{N!}{a_1! \cdots a_k!}.$$

To see this, let H_j be as in Corollary 19, i.e., the first Chern class of the line bundle of multidegrees 0 in all places but the j^{th} where it is degree 1. Then, we have

$$c_1(K_X \otimes \det(\mathcal{E})) = \sum_{j=1}^{k} (N d_j - a_j - 1) H_j, \quad c_{N-1}(\mathcal{E}) = N \left(\sum_{j=1}^{k} d_j H_j \right)^{N-1}, \quad c_N(\mathcal{E}) = \left(\sum_{j=1}^{j} d_j H_j \right)^{N}.$$

As in the proof of Corollary 19, if $(b_1, \ldots, b_k)$ are nonnegative integers summing to N, then

$$H_1^{b_1} \cdots H_k^{b_k} = \begin{cases} 1 & (a_1, \ldots, a_k) = (b_1, \ldots, b_k), \\ 0 & \text{otherwise.} \end{cases}$$

As always, we identify the generator of $H^{2N}(X, \mathbb{Z}) = \mathbb{Z}$ corresponding to X with 1. Then, we have

$$c_1(K_X \otimes \det(\mathcal{E}))c_{N-1}(\mathcal{E}) = N \sum_{j=1}^{k} \left((Nd_j - a_j - 1) \frac{(N-1)!}{(a_j - 1)! \prod_{i \neq j} a_i!} d_j^{a_j - 1} \prod_{i \neq j} d_i^{a_i} \right)$$

and

$$c_N(\mathcal{E}) = \binom{N}{a_1, \ldots, a_k} d_1^{a_1} \cdots d_k^{a_k}.$$

Thus, (12) becomes

$$N \sum_{j=1}^{k} \left((Nd_j - a_j - 1) \frac{(N-1)!}{(a_j - 1)! \prod_{i \neq j} a_i!} d_j^{a_j - 1} \prod_{i \neq j} d_i^{a_i} \right) + N \binom{N}{a_1, \ldots, a_k} d_1^{a_1} \cdots d_k^{a_k}$$

$$= N \binom{N}{a_1, \ldots, a_k} d_1^{a_1} \cdots d_k^{a_k} \left(\sum_{j=1}^{k} \left((Nd_j - a_j - 1) \frac{a_j}{Nd_j} \right) + 1 \right), \quad (17)$$

which is equivalent to (16) since $N = a_1 + \cdots + a_k$.

The case with $k = 2$ was considered in [12, Thm. 4]. With $N = a_1 + a_2$, the multinomial coefficient becomes the binomial coefficient so that (16) becomes

$$\binom{a_1 + a_2}{a_1} d_1^{a_1} d_2^{a_2} \left((a_1 + a_2)(a_1 + a_2 + 1) - \frac{a_1(a_1 + 1)}{d_1} - \frac{a_2(a_2 + 1)}{d_2} \right). \quad (18)$$

which is equal to the formula reported in [12, Thm. 4], namely

$$2 \binom{a_1 + a_2}{a_1} d_1^{a_1 - 1} d_2^{a_2 - 1} \left(\binom{a_1 + a_2 + 1}{2} d_1 d_2 - \binom{a_1 + 1}{2} d_2 - \binom{a_2 + 1}{2} d_1 \right). \quad (19)$$

Consider the case when $a_1 = \cdots = a_k = a$ and $d_1 = \cdots = d_k = d$, i.e., polynomials of multidegree $(d, \ldots, d)$ on $X = (\mathbb{P}^a)^k$ with $N = k \cdot a$. Then, (16) and (17) become

$$N \frac{N!}{(a!)^k} d^N \left(k(Nd - a - 1) \frac{a}{Nd} + 1 \right) = \frac{N!}{(a!)^k} d^N \left(N(N+1) - \frac{N(a+1)}{d} \right).$$

The case with $k = 1$, i.e., $N = a$, was consider in [12, Thm. 2], which yields

$$Nd^N \left(\frac{Nd - N - 1}{d} + 1 \right) = (N+1)N(d-1)d^{N-1} = 2 \binom{N+1}{2} (d-1) d^{N-1}$$

as reported in [12, Thm. 2]. Since d^N is the Bézout number, the singularity ratio is

$$r = N \left(\frac{Nd - N - 1}{d} + 1 \right) = N(N+1) \left(1 - \frac{1}{d} \right).$$

Thus, the singularity ratio grows quadratically in N when $d \geq 2$.

3.2 What are the multiplicities of the singular points?

The short answer is that singularities almost always have multiplicity two.

Theorem 22. *Let X be a projective variety with singularities of codimension at least two. Let $\mathcal{E} = \mathcal{G} \oplus \mathcal{L}$ where $\mathcal{G}$ is a rank $N-1$ vector bundle on X and $\mathcal{L}$ is a line bundle on X. Let $V = V_{\mathcal{G}} \oplus V_{\mathcal{L}}$ where $V_{\mathcal{G}}$ spans $\mathcal{G}$, $V_{\mathcal{L}}$ spans $\mathcal{L}$, and the associated map of X to $\mathbb{P}(V_{\mathcal{L}})$ is an embedding. Assume that at least one section of $\mathcal{E}$ coming from V has isolated solutions. Let ℓ be a general line on $\mathbb{P}(V^*)$. Then, the following hold:*

1. *the curve C of systems parameterized by ℓ is smooth and connected; and*

2. *the singularities of the systems parameterized by ℓ (which are the branch points of the projection $C \to \ell$) are of multiplicity two (and ramification index one).*

Proof. The first statement was shown in Theorem 16.

