This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are. Show all work. If something is not clear, ASK ME!! Good luck!

1. (10 points) Let $G$ be a group, $H$ a subgroup, and $g \in G$. Show that $gHg^{-1}$ is also a subgroup of $G$.

   Answer:

   To show that a subset $H$ of a group $G$ is a subgroup, it’s enough to show that it is closed under the group operation and closed under inverses. For the first, if $ghg^{-1}$ and $gh'g^{-1}$ are in $gHg^{-1}$ then $(ghg^{-1})(gh'g^{-1}) = g( hh' ) g^{-1} \in gHg^{-1}$ since $H$ is closed under the group operation, so $hh' \in H$. For the second, since $H$ is a subgroup, it is closed under inverses, so if $h \in H$ then $h^{-1} \in H$. Then note that $(ghg^{-1})(gh^{-1}g^{-1}) = e$ so $(ghg^{-1})^{-1} = gh^{-1}g^{-1} \in gHg^{-1}$.

2. Let $G = \mathbb{Z}_6 \times \mathbb{Z}_9$ and let $H$ be the cyclic subgroup $\langle (2,2) \rangle$.

   (a) (5 points) Explicitly write the elements of $H$.

   Answer:

   $\langle (2,2) \rangle = \{(2,2), (4,4), (0,6), (2,8), (4,1), (0,3), (2,5), (4,7), (0,0)\}$.

   (b) (5 points) Find $|G|$, $|H|$ and $|G/H|$.

   Answer:

   $|G| = 54$, $|H| = 9$, $|G/H| = 6$.

   (c) (5 points) According to the Fundamental Theorem of Finitely Generated Abelian Groups, to what group is $G/H$ isomorphic? Explain your answer. (Hint: from the information contained in (a) and (b), you can already answer this part without any further computation.)

   Answer:

   The FTFGAG gives us that $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, which is also isomorphic to $\mathbb{Z}_6$.

3. Let $G = \mathbb{Z}_8 \times \mathbb{Z}_{12}$ and let

   $H = \{(6,6), (4,0), (2,6), (0,6), (6,0), (4,6), (2,0), (0,0)\}$,

   so $|G| = 96$, $|H| = 8$ and $|G/H| = 12$. You do not have to prove that $H$ is actually a subgroup, or justify these numbers – take it as a fact.

   (a) (5 points) According to the Fundamental Theorem of Finitely Generated Abelian Groups, what are the two possibilities for $G/H$ up to isomorphism?

   Answer:

   $\mathbb{Z}_3 \times \mathbb{Z}_4$ or $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. 

(b) (5 points) I don’t want you wasting time with a tedious calculation, so let’s just see what you **would** do to determine which of the two possibilities in (a) is the correct one. Without actually making any computations, what isomorphism invariant would you look for to distinguish between the two possibilities? (Your answer should not take more than a line or two.)

**Answer:**

Look for an element of order 4, or an element of order 12. If either exists, the answer is \( \mathbb{Z}_3 \times \mathbb{Z}_4 \). If neither exists, it’s \( \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \).

(c) (5 points) Compute the order of the element \( (3, 5) + H \) in \( G/H \), showing your work.

**Answer:**

We want to count how many times we can add \( (3, 5) \) to itself before we get to an element in \( H \).

\[
\begin{align*}
1 \cdot [(3, 5) + H] &= (3, 5) + H \\
2 \cdot [(3, 5) + H] &= (6, 10) + H \\
3 \cdot [(3, 5) + H] &= (1, 3) + H \\
4 \cdot [(3, 5) + H] &= (4, 8) + H \\
5 \cdot [(3, 5) + H] &= (7, 1) + H \\
6 \cdot [(3, 5) + H] &= (2, 6) + H = (0, 0) + H
\end{align*}
\]

so the order is 6.

(d) **Extra credit (2 points): After you finish all the other problems on the exam, come back to this one and tell me which of the two possibilities for \( G/H \) is the correct one, justifying your answer.** [Warning: part (c) is not intended to help with this part.]

**Answer:**

Let’s list all the cosets and their orders as elements of \( G/H \), just like in part (c).

