RESEARCH STATEMENT

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My primary interest is in unstable homotopy theory, operads and Koszul duality of these, and Goodwillie calculus.

The fundamental approach to algebraic topology is to access topological data from a space by computing algebraic invariants. The strongest invariant we have are the homotopy groups π_*, and we call two spaces X and Y weakly equivalent if there is a map $X \to Y$ that induces an isomorphism on π_*. Unfortunately, the homotopy groups prove supremely difficult to compute, and in many ways we have not been able to move much further than Toda's amazing work, [Tod16], from 1962, wherein he computed $\pi_k(S^{n+k})$ for all $k \leq 19$.

In [Whi41], Whitehead defined the Whitehead bracket on a space

$$[\ ,\ ] : \pi_k X \otimes \pi_l X \to \pi_{k+l-1} X$$

giving $\pi_* X$ the structure of a shifted Lie algebra. Quillen leveraged this structure to give a complete description of rational homotopy theory of 1-connected spaces in terms of differential graded Lie algebras [Qui69]. A different approach is applying rational cochains to the space to get a differential graded commutative algebra. This also gives a complete description of rational homotopy theory, as seen in [Sul77]. Given a differential graded algebra A, the way to move between the two models is to replace A by a minimal model $\Lambda(A)$, and then take the dual of its indecomposables. This induces an equivalence of categories between commutative differential algebras and differential Lie algebras, which is refered to as Koszul duality.

For non-rational coefficients, the cochains of a space do not have a model which is a commutative algebra, rather only a commutative algebra up to all higher homotopies. Several similar examples occur in algebraic topology, where algebraic formulas are only satisfied up to some coherent homotopies. One language for encoding this sort of structure is operads, as developed by Boardman and Vogt [BV06], and May, see for example [May72]. The cochains on a space forms an $\mathbb{E}_\infty$-algebra, encoding the fact that the multiplication on cochains are commutative up to all higher homotopies. The weakening of the notion of commutativity allows one to construct interesting power operations, see [May70], such as the Steenrod operations in mod p homology or Adams operations in K-theory, which in practice often help with computations.

A less intrusive simplification of the homotopy groups than rationalizing are the stable homotopy groups $\pi_*^S(X) = \operatorname{colim}_l \pi_{*+l} \Sigma^l X$. These groups still turn out to be far more computable than the ordinary unstable homotopy groups. Further when $n \leq 2 \cdot \operatorname{conn}(X)$ we get $\pi_n(X) \cong \pi_n^S(X)$, according to the Freudenthal theorem. Stable homotopy groups are more naturally studied in the category of spectra, Sp. In the category of spectra we have

$\pi_*^S(X) \cong \pi_*(\Sigma^\infty X)$, where $\Sigma^\infty : Top_* \to Sp$ is the suspension spectrum functor, and Top_* is the category of pointed spaces.

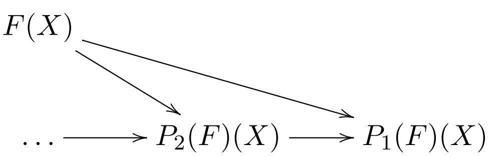
Similarly to rings we can localize the stable homotopy groups at any prime ideal of the ring $\pi_0\mathbb{S} \cong \mathbb{Z}$, where $\mathbb{S}$ is the sphere spectrum. As a ring spectrum, $\mathbb{S}$ has more prime ideals, coming from the polynomial generators of $\pi_* BP \cong \mathbb{Z}_{(p)}[\nu_1, \nu_2, \ldots]$, allowing us to form the ν_h-periodic stable homotopy groups $\nu_h^{-1}\pi_*^S(X)$. If X is a finite cell complex we get a chromatic spectral sequence $\bigoplus_h \nu_h^{-1}\pi_*^S(X) \Rightarrow \pi_*^S(X)_{(p)}$. This has facilitated many stable computations, despite the fact we only have some computations of $\nu_h^{-1}\pi_*^S(X)$ for $h = 0, 1$, see [MRW77].