Since the multiplicity two condition is an open condition, it suffices to show that this is true for one system. Choose a general element g of $V_{\mathcal{G}}$. By the fact that $V_{\mathcal{G}}$ spans $\mathcal{G}$ and Bertini's Theorem, the solution set $\mathcal{C}$ of g is either smooth of dimension one or empty. It cannot be empty since a general element of V has isolated solutions. Thus, $\mathcal{C}$ is a union of a finite number of smooth curves $\mathcal{C}_1, \ldots, \mathcal{C}_k$. By the existence of Lefschetz pencils, e.g., [18, Chap. 2.1.1], it follows that, for almost every choice of two sections ℓ_2, ℓ_3 of $V_{\mathcal{L}}$, the singularities of the systems on $\mathcal{C}$ parameterized by

$$\left\{\lambda\ell_2 + \mu\ell_3 \mid [\lambda, \mu] \in \mathbb{P}^1\right\}$$

are multiplicity two. Therefore, the linear $\mathbb{P}^1$ on $\mathbb{P}(V^*)$ of systems

$$\begin{bmatrix} g \\ \lambda\ell_2 + \mu\ell_3 \end{bmatrix}$$

has the property that all systems parameterized by the $\mathbb{P}^1$ have multiplicity two solutions. □

Corollary 23. *Under the hypotheses of Theorem 22, the multiplicities of the components of R are all one. Therefore, by Lemma 15, given an arbitrary point $y \in \mathbb{P}(V^*)$ and a general $x \in \mathbb{P}(V^*)$, the singularities of the systems parameterized by the line containing x and y on $\mathbb{P}(V^*)$ are multiplicity two with the possible exception of the singular points of the system y.*

Remark 24. These assumptions in Theorem 22 are actually quite mild. For example, the hypotheses hold on $\mathbb{P}^N$ when the degrees $d_1, \ldots, d_N$ satisfy $d_1 \cdots d_N > 0$. More generally, the conditions of the theorem hold on $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_k}$ with $N = a_1 + \cdots + a_k$ equations having corresponding multidegrees $d_1 = (d_{1,1}, \ldots, d_{1,k}), \ldots, d_N = (d_{N,1}, \ldots, d_{N,k})$ satisfying:

1. there exists $i \in \{1, \ldots, N\}$ such that $d_{i,j} > 0$ for every $j = 1, \ldots, k$; and

2. a general system with these multidegrees has isolated solutions.

3.3 What is the maximum number of singularities in a fiber?

Consider the hypotheses of Theorem 22 and the curve $\mathcal{C}$ arising in its proof. If we knew that $\mathcal{C}$ was connected, then it would immediately follow that there was at most one singular solution in a fiber of a general pencil of systems. For instance, if besides being spanned, suppose that X was smooth and $\mathcal{G}$ was ample of rank $N-1$, then this would follow from a generalization of the first Lefschetz Theorem, e.g., [13, Thm. 7.1.1]. For direct sums of line bundles, a much stronger statement may be made.

Theorem 25. *Let X be an N-dimensional projective variety with singular set of codimension at least two and let H_X be an ample line bundle on X. Let $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_N$ where each $\mathcal{L}_j$ is a line bundles on X with V_j spanning $\mathcal{L}_j$. Assume that the map of X to $\mathbb{P}(V_N)$ associated to $\mathcal{L}_N$ is an embedding off the singular set of X. Let $V = V_1 \oplus \cdots \oplus V_N$ and assume that, for $j = 1, \ldots, N-1$,*

$$\left(\prod_{i=1}^{j-1} c_1(\mathcal{L}_i)\right) c_1(\mathcal{L}_j)^2 c_1(H_X)^{N-j-1} \neq 0 \tag{20}$$

Then, there is at most one singularity for every system parameterized by a general line on $\mathbb{P}(V^*)$.

Proof. A Lefschetz pencil [18, Chap. 2.1.1] has at most one singularity in a fiber. As noted above, if the curve $\mathcal{C}$ in the proof of Theorem 22 was connected, we would have the requisite conclusion. For this, one can take $\mathcal{G} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{N-1}$ and utilize Lemma 13. □

In terms of R, the ramification locus of the map $q_{\mathcal{U}} : \mathcal{U} \to \mathcal{V}$, Theorem 25 says that two distinct components of R cannot have the same image under q. This yields the following.

Corollary 26. *Under the hypotheses of Theorem 25, given an arbitrary point $y \in \mathbb{P}(V^*)$ and a general $x \in \mathbb{P}(V^*)$, there is at most one singularity for any of the systems parameterized by the line containing x and y on $\mathbb{P}(V^*)$ with the possible exception of the singular points of the system y.*

The following is an immediate consequence when applied to $\mathbb{P}^N$.

