4. Recall that in class we defined a *topology* on a set $X$ to be a designation of which subsets are the open subsets, provided that
- both $X$ itself and $\emptyset$ are open;
- any finite intersection of open sets is open;
- an arbitrary union of open sets is open.

We also said that a *closed* set is defined to be the complement of an open set; i.e. $V$ is closed if and only if

$$U = X - V = \{ P \in X \mid P \notin V \}$$

is open.

a) For any collection of subsets $W_\alpha \subset X$, where $\alpha$ ranges over some indexing set, prove that

$$X - \bigcap_{\alpha \in A} W_\alpha = \bigcup_{\alpha \in A} (X - W_\alpha)$$

Note: For the rest of this problem, you can also use the fact that

$$X - \bigcup_{\alpha \in A} W_\alpha = \bigcap_{\alpha \in A} (X - W_\alpha)$$

You do not have to prove this latter fact. Note that in either case, $A$ may or may not be finite.

b) Prove that in any topology, the following hold:
- both $X$ itself and $\emptyset$ are closed;
- an arbitrary intersection of closed sets is closed;
- any finite union of closed sets is closed.

(Hint: for each open set $U$ there is a closed set $V$ such that $U = X - V$, and for each closed set $V$ there is an open set $U$ such that $V = X - U$.)

c) In the Zariski topology, give an example of a union of closed sets that is not closed. Justify your answer – here “justify” means find the Zariski closure of your union and show that it is not equal to the union of closed sets you started with. In justifying your answer, you are allowed to quote (accurately!) from class or the first problem set without reproving the results you’re quoting.

5. This problem is designed to complement Example 1.4.8 of Schenck. Find

$$\langle x^2 - x, y^2 - y \rangle : \langle x + y - 1, y^2 - y \rangle.$$ 

Justify your answer geometrically! (You don’t need a lot of computation, and certainly don’t need a computer algebra program.)

[Hint: you can take as a fact that the answer will be an ideal generated by one polynomial of degree 1 and one polynomial of degree 2. You just have to find these polynomials. You can freely use anything from Schenck and anything from class.]
6. Let
\[ I = \langle x^{2728}, y^{154}, 113x^{10}, y^{17} + 285x^{16}y + 285x^{210}y^{22} - 7 \rangle \subset \mathbb{C}[x, y], \]
where \( \mathbb{C} \) represents the complex numbers. Prove that
\[ 6199x^{1994} + 3450x^{12}y^{53} - \pi x^{299}y^{333} - 1234 \in I. \]

[Hint: there is a theorem from class that lets you do this in just a few lines. The question is not asking for any specific linear combination of the generators. If you quote a theorem from class, make sure that you explain why all of its hypotheses are satisfied, and how the question here follows from the theorem.]

7. Cox-Little-O’Shea page 182, #11. Assume that the field is \( \mathbb{C} \), the complex numbers and that the ring containing this ideal is \( \mathbb{C}[x] \). Part of the problem is to compare with Exercise 17 of Chapter 1, §5. You do not have to solve Exercise 17 separately, but explain what the connection is (by quoting a theorem). [Hint: it might help to factor these polynomials.]