Bousfield and Kuhn defined functors $\Phi_h : Top_* \to Sp$ (see [K$^+$08]) with the property that for E a spectrum,

$$\pi_*(\Phi_h(\Omega^\infty(E))) \cong \nu_h^{-1}\pi_*(E),$$

where $\Omega^\infty : Sp \to Top_*$ takes the underlying infinite loopspace of a spectrum. For a space X this allows us to define the unstable ν_h-periodic homotopy groups $\nu_h^{-1}\pi_* X$ to be $\pi_*\Phi_h(X)$.[1]

A very different approach to general homotopical calculations is Goodwillie calculus as developed in [Goo03]. Given a nice functor $F : Top_* \to Top_*$, Goodwillie developed a tower of functors $P_n(F) : Top_* \to Top_*$, under F:



Under certain conditions on both F and X, one gets an equivalence $F(X) \simeq \varprojlim_n P_n(F)(X)$. Further the layers of the tower

$$D_n(F)(X) := \mathrm{Fib}\big[P_n(F)(X) \to P_{n-1}(F)(X)\big]$$

have the form $\Omega^\infty(\partial_n(F) \wedge_{h\Sigma_n} X^{\wedge n})$, for some Borel-$\Sigma_n$-equivariant spectrum ∂_n. This implies that when the tower converges, i.e., when $F(X) \simeq \varprojlim_n P_n(F)(X)$, then we get a spectral sequence computing $\pi_* F(X)$ with input only dependent on stable information.

When $F = id$, we repress it from our notation, and write $P_n := P_n(id)$, and $\partial_n := \partial_n(id)$. Then $P_1(X) = \Omega^\infty\Sigma^\infty X$, and $\pi_* P_2(X)$ are the metastable homotopy groups, see [Mah67]. The tower thus filters the homotopy groups, starting with some classical notions. Further it was shown in [Chi05] that $\partial_* := \{\partial_n\}_n$ forms an operad. This was shown using the fact that all spaces are commutative comonads, coming from the diagonal, and then using the Koszul duality for operads identifying ∂_* as the shifted Lie operad in spectra.

If $F = \Phi_h$ as above, it was proven in [BR17] that $P_n(\Phi_h) \simeq \Phi_h \circ P_n$. This implies that when the Goodwillie tower converges, we can calculate unstable ν_h-periodic homotopy groups from stable ν_h-periodic homotopy groups.

[1] Note that there are several variants of ν_h-periodic homotopy groups and the Bousfield-Kuhn functor. We are considering the $K(h)$-local version discussed on p. 19 of [BR17]

My Work

Lie Power operations for mod p-homology. In [Beh11] Behrens defined power operations for the shifted Lie operad on mod 2 homology. This was done in an effort to provide a new proof of the Kuhn's theorem, [Kuh82]. These power operations also allowed for a restatement of the main computation of [AM99], where the homology of the layers of the Goodwillie tower for the identity evaluated on a sphere were computed. The mod 2 homology of a free algebra over the shifted Lie algebra was computed by Antolín Camarena in his thesis [AC15].

In my paper, [Kja16], the analogous computation was carried out for odd primary homology.

Theorem 1. *If L is a ∂_*-algebra, and $H_*(\ )$ is mod p homology for an odd prime, then there are power operations:*

- $\overline{\beta^\epsilon Q^i} : H_k(L) \to H_{k+2(p-1)i-\epsilon-1}(L)$,
- $[\ ,\ ] : H_i(L) \otimes H_j(L) \to H_{i+j-1}(L)$.

Further if L is a free ∂_*-algebra on a finite spectrum X, then $H_*(L)$ can be given an explicit basis in terms of these operations, and a basis for $H_*(X)$.

The construction of these power operations relies almost exclusively on the cellular model for the spaces of ∂_* as the bar construction on the commutative operad given in [Chi05], some knowledge from [Beh12] on how the Goodwillie tower interacts with the EHP-sequence, as well as knowledge of how power operations were classically constructed from cellular filtrations of $B\Sigma_p$ as in [May70].

Unstable ν_1-periodic homotopy groups from calculus. In [Beh12] Behrens is able to recompute the Toda range, [Tod16], of unstable homotopy groups using Goodwillie calculus. In his thesis, [Bra17], Brantner calculates the $K(h)$-completed E_h Lie power operations, where $K(h)$ is Morava K-theory at height h and E_h is the Lubin-Tate theory at the same height. These power operations are given by being Koszul dual to the Dyer-Lashof operations for E_h. At height $h = 1$, these Dyer-Lashof operations are classically well understood, [M$^+$83], in terms of Adams operations.