Corollary 27. *Let $\mathcal{E} = [d_1]_{\mathbb{P}^N} \oplus \cdots \oplus [d_N]_{\mathbb{P}^N}$ and $V = H^0(\mathcal{E})$. Then, the conditions of Theorem 25 and Corollary 26 hold if (and only if) $d_j > 0$ for $j = 1, \ldots, N$.*

As the discussion after Lemma 13 noted, the failure of the condition imposed by (20) does not preclude the result from still being true. Thus, this is a sufficient condition.

When considering a product of projective spaces, (20) is very natural. Just as with counting summarized in Remark 20, it is easy to use a computer algebra system to check the condition. Similar with Remark 20, we have created a Maple worksheet for this as well.

Although (20) holds for many common situations, e.g., Corollary 27, it is interesting to consider cases for which it does not hold. Even when using all the sections of the line bundles, the Chern class condition (20) for connectedness can fail and/or $\mathcal{L}_N$ can fail to be ample.

Example 28. Consider two cases of general pencils with $X = \mathbb{P}^1 \times \mathbb{P}^1$:

1. bihomogeneous polynomials of bidegrees $(d, 0)$ and $(0, d)$; and
2. bihomogeneous polynomials of bidegrees $(3, 0)$ and $(2, 2)$.

In the first case, Theorem 18 yields that the total number of singularities of the solution set (all of order two) is $\sigma_d = 4d(d-1)$ with Bézout number being d^2. The case $d = 1$ has a Bézout number of one (as can easily be seen directly) and $\sigma_1 = 0$, i.e., no singularities, and thus is not interesting. Using `Bertini` [2, 3], we obtain the following:

1. for $d = 2$: $\sigma_2 = 8 = 4 \cdot 2$ as predicted by our formula but the singularities arise from four pairs, i.e., fibers with singularities have two singular points; and
2. for $d = 3$, $\sigma_3 = 24 = 8 \cdot 3$ as predicted by our formula but the singularities arise from eight triplets, i.e., fibers with singularities have three singular points.

In the second case, $\mathcal{L}_2$ is ample and Theorem 18 yields the total number of singularities of the solution set (all of order two) is $\sigma = 18$ with Bézout number of the system being 6. Using `Bertini` [2, 3], we find 10 singularities which arise from a unique fiber *and* eight appearing in pairs (four fibers with two per fiber) with $18 = 10 + 4 \cdot 2$.

Remark 29. The second case in Example 28 shows that the assumption of connectedness in the existence theorem for Lefschetz pencils cannot be dropped.

4 Examples

The following collects some interesting and examples. The first, in Section 4.1, considers the sections of the tangent bundle of $\mathbb{P}^N$ twisted by $[d]_{\mathbb{P}^N}$, i.e., $T_{\mathbb{P}^N}(d) = T_{\mathbb{P}^N} \otimes_{\mathbb{C}} [d]_{\mathbb{P}^N}$, for $d \geq -1$. This is a example of a bundle that is not a direct sum of line bundles. It leads to systems with lower Bézout numbers than one would expect using multihomogeneous counts.

The second, in Section 4.2 considers Alt's nine-point path synthesis problem for four-bar linkages [1, 19]. This problem has had a major influence on our view of continuation and branch points. The realization in [4, §3.3] that 0.83% of the paths of the homotopy to solve it passed near enough to singularities of the pencil to require precision higher than double was one of the inspirations of this article and the companion article [12]. Therefore, the numbers of singularities for systems like this are of particular interest to us. In particular, Table 1 compares four different formulations.

Finally, Section 4.3 considers a closely related family, namely the Alt-Burmester synthesis problems for four-bar linkages [7]. Table 2 compares two different formulations on the collection of Alt-Burmester problems.

4.1 Twists of the tangent bundle of projective space

The tangent bundle of $\mathbb{P}^N$, $T_{\mathbb{P}^N}$, is not a direct sum of line bundles as can be checked from the Chern classes of the bundle computed in Remark 7. Nonetheless, it gives rise to a very interesting class of polynomial systems.

Theorem 30. *Let $(x_1, \ldots, x_N)$ be coordinates on $\mathbb{C}^N$ and let $d \geq -1$ be an integer. Consider systems of the form*

$$\begin{bmatrix} p_1(x) - x_1 q(x) \\ \vdots \\ p_N(x) - x_N q(x) \end{bmatrix} \tag{21}$$

where $q(x)$ is a homogeneous polynomial of degree $d+1$ in $x_1, \ldots, x_N$ and each $p_j(x)$ is a polynomial (not necessarily homogeneous) of degree $d+1$. Each such system extends to an algebraic section of $T_{\mathbb{P}^N}(d)$, the tangent bundle of $\mathbb{P}^N$ twisted by a integer, i.e., $T_{\mathbb{P}^N} \otimes_{\mathbb{C}} [d]_{\mathbb{P}^N}$. Moreover, the correspondence between these systems and the sections of $T_{\mathbb{P}^N}(d)$ is one-to-one and onto.