<table>
<thead>
<tr>
<th>Element</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0, 0) + H = {(6, 6), (4, 0), (2, 6), (0, 6), (6, 0), (4, 6), (2, 0), (0, 0)} )</td>
<td>1</td>
</tr>
<tr>
<td>( (1, 1) + H = {(7, 7), (5, 1), (3, 7), (1, 7), (7, 1), (5, 7), (3, 1), (1, 1)} )</td>
<td>6</td>
</tr>
<tr>
<td>( (2, 2) + H = {(0, 8), (6, 2), (4, 8), (2, 8), (0, 2), (6, 8), (4, 2), (2, 2)} )</td>
<td>3</td>
</tr>
<tr>
<td>( (3, 3) + H = {(1, 9), (7, 3), (5, 9), (3, 9), (1, 3), (7, 9), (5, 3), (3, 3)} )</td>
<td>2</td>
</tr>
<tr>
<td>( (1, 0) + H = {(7, 6), (5, 0), (3, 6), (1, 6), (7, 0), (5, 6), (3, 0), (1, 0)} )</td>
<td>2</td>
</tr>
<tr>
<td>( (0, 1) + H = {(6, 7), (4, 1), (2, 7), (0, 7), (6, 1), (4, 7), (2, 1), (0, 1)} )</td>
<td>6</td>
</tr>
<tr>
<td>( (0, 3) + H = {(6, 9), (4, 3), (2, 9), (0, 9), (6, 3), (4, 9), (2, 3), (0, 3)} )</td>
<td>2</td>
</tr>
<tr>
<td>( (0, 4) + H = {(6, 10), (4, 4), (2, 10), (0, 10), (6, 4), (4, 10), (2, 4), (0, 4)} )</td>
<td>3</td>
</tr>
<tr>
<td>( (5, 5) + H = {(3, 11), (F), (7, 11), (5, 11), (3, 5), (1, 11), (7, 5), (5, 5)} )</td>
<td>6</td>
</tr>
<tr>
<td>( (1, 2) + H = {(7, 8), (5, 2), (3, 8), (1, 8), (7, 2), (5, 8), (3, 2), (1, 2)} )</td>
<td>6</td>
</tr>
<tr>
<td>( (1, 4) + H = {(7, 10), (5, 4), (3, 10), (1, 10), (7, 4), (5, 10), (3, 4), (1, 4)} )</td>
<td>6</td>
</tr>
<tr>
<td>( (2, 5) + H = {(0, 11), (6, 5), (4, 11), (2, 11), (8, 5), (6, 11), (4, 5), (2, 5)} )</td>
<td>6</td>
</tr>
</tbody>
</table>

There are no elements of order 4 or of order 12. So the correct answer is \( \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \).

4. **Short** proofs. Each of the following proofs should only take a couple of lines.

(a) (8 points) Let \( G \) be a group and let \( H \) be a subgroup. Let \( a \in G \) and let \( h \in H \). Prove that we have an equality of cosets \( aH = (ah)H \).

**Answer:**

We know that we have an equality of cosets \( aH = bH \) if and only if \( a^{-1}b \in H \). So here, since \( a^{-1}(ah) = h \in H \), we have the desired equality.
(b) (8 points) Let \( G = \langle a \rangle \) be a cyclic group. Let \( H \) be a subgroup. Explain why \( H \) is normal, and prove that \( G/H \) is again cyclic.

Answer:

\( H \) is normal because every cyclic group is abelian, and any subgroup of an abelian group is normal. We claim \( G/H = \langle aH \rangle \). Let \( bH \in G/H \). Then \( b \in G \), so \( b = a^n \) for some \( n \). Thus \( bH = a^nH = (aH)^n \).

(c) (8 points) Assume that \( R \) is a ring with unity. If \( a \) and \( b \) are non-zero elements of \( R \) such that \( ab = 0 \), prove that \( a \) is not a unit.

Answer:

Suppose \( a \) were a unit. Then there exists \( c \in R \) such that \( ac = ca = 1 \). Thus

\[
0 = c \cdot 0 = ab = c \cdot (ab) = (ca) \cdot b = 1 \cdot b = b,
\]

so \( b = 0 \), contradicting the assumption that \( a \) and \( b \) are not zero.

5. Examples.

(a) (5 points) Give an example of an infinite ring that contains zero-divisors. In your answer, be sure that your ring is infinite, and be sure to include an example of two non-zero elements of this ring whose product is zero.

Answer:

\( M_2(\mathbb{R}) \). For example,

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

(b) (5 points) Give an example of three groups \( G_1, G_2, G_3 \) such that the direct product \( G_1 \times G_2 \times G_3 \) is cyclic. How do you know it’s cyclic?