In work to appear I recover the odd primary case of $\nu_1^{-1}\pi_* SU(d)$, for all d. These were first computed by Davis, [Dav91], using the unstable Adams-Novikov-spectral sequence, and independently computed by Bousfield, [Bou99]. More generally I prove an analogous result to Bousfield's.

Theorem 2. *Let p be an odd prime, and X be a space such that $E_1^*(X)$ is finitely generated as a E_1^*-module, and free as a E_1^*-algebra. Assume further that for all k, the Adams operations $\psi^k : E_1^*(X) \to E_1^*(X)$ are injective. Lastly, assume that the Goodwillie tower for the Bousfield- Kuhn fuctor converges for X (see* [BR17] *for the known cases). Let $M = Q(E_1^*(X))/\psi^p$, where $Q(\ )$ denotes the indecomposables of the algebra, and let $(\ )^\#$ be the Pontryagin dual. Then*

$$\nu_1^{-1}\pi_* X \cong \left(\ker(M \overset{1-\psi^l}{\to} M) \oplus \operatorname{coker}(M \overset{1-\psi^l}{\to} M)\right)^\#,$$

where l is a topological generator of $\mathbb{Z}_p^\times$.

Theorem 2 is proven from a stronger result, which describes $\pi_*\Phi_1(X)$, under the conditions on the space X above. In retrospect the case of X an odd sphere was carried out by Mahowald, [Mah82], and Thompson, [Tho90]. This thus represents the only computation of the ν_1-periodic Goodwillie tower of a space which is not the sphere.

I will now give a brief account of the work that goes into proving Theorem 2 above. Let $\Phi := \Phi_1$, and recall that we defined $\nu_1^{-1}\pi_*X := \pi_*\Phi(X)$. Behrens and Rezk showed that there is an equivalence

$$P_k(\Phi) \to B(1, \mathcal{F}, S^X_{K(1)})[\leq k]^\vee$$

where $\mathcal{F}(X) = \bigvee_i X^{\wedge i}_{h\Sigma_i}$, and $[\leq k]$ refers to the filtration of the bar construction coming from the filtration of $\mathcal{F}$, given by

$$\mathcal{F}(X)[\leq k] = \bigvee_i^k X^{\wedge i}_{h\Sigma_i}.$$

The Goodwillie spectral sequence for Φ thus takes the form

$$E_1^{k,*} = \pi_*B(1, \mathcal{F}, S^X_{K(1)})[= k]^\vee \Rightarrow \nu_1^{-1}\pi_*X.$$

The E_1-page of this spectral sequence should, by Koszul duality, be the "free spectral Lie algebra on $\pi_*^S X$," but unfortunately we do not have a description of what this is. We therefore apply E_1-homology to the spectral sequence. Using Brantner's result [Bra17] we get a spectral sequence

$$\mathrm{Lie}_{\mathcal{H}}\big((E_1)_*X\big) \Rightarrow (E_1)_*(\Phi(X))$$

where $\mathrm{Lie}_{\mathcal{H}}(\ )$ is the free Hecke Lie algebra. For A a free $(E_1)_*$-module we have

$$\mathrm{Lie}_{\mathcal{H}}(A) = \mathrm{Lie}(A) \oplus \overline{\theta} \cdot \mathrm{Lie}(A),$$

where $\mathrm{Lie}(\ )$ is the free shifted Lie algebra, and $\overline{\theta}$ is the Koszul dual to the power operation θ on K-theory.