Proof. To see this, fix coordinates $(z_0, \ldots, z_N)$ on $\mathbb{C}^{N+1}$. Consider the usual map

$$\pi : \mathbb{C}^{N+1} \setminus \{(0, \ldots, 0)\} \to \mathbb{P}^N \quad \text{given by} \quad (z_0, \ldots, z_N) \to [z_0, z_1, \ldots, z_N]$$

where we regard the $[z_0, z_1, \ldots, z_N]$ as homogeneous coordinates. The vector fields on $\mathbb{C}^{N+1}$ that are mapped to vector fields on $\mathbb{P}^N$ are precisely those of the form

$$\sum_{j=0}^{N} L_j(z) \frac{\partial}{\partial z_j} \tag{22}$$

with $L_j(z)$ homogeneous linear in the z variables. The only one of these vector fields that goes to zero on $\mathbb{P}^N$ is

$$z_0 \frac{\partial}{\partial z_0} + \cdots + z_N \frac{\partial}{\partial z_N}$$

which is tangent to the fibers of the map π and goes to 0.

Consider the long exact cohomology sequence associated to (7) tensored with $[d]_{\mathbb{P}^N}$. On $\mathbb{P}^N$, $H^j(\mathbb{P}^N, [k]_{\mathbb{P}^N}) = 0$ for all k with $0 < j < N$. Thus we have

$$0 \to H^0(\mathbb{P}^N, [d]_{\mathbb{P}^N}) \to \bigoplus_{j=0}^{N} H^0(\mathbb{P}^N, [d+1]_{\mathbb{P}^N}) \to H^0(\mathbb{P}^N, T_{\mathbb{P}^N}(d)) \to 0.$$

The first term is exactly the vector space of homogeneous polynomials of degree d. The second term is the vector space of $(N+1)$-tuples of homogeneous polynomials of degree $d+1$, with the map from the first to the second term given by

$$g(z) \to (z_0 g(z), \ldots, z_N g(z)).$$

The third term is the sections of the algebraic bundle $T_{\mathbb{P}^N}(d)$. The map from the second term to the third term is given by

$$(p_0(z), \ldots, p_N(z)) \to p_0(z) \frac{\partial}{\partial z_0} + \cdots + p_N(z) \frac{\partial}{\partial z_N}.$$

Exactness comes down to $z_0 \frac{\partial}{\partial z_0} + \cdots + z_N \frac{\partial}{\partial z_N}$ being the only vector field of those in (22) that is zero on $\mathbb{P}^N$.

We map $\mathbb{C}^N \to \mathbb{P}^N$ by sending $(x_1, \ldots, x_N) \to [1, x_1, \ldots, x_N]$. We will now proceed to see what the sections $p_0 \frac{\partial}{\partial z_0} + \cdots + p_N \frac{\partial}{\partial z_N}$ give rise to when restricted to this $\mathbb{C}^N$. Given

$$p_0(z) \frac{\partial}{\partial z_0} + \cdots + p_N(z) \frac{\partial}{\partial z_N},$$

we can subtract

$$g(z) \left(z_0 \frac{\partial}{\partial z_0} + \cdots + z_N \frac{\partial}{\partial z_N} \right),$$

where $g(z)$ is of degree d without changing the vector field on $\mathbb{P}^N$. In this way, we can assume that $p_0(z)$ is homogeneous of degree $d+1$ in the variables $z_1, \ldots, z_N$ and each $p_j(z)$ for $j = 1, \ldots, N$ are arbitrary homogeneous polynomials of degree $d+1$. Using

$$z_0 p_0(z) \frac{\partial}{\partial z_0} = -z_1 p_0(z) \frac{\partial}{\partial z_1} \cdots - z_N p_0(z) \frac{\partial}{\partial z_N},$$

we see that the algebraic sections of $T_{\mathbb{P}^N}(d)$ are precisely of the form

$$(p_1(z) - z_1 p_0(z)) \frac{\partial}{\partial z_1} + \cdots + (p_N(z) - z_N p_0(z)) \frac{\partial}{\partial z_N}.$$

On the $\mathbb{C}^N$ with the x coordinates, these are exactly the systems in (21). □

The Bézout number for $T_{\mathbb{P}^N}(d)$ is

$$B_{T_{\mathbb{P}^N}(d)} := \sum_{j=0}^{N} \binom{N+1}{j} d^{N-j} = \frac{(d+1)^{N+1} - 1}{d} = \sum_{j=0}^{N} (d+1)^j \tag{23}$$

which is precisely the number of isolated smooth solutions of a generic algebraic section of $T_{\mathbb{P}^N}(d)$. In (23), the first equality follows from combining the formulae for the Chern classes of $T_{\mathbb{P}^N}$, e.g., Remark 7, and the formula for the Chern classes of a bundle twisted by a line bundle. The solution count of (23) is lower than the count of

$$(d+2)^N = ((d+1)+1)^N = \sum_{j=0}^{N} \binom{N}{j} (d+1)^j$$

for N general degree $d+2$ polynomials. In [14, pg. 145], a different, but incorrect, system of two degree $d+2$ polynomials is stated to have the properties of the above system when $N = 2$.