Answer:

\( \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \). It’s cyclic since 2, 3 and 5 are relatively prime, so the least common multiple is 30, and \((1,1,1)\) generates the group.

6. In this problem we want to look at homomorphisms, both in the context of group homomorphisms and in the context of ring homomorphisms.

(a) (7 points) Let \( G = \langle a \rangle \) be a cyclic group. If \( \phi : G \to G' \) is a group homomorphism, explain how you can find \( \phi(x) \) for any \( x \in G \) as long as you know \( \phi(a) \).

Answer:

If \( x \in G \) then \( x = a^n \) for some \( n \), and \( \phi(x) = \phi(a^n) = \phi(a)^n \), so you know \( \phi(x) \) if you know \( \phi(a) \).

(b) (7 points) What are the possible group homomorphisms \( \phi : \mathbb{Z}_6 \to \mathbb{Z}_6 \)? Just tell me what the possible values for \( \phi(1) \) are. Explain your answer.

Answer:

Since

\[
0 = \phi(0) = \phi(1 + 1 + 1 + 1 + 1 + 1) = \phi(1) + \phi(1) + \phi(1) + \phi(1) + \phi(1) + \phi(1),
\]

the order of \( \phi(1) \) has to divide 6. Since \( \phi(1) \) is in \( \mathbb{Z}_6 \), we again get that its order has to divide 6, which is not a new condition. So \( \phi(1) \) can be anything: 0, 1, 2, 3, 4 or 5.
(c) (5 points) Now we want to focus on ring homomorphisms. Show that if \( \phi : \mathbb{Z}_6 \to \mathbb{Z}_6 \) is a ring homomorphism then for any \( x \in \mathbb{Z}_6 \) we must have \( \phi(x)[\phi(1) - 1] = 0 \). [Hint: \( x \cdot 1 = x \).]

Answer:
We know that \( \phi(x) = \phi(x \cdot 1) = \phi(x) \cdot \phi(1) \) since \( \phi \) is a ring homomorphism. Thus
\[
0 = \phi(x) \cdot \phi(1) - \phi(x) = \phi(x)[\phi(1) - 1].
\]

(d) (7 points) What are the possible values for \( \phi(1) \) in order for the formula in (c) to be correct, for \( \phi : \mathbb{Z}_6 \to \mathbb{Z}_6 \)?

Answer:
In the formula from (c), let \( x = 1 \). Then we get \( \phi(1)[\phi(1) - 1] = 0 \). So what can \( \phi(1) \) be equal to? Plugging in the possible values of \( \phi(1) \) from \( \mathbb{Z}_6 \), we see that
\[
\begin{align*}
0[0 - 1] &= 0 \quad \text{is true} \\
1[1 - 1] &= 0 \quad \text{is true} \\
2[2 - 1] &= 0 \quad \text{is false} \\
3[3 - 1] &= 0 \quad \text{is true} \\
4[4 - 1] &= 0 \quad \text{is true} \\
5[5 - 1] &= 0 \quad \text{is false}
\end{align*}
\]
so only \( \phi(1) = 0, \phi(1) = 1, \phi(1) = 3 \) and \( \phi(1) = 4 \) are possible.

Up to here got full credit, but let’s see if our four possibilities actually give ring homomorphisms. We know from (b) that \( \phi(1) \) can take any value from the point of view of group homomorphisms, so we just have to focus on the multiplicative structure. Of these clearly \( \phi(1) = 0 \) (the trivial homomorphism) and \( \phi(1) = 1 \) (the identity) are homomorphisms, so we just have to check the other two. We know that
\[
\phi(x) = \underbrace{\phi(1 + \cdots + 1)}_{x \text{ times}} = \underbrace{\phi(1) + \cdots + \phi(1)}_{x \text{ times}} = x \cdot \phi(1).
\]
So
\[
\phi(xy) = xy \cdot \phi(1) \quad \text{and} \quad \phi(x)\phi(y) = x \cdot \phi(1) \cdot y \cdot \phi(1) = xy\phi(1)^2.
\]
So \( \phi(xy) = \phi(x)\phi(y) \) if and only if \( \phi(1)^2 = \phi(1) \). This is true for \( \phi(1) = 0, 1, 3 \) and \( 4 \), so all four values do indeed give ring homomorphisms.

7. In the quaternions, \( \mathbb{H} \), find the multiplicative inverse of \( 1 + i + j + k \).

Answer:
\[
\frac{1}{4} - \frac{i}{4} - \frac{j}{4} - \frac{k}{4}.
\]