To study differentials in this spectral sequence we recall the work of Rezk in [R$^+$09]. There Rezk defined a monad $\mathbb{T} : (E_1)_*\text{-}mod \to (E_1)_*\text{-}mod$, such that if X is a E_1-algebra, with π_*X free and finitely generated as a E_1^*-module, then $\pi_*\mathcal{F}(X) \cong \mathbb{T}\pi_*X$. Now filtering $B(1, \mathcal{F}, E_1^X)^\vee$ by the bar filtration we get a square of spectral sequences:

$$\begin{array}{ccc} \mathrm{Lie}_{\mathcal{H}}\big((E_1)_*(X)\big) & \Longrightarrow & (E_1)_*\Phi(X) \\ \Uparrow & & \Uparrow \\ \bigoplus_k H^*B(1, \mathbb{T}, E_1^*(X))[= k] & \Longrightarrow & H^*B(1, \mathbb{T}, E_1^*(X)) \end{array}$$

Here the lower horizontal spectral sequence is completely algebraic, and thus the differentials are computable. We can then use the vertical spectral sequences to obtain differentials in the upper spectral sequence. Under the hypotheses, there is no room for non-algebraic differentials in the upper horizontal spectral sequence. This allows us to compute $(E_1)_*\Phi(X) \cong M$, with M as in the theorem above.

Future Vistas

Koszul Duality of Operads.

Question 1. *Is the $\mathbb{E}_n$ operad in spectra Koszul dual to an operadic desuspension of it self?*

This is known to be true for $n = 1$. This can be seen from the construction in [Chi05], and for operads in the algebraic world, see [Fre11]. One way one could give further evidence would be to calculate the power operations for algebras over the Koszul dual operad of $\mathbb{E}_n$ in mod p-homology, as this should be the Koszul dual of the ordinary algebra of power operations for algebras over $\mathbb{E}_n$, which is known to be Koszul self dual.

A further reason for attempting to study this is related to Theorem 1 above. I was unable to prove the analogs of the Adem relations (conjecturely they would satisfy the mixed Adem relations). The filtration of $\mathbb{E}_\infty$ by $\mathbb{E}_n$ gives a cofiltration of ∂_* by the Koszul duals of $\mathbb{E}_n$, and maybe this will allow a better understanding of the Lie power operations.

Question 2. *For which other operads of spectra can we find its Koszul dual?*

There exist many operads, encoding different interesting algebraic structures, and it would be interesting to see for which of the operads in spaces or spectra one would be able to compute the Koszul dual of them.

For example Robinson introduced a filtration of $\mathbb{E}_\infty$ in the context of his obstruction theory, [Rob13]. Here $\mathbb{E}_\infty$ is filtered not only by how homotopy commutative the operations are, but also by the highest arity of operations. One could hope that the latter allows for a good description of the Koszul duals.

Question 3. *What is the realtionship between the point set and ∞-categorical cooperad structure on the bar construction of an operad?*

In [Lur16], section 5.2, Lurie shows that in the world of quasi-categories the bar construction on a augmented monoid is a comonoid. Since we can view operads as monoids with respect to a certain monoidal product on the category symmetric sequences, this shows that the bar construction on an operad is a something like a cooperad up to all higher homotopies, [FG12]. An obvious question is how does this compares to Ching's cooperadic model of the bar construction of an operad in spectra, [Chi05]?

Recall that for an augmented monad M, Lurie shows that $M \to BM$ is an algebra in the quasi-category of twisted arrows, and that it has a certain universal property as such. So if we want to compare that to Ching's construction we need, for any m-simplex $[k_m] \overset{\alpha^m}{\leftarrow} [k_{m-1}] \overset{\alpha^{m-1}}{\leftarrow} \dots \overset{\alpha^1}{\leftarrow} [1]$ of $N(\Delta^{op})$, coherent homotopies filling out the diagram from Figure 1. Here $\alpha^{\mathcal{O}}$ and $\alpha_{B\mathcal{O}}$ are the structure maps for $\mathcal{O}$ an operad, respectively $B\mathcal{O}$ a cooperad. This is something I in fact can do. Unfortunately this does not show what was needed due to technicalities pertaining to how Lurie constructs a monoidal structure on the quasi-category of twisted arrows, and the fact that a cooperad is not an algebra with respect to any monoidal product in ordinary categories.