Further,

$$c_{N-1}\left(T_{\mathbb{P}^N}(d)\right) = \left(\sum_{j=0}^{N-1} (N-j) \binom{N+1}{j} d^{N-1-j} \right) H_1^{N-1} = \left(\frac{(Nd-1)(d+1)^N + 1}{d^2} \right) H_1^{N-1},$$

where $H_1 = c_1([1]_{\mathbb{P}^N})$. This follows from the same sort of algebra as the computation of the Bézout number above. An easier computation shows that

$$c_1(K_{\mathbb{P}^N}) + c_1\left(T_{\mathbb{P}^N}(d)\right) = NdH_1.$$

Thus, Theorem 18 shows that a general pencil of systems has

$$\sigma_{T_{\mathbb{P}^N}(d)} := Nd \left(\frac{(Nd-1)(d+1)^N + 1}{d^2} \right) + N \frac{(d+1)^{N+1} - 1}{d} = N(N+1)(d+1)^N \tag{24}$$

singularities. In particular, the singularity ratio is

$$\frac{\sigma_{T_{\mathbb{P}^N}(d)}}{B_{T_{\mathbb{P}^N}(d)}} = \frac{N(N+1)d(d+1)^N}{(d+1)^{N+1} - 1} = N(N+1)\frac{d}{d+1} + O\left(\frac{1}{(d+1)^N}\right)$$

where $O\left(\frac{1}{(d+1)^N}\right)$ is the usual Hardy's O, i.e., the remainder to the approximation $N(N+1)\dfrac{d}{d+1}$ bounded by a constant independent of d (depending on N) times $\dfrac{1}{(d+1)^N}$.

Consider comparing this to the more general case of systems of N polynomials of degree $d+2$. The bundle is

$$\mathcal{G} = \bigoplus_{j=1}^{N} [d+2]_{\mathbb{P}^N}$$

The Bézout number is $B_{\mathcal{G}} = (d+2)^N$ and number of singularities is $\sigma_{\mathcal{G}} = (d+2)^{N-1}N(N+1)(d+1)$. We already saw that $B_{\mathcal{G}}$ is larger than $B_{T_{\mathbb{P}^N}(d)}$. Similarly, $\sigma_{\mathcal{G}}$ is always larger than $\sigma_{T_{\mathbb{P}^N}(d)}$ since

$$\frac{\sigma_{\mathcal{G}}}{\sigma_{T_{\mathbb{P}^N}(d)}} = \left(\frac{d+2}{d+1}\right)^{N-1}.$$

In particular,

$$\frac{\frac{\sigma_{\mathcal{G}}}{D_{\mathcal{G}}}}{\frac{\sigma_{T_{\mathbb{P}^N}(d)}}{D_{T_{\mathbb{P}^N}(d)}}} = 1 + \frac{1 - \frac{1}{(d+1)^{N-1}}}{(d+2)d} > 1.$$

4.2 Alt's nine-point path synthesis problem

To keep notation simple, we use a compact description of the systems and leave details regarding the actual systems to [19]. The first formulation of Alt's problem consists of solving sixteen cubics and eight quartics on $\mathbb{P}^{24}$, which will be represented by

$$(3H)^{16}(4H)^8 \text{ on } \mathbb{P}^{24}.$$

The Freudenstein and Roth formulation, which was the starting point in [19], yields a system of eight septics on $\mathbb{P}^8$, which will be represented by

$$(7H)^8 \text{ on } \mathbb{P}^8.$$

In [19], four new variables were added to the Freudenstein and Roth formulation along with four new polynomials which reduced the system to four quadrics and eight quartics on $\mathbb{P}^{12}$. We represent this by

$$(2H)^4(4H)^8 \text{ on } \mathbb{P}^{12}.$$

Finally, one can view this system naturally on $\mathbb{P}^6 \times \mathbb{P}^6$ consisting of

- two bihomogeneous polynomials of multidegree $(2, 0)$;
- two bihomogeneous polynomials of multidegree $(0, 2)$; and
- eight bihomogeneous polynomials of multidegree $(2, 2)$.

This will be represented by

$$(2H_1)^2(2H_2)^2(2H_1 + 2H_2)^8 \text{ on } \mathbb{P}^6 \times \mathbb{P}^6.$$

With these four formulations of Alt's problem, Table 1 summarizes the Bézout numbers (B), number of singularities (σ), and the singularity ratio ($r = B/\sigma$).

The only nontrivial one of these is the bihomogeneous formulation. Letting $[a, b]$ be the line bundle of bidegree (a, b) on $\mathbb{P}^6 \times \mathbb{P}^6$, we have

$$\mathcal{E} = [2, 0]^{\oplus 2} \oplus [0, 2]^{\oplus 2} \oplus [2, 2]^{\oplus 8}.$$

The total Chern class is

$$c_t(\mathcal{E}) = (1 + 2H_1t)^2(1 + 2H_2t)^2(1 + (2H_1 + 2H_2)t)^8.$$

Thus, $c_1(\det(\mathcal{E})) = c_1(\mathcal{E})$ is equal to the coefficient of t in $c_t(\mathcal{E})$, namely $20(H_1 + H_2)$. Similarly, $c_{11}(\mathcal{E})$ is equal to the coefficient of t^{11}. Remembering that $H_1^j H_2^{12-j}$ equals 1 if $j = 6$ and 0 otherwise, we have

$$c_{11}(\mathcal{E}) = 1{,}089{,}536 H_1^5 H_2^5(H_1 + H_2).$$

Version	Degree Structure	B	σ	$r = B/\sigma$
Original	$(3H)^{16}(4H)^8$ on $\mathbb{P}^{24}$	11,019,960,576	4,275,744,703,488	388
Freudenstein-Roth	$(7H)^8$ on $\mathbb{P}^8$	5,764,801	355,770,576	≈ 61.7
Total Degree	$(2H)^4(4H)^8$ on $\mathbb{P}^{12}$	1,048,576	125,829,120	120
Bihomogeneous	$(2H_1)^2(2H_2)^2(2H_1 + 2H_2)^8$ on $\mathbb{P}^6 \times \mathbb{P}^6$	286,720	31,768,576	110.8

Table 1: Summary of different formulations of Alt's nine-point path synthesis problem

Similarly, the Bézout number is $c_{12}(\mathcal{E})$ which is equal to 286,720 under the usual identification for connected compact manifolds M of $H^{\dim_{\mathbb{R}} M}(X,\mathbb{Z})$ with $\mathbb{Z}$. The canonical bundle K_X equals $[-7,-7]$ and therefore $c_1(K_X) = -7H_1 - 7H_2$. Putting everything together with Theorem 18 yields

$$\begin{aligned}(c_1(K_X) + c_1(\mathcal{E}))\, c_{11}(\mathcal{E}) + 12c_{12}(\mathcal{E}) &= 13(H_1 + H_2) \cdot 1{,}089{,}536 H_1^5 H_2^5 (H_1 + H_2) + 12 \cdot 286{,}720 \\ &= 14{,}163{,}968 \cdot H_1^5 H_2^5 (H_1 + H_2)^2 + 3{,}440{,}640 \\ &= 14{,}163{,}968 \cdot 2 + 3{,}440{,}640 = 31{,}768{,}576.\end{aligned}$$

For all four formulations summarized in Table 1, using all sections of $\mathcal{E}$ in each case, the conditions of Theorem 25 and Corollary 26 are satisfied.

4.3 Alt-Burmester systems

Alt's nine-point path synthesis problem [1] in Section 4.2 and Burmester's five-pose path synthesis problem [8] can be considered as part of a family of four-bar synthesis problems called Alt-Burmester problems [7]. This family of zero-dimensional problems is parameterized by nonnegative integer pairs (m,n) such that $2m+n = 10$ with $m \geq 1$. The (m,n) synthesis problem aims to compute four-bar linkages satisfying m poses (position and orientation) and n precision points (position only). The reason for $m \geq 1$ is that one can always trivially match any orientation at one point simply by setting the frame of reference. In particular, the two extremes of the Alt-Burmester problems are Burmester's problem corresponding with $(5,0)$ and Alt's problem corresponding to $(1,8)$, where an orientation is added to one of the nine points to trivially set the frame of reference.

While the formulation in Section 4.2 was highly specialized to Alt's problem, here we follow the "standard" formulation from [7]. In this "standard" formulation, an (m,n) Alt-Burmester problem (with $2m + n = 10$) corresponds with a system with degree structure denoted as

$$\left\{ \prod_{j=1}^{n} [(H_{2j-1} + H_{2j} + H_{2n+1} + H_{2n+2})(H_{2j-1} + H_{2j} + H_{2n+3} + H_{2n+4})(H_{2j-1} + H_{2j})] \right\}$$
$$\cdot (H_{2n+1} + H_{2n+2})^{m-1}(H_{2n+3} + H_{2n+4})^{m-1} \quad \text{on} \quad (\mathbb{P}^1)^{2n} \times (\mathbb{P}^2)^4.$$

Note that $2n+8 = 3n+2(m-1)$. In this formulation, rotations of the coupler link at each precision point are variables of the system, cast onto $\mathbb{P}^1 \times \mathbb{P}^1$ for each of the n precision points via isotropic coordinates. When $n \geq 1$, the corresponding rotation variables can easily be eliminated to produce an "alternate" formulation with degree structure

$$(2H_1 + 2H_2 + 2H_3 + 2H_4)^n (H_1 + H_2)^{m-1} (H_3 + H_4)^{m-1} \quad \text{on} \quad (\mathbb{P}^2)^4.$$

In particular, the $(1,8)$ "standard" and "alternate" systems provide two more presentations of Alt's nine-point problem. The Bézout number for both of these formulations is 645,120 which falls between the last two entries in Table 1.