One could hope to construct an explicit monoidal quasi-category, where the data above defines an algebra, and then compare this category with

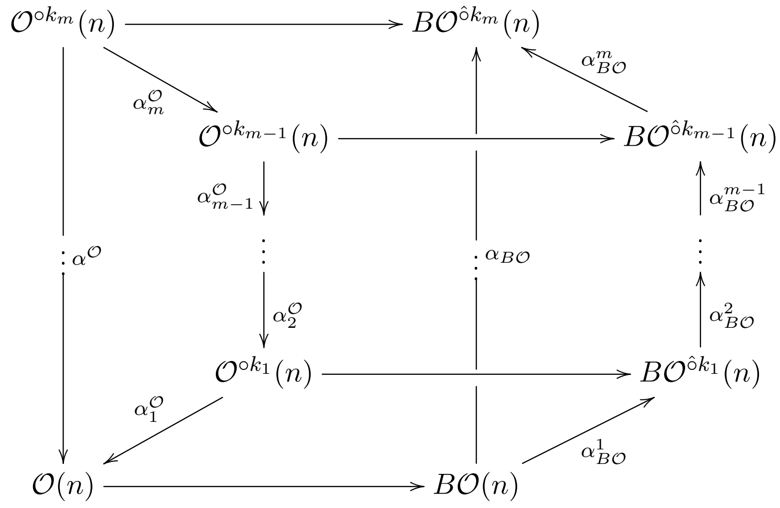


FIGURE 1.

Lurie's monoidal structure on the quasi-category of twisted arrows to get a positive answer to the question. This is something I plan to continue to pursue.

Further computations.

Question 4. *Can we get similar results to Theorem 2 at other heights?*

The results need for the computation from [Bra17] and [BR17] holds at all heights. The Dyer-Lashof algebra at higher heights gets more complicated, and therefore, so does the Koszul dual Lie Hecke algebra. We therefore would get a spectral sequence with a more complicated algebraic input, and thus more differentials. But it is likely that we would still be able to carry through the computations needed.

In Theorem 2 we can see that $X = SU(d)$ satisfies the hypotheses, and we can thus use it to compute its unstable ν_1-periodic homotopy groups. If we are able to get an analogue for Theorem 2 at height 2, can we find a space that would satisfy those hypotheses, whatever they might be? The primary issue would presumably be finding a space whose E_2 Dyer-Lashof algebra action is known, and nice enough.

Question 5. *Can we get results similar to Theorem 2 for other classes of spaces?*

Theorem 2 above has some fairly strong restrictions on the space X. Behrens' and Rezk's work, [BR17], as well as Brantner's, [Bra17], require $(E_h)^*(X)$ to be free both as a module and an algebra for their Koszul duality arguments to work. It would be very interesting to see if we can get similar results for spaces like $SO(d)$, which are very well behaved, but whose E_1-cohomology has 2-torsion. The main obstacle would be to work out how the Koszul duality interacts with the presence of torsion elements.

Question 6. *What can we say about the Goodwillie tower for the identity based on Theorem 2?*

In [Beh12] Behrens used the Goodwillie tower for the identity to recover the Toda range, [Tod16], of computation of the unstable homotopy groups of the spheres. This was done by analysing the Goodwillie tower of the EHP sequence. One could hope to get a better understanding of the Goodwillie tower for the identity evaluated on spheres by studying it for $SU(d)$, and using the fiber sequence $SU(d-1) \to SU(d) \to S^{2d+1}$. Theorem 2 could hopefully give us access to periodic of families of differentials in the Goodwillie spectral sequence for the identity for spheres.

Question 7. *Is the Goodwillie spectral sequence a spectral sequence of shifted Lie algebras?*

The E_1-page of the Goodwillie spectral sequence for the identity has a shifted Lie algebra action coming from the action on the differentials by ∂_*. The E_∞-page receives a shifted Lie algebra action coming from the Whitehead bracket. The obvious conjecture is that the Goodwillie spectral sequence is a spectral sequence of Lie algebras. We would hope to be able to prove this by comparing with the spectral sequence arising from the more algebraic fake Taylor tower as defined in [AC11].

References

[AC11] Gregory Arone and Michael Ching, *Operads and chain rules for the calculus of functors*, Société mathématique de France, 2011.

[AC15] Omar Antolin-Camarena, *The mod 2 homology of free spectral lie algebras*, Ph.D. thesis, 2015.

[AM99] Greg Arone and Mark Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Inventiones mathematicae **135** (1999), no. 3, 743–788.

[Beh11] Mark Behrens, *The Goodwillie tower for S^1 and Kuhn's theorem*, Algebraic & Geometric Topology **11** (2011), no. 4, 2453–2475.