For concreteness, consider the $(4,2)$ problem. The "standard" formulation degree structure is

$$\begin{aligned}&(H_1 + H_2 + H_5 + H_6)(H_1 + H_2 + H_7 + H_8)(H_1 + H_2) \\ &\quad \cdot (H_3 + H_4 + H_5 + H_6)(H_3 + H_4 + H_7 + H_8)(H_3 + H_4) \\ &\quad\quad \cdot (H_5 + H_6)^3 (H_7 + H_8)^3 \quad \text{on} \quad (\mathbb{P}^1)^4 \times (\mathbb{P}^2)^4. \qquad (25)\end{aligned}$$

with (26) providing explicit polynomials in affine coordinates for simplicity. Meanwhile, the "alternate" formulation degree structure is

$$(2H_1 + 2H_2 + 2H_3 + 2H_4)^2 (H_1 + H_2)^3 (H_3 + H_4)^3 \quad \text{on} \quad (\mathbb{P}^2)^4.$$

(m,n)	Formulation	B	σ	$r=\sigma/B$
$(5,0)$	Standard	36	576	16
$(4,2)$	Standard	288	15,840	55
	Alternate	288	13,824	48
$(3,4)$	Standard	3,456	345,600	100
	Alternate	3,456	237,312	≈ 68.7
$(2,6)$	Standard	46,080	6,958,080	151
	Alternate	46,080	3,363,840	73
$(1,8)$	Standard	645,120	134,184,960	208
	Alternate	645,120	38,707,200	60

Table 2: Comparison of "standard" and "alternate" formulations of the zero-dimensional Alt-Burmester problems

Table 2 provides the Bézout number (B), number of singularities (σ), and the singularity ratio ($r=\sigma/B$) for each of the zero-dimensional Alt-Burmester systems. Note that the Bézout number is the same for both the "standard" and "alternate" formulations, but the "alternate" formulation has fewer singularities in a smaller dimensional ambient space.

5 A final question

The standard setting used in this article was that X is an N-dimensional projective variety with $\dim \mathrm{Sing}\,(X) \leq N-2$ and $\mathcal{E}$ is a rank N vector bundle spanned by a vector space V of sections such there is at least one section $s \in V$ with at least one isolated solution on X. Then, the total number, $\sigma_{\mathcal{E}}$, of singularities of the solution sets of systems parameterized by a general line in $\mathbb{P}\,(V^*)$, was computed in Theorem 18. In particular, under modest conditions, $\sigma_{\mathcal{E}}$ can be computed with a straight-forward formula. Thus, one can compare the Bézout number, $B_{\mathcal{E}}$, with the number of singularities, $\sigma_{\mathcal{E}}$, by way of the singularity ratio $r_{\mathcal{E}} = \sigma_{\mathcal{E}}/B_{\mathcal{E}}$.

Rather than considering a general pencil, suppose that one takes a pencil $\mathcal{H} = \langle g, f\rangle$ such that $g \in \mathbb{P}\,(V^*)$ is general and $f \in \mathbb{P}\,(V^*)$ is arbitrary. Let $\sigma_{\mathcal{E},\mathcal{H},f}$ denote the number of such singularities away from f. Clearly, $\sigma_{\mathcal{E},\mathcal{H},f} \leq \sigma_{\mathcal{E}}$, but it would be nice to have a reasonable lower bound for $\sigma_{\mathcal{E},\mathcal{H},f}$ to understand how the singularities at f impact the singularities away from f. In practice, these two numbers are relatively close, which, combined with the study in [12], justifies using $\sigma_{\mathcal{E}}$ as a measure of quality of the homotopy $\mathcal{H}$.

Question: Is there a good upper bound (independent of the choice of f) for the *deficiency ratio*

$$\frac{\sigma_{\mathcal{E}} - \sigma_{\mathcal{E},\mathcal{H},f}}{B_{\mathcal{E}}}?$$

Thinking (loosely), it might be reasonable to expect that you could lose at most the Bézout number of singularities at f. However, this is false as shown in the following two simple examples of three quadrics on $\mathbb{P}^3$. Here,

$$\mathcal{E} = [2]_{\mathbb{P}^3} \oplus [2]_{\mathbb{P}^3} \oplus [2]_{\mathbb{P}^3}$$

with $B_{\mathcal{E}} = 8$ and $\sigma_{\mathcal{E}} = 48$. First, when f is the system (xy, xz, yz) giving the coordinate lines and g general, $\sigma_{\mathcal{E},\mathcal{H},f} = 33$. Hence, $\sigma_{\mathcal{E}} - \sigma_{\mathcal{E},\mathcal{H},f} = 48 - 33 = 15 > 8 = B_{\mathcal{E}}$ with

$$\frac{\sigma_{\mathcal{E}} - \sigma_{\mathcal{E},\mathcal{H},f}}{B_{\mathcal{E}}} = \frac{15}{8} > 1.$$

Similarly, when f is the system $(x^2 - y, xy - z, xz - y^2)$ giving a twisted cubic and g general, $\sigma_{\mathcal{E},\mathcal{H},f} = 34$. Hence, $\sigma_{\mathcal{E}} - \sigma_{\mathcal{E},\mathcal{H},f} = 48 - 34 = 14 > 8 = B_{\mathcal{E}}$ with

$$\frac{\sigma_{\mathcal{E}} - \sigma_{\mathcal{E},\mathcal{H},f}}{B_{\mathcal{E}}} = \frac{7}{4} > 1.$$

Finally, let us give one substantial example. In §4.3, we gave to counts for general lines with the same multihomogeneous structure as the Alt-Burmester problems. Let g be a general system of the type specified in (25) and f be a general Alt-Burmester system of type $(4, 2)$ in $(\mathbb{P}^1)^4 \times (\mathbb{P}^2)^4$, i.e., synthesizing a four-bar linkage with four given poses and two precision points selected generically. In particular, for simplicity, the following shows f in affine coordinates:

$$\left[\begin{array}{ll} (p_j z_1 + q_j - x_1)\,(\widehat{p_j}\widehat{z_1} + \widehat{q_j} - \widehat{x_1}) - (p_1 z_1 + q_1 - x_1)\,(\widehat{p_1}\widehat{z_1} + \widehat{q_1} - \widehat{x_1}) & \text{for } j = 2,3,4 \\ (p_j z_2 + q_j - x_2)\,(\widehat{p_j}\widehat{z_2} + \widehat{q_j} - \widehat{x_2}) - (p_1 z_2 + q_1 - x_2)\,(\widehat{p_1}\widehat{z_2} + \widehat{q_1} - \widehat{x_2}) & \text{for } j = 2,3,4 \\ (\theta_1 z_1 + q_j - x_1)\left(\widehat{\theta_1}\widehat{z_1} + \widehat{q_j} - \widehat{x_1}\right) - (p_1 z_1 + q_1 - x_1)\,(\widehat{p_1}\widehat{z_1} + \widehat{q_1} - \widehat{x_1}) & \text{for } j = 5,6 \\ (\theta_2 z_2 + q_j - x_2)\left(\widehat{\theta_2}\widehat{z_2} + \widehat{q_j} - \widehat{x_2}\right) - (p_1 z_2 + q_1 - x_2)\,(\widehat{p_1}\widehat{z_2} + \widehat{q_1} - \widehat{x_2}) & \text{for } j = 5,6 \\ \theta_1\widehat{\theta_1} - 1 & \\ \theta_2\widehat{\theta_2} - 1 & \end{array}\right] \tag{26}$$

where

- affine variables on the four $\mathbb{C}$'s are given by $\{\theta_1\}$, $\{\widehat{\theta_1}\}$, $\{\theta_2\}$, $\{\widehat{\theta_2}\}$ defining rotations;
- the affine variables on the four $\mathbb{C}^2$'s are given by $\{x_1, z_1\}$, $\{\widehat{x_1}, \widehat{z_1}\}$, $\{x_2, z_2\}$, $\{\widehat{x_2}, \widehat{z_2}\}$ defining the four-bar linkage with pivots $(x_j, \hat{x_j})$ and legs $(z_j, \hat{z_j})$;
- eight parameters $\{p_j, \widehat{p_j}\}$ for $j = 1, \ldots, 4$ with $p_j \cdot \widehat{p_j} = 1$ defining the orientations of the four poses; and
- twelve parameters $\{q_j, \widehat{q_j}\}$ for $j = 1, \ldots, 6$ defining the points (poses correspond with $j = 1, \ldots, 4$ and precision points correspond with $j = 5, 6$).

For a general linear homotopy (as computed in Table 2), we have $B_{\mathcal{E}} = 288$ and $\sigma_{\mathcal{E}} = 15{,}840$. Letting $\mathcal{H} = \langle g, f\rangle$, `Bertini` [2, 3] computed $\sigma_{\mathcal{E},\mathcal{H},f} = 15{,}064$, i.e, that there are 15,064 singularities for the systems (excluding those of f) parameterized by $\mathcal{H}$. Therefore

$$\frac{\sigma_{\mathcal{E}} - \sigma_{\mathcal{E},\mathcal{H},f}}{B_{\mathcal{E}}} = \frac{15{,}840 - 15{,}064}{288} = \frac{776}{288} = \frac{97}{36} > 2.694.$$

Hence, it would be interesting to know how large this deficiency ratio can be.

We note that the solution set of f for the "standard" formulation of the general Alt-Burmester problem of type $(4, 2)$ as specified above consists of the following:

- four irreducible components of dimension two
  - all at "infinity" and isomorphic to the vanishing of the last two polynomials in $(\mathbb{P}^1)^4$;
- five irreducible components of dimension one
  - one corresponds with the four-bar linkage degenerating to a $2R$ linkage, i.e., $x_1 = x_2$, $\widehat{x_1} = \widehat{x_2}$, $z_1 = z_2$, $\widehat{z_1} = \widehat{z_2}$,
  - four arising from having one of the pivots $(x_j, \widehat{x_j})$ or legs $(z_j, \widehat{z_j})$ at "infinity"; and

- 64 nonsingular isolated solutions
  - 60 "finite" (which solve the (4,2) Alt-Burmester problem as in [7]) and 4 at "infinity."

The solution set decomposition is the same for a nonempty Zariski open set of the parameter space.

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