[Beh12] ———, *The Goodwillie tower and the EHP sequence*, vol. 218, American Mathematical Society, 2012.

[Bou99] AK Bousfield, *The K-theory localizations and v_1-periodic homotopy groups of H-spaces*, Topology **38** (1999), no. 6, 1239–1264.

[BR17] Mark Behrens and Charles Rezk, *Spectral algebra models of unstable v_n-periodic homotopy theory*, arXiv preprint arXiv:1703.02186 (2017).

[Bra17] Lukas Brantner, *The Lubin-Tate theory of spectral lie algebras*, Ph.D. thesis, 2017.

[BV06] John Michael Boardman and Rainer M Vogt, *Homotopy invariant algebraic structures on topological spaces*, vol. 347, Springer, 2006.

[Chi05] Michael Ching, *Bar constructions for topological operads and the Goodwillie derivatives of the identity*, Geometry & Topology **9** (2005), no. 2, 833–934.

[Dav91] Donald M Davis, *The ν_1-periodic homotopy groups of SU (n) at odd primes*, Journal of the London Mathematical Society **2** (1991), no. 3, 529–544.

[FG12] John Francis and Dennis Gaitsgory, *Chiral Koszul duality*, Selecta Mathematica, New Series **18** (2012), no. 1, 27–87.

[Fre11] Benoit Fresse, *Koszul duality of E_n-operads*, Selecta Mathematica **17** (2011), no. 2, 363–434.

[Goo03] Thomas G Goodwillie, *Calculus III: Taylor series*, Geometry & Topology **7** (2003), 645–711.

[K$^+$08] Nicholas J Kuhn et al., *A guide to telescopic functors*, Homology, Homotopy and Applications **10** (2008), no. 3, 291–319.

[Kja16] Jens Jakob Kjaer, *On the odd primary homology of free algebras over the spectral lie operad*, Journal of Homotopy and Related Structures (2016), 1–17.

[Kuh82] Nicholas J Kuhn, *A Kahn-Priddy sequence and a conjecture of GW Whitehead*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 92, Cambridge University Press, 1982, pp. 467–483.

[Lur16] Jacob Lurie, *Higher algebra. 2014*, Preprint, available at http://www. math. harvard. edu/˜ lurie (2016).

[M+83] James E McClure et al., *Dyer-Lashof operations in k-theory*, Bulletin (New Series) of the American Mathematical Society **8** (1983), no. 1, 67–72.

[Mah67] Mark E Mahowald, *The metastable homotopy of S^n*, no. 72, American Mathematical Soc., 1967.

[Mah82] Mark Mahowald, *The image of J in the EHP sequence*, Annals of Mathematics (1982), 65–112.

[May70] J Peter May, *A general algebraic approach to Steenrod operations*, The Steenrod Algebra and its Applications: a conference to celebrate NE Steenrod's sixtieth birthday, Springer, 1970, pp. 153–231.

[May72] ———, *Iterated loop spaces and the*, The Geometry of Iterated Loop Spaces, Springer, 1972, pp. 39–49.

[MRW77] Haynes R Miller, Douglas C Ravenel, and W Stephen Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence*, Annals of Mathematics **106** (1977), no. 3, 469–516.

[Qui69] Daniel Quillen, *Rational homotopy theory*, Annals of Mathematics (1969), 205–295.

[R+09] Charles Rezk et al., *The congruence criterion for power operations in Morava E-theory*, Homology, Homotopy and Applications **11** (2009), no. 2, 327–379.

[Rob13] Alan Robinson, *E-infinity obstruction theory*, arXiv preprint arXiv:1301.1572 (2013).

[Sul77] Dennis Sullivan, *Infinitesimal computations in topology*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **47** (1977), no. 1, 269–331.

[Tho90] Robert D Thompson, *The ν_1-periodic homotopy groups of an unstable sphere at odd primes*, Transactions of the American Mathematical Society **319** (1990), no. 2, 535–559.

[Tod16] Hiroshi Toda, *Composition methods in homotopy groups of spheres.(am-49)*, vol. 49, Princeton University Press, 2016.

[Whi41] John Henry Constantine Whitehead, *On adding relations to homotopy groups*, Annals of Mathematics (1941), 409–428.