APPLICATIONS OF LIAISON

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Abstract. Over the course of more than 150 years a beautiful theory of liaison has emerged. Classically, complete intersections were used for the links. A systematic study of liaison theory where one uses, more generally, arithmetically Gorenstein schemes was begun only in the last few decades. It led to a flurry of new insights and applications. After reviewing some needed concepts and results, several of these applications are discussed. Topics include Hilbert functions and free resolutions, hyperplane arrangements, Gröbner bases, Rees algebras, simplicial complexes and more.

Contents

1. Introduction

Liaison theory has a long and rich history, with several periods of pronounced activity in the last century and a half. Many important questions have been answered, and important questions still remain. We refer the reader to [61], [63] and [67] for detailed treatments of liaison theory, and the authors of this paper hope to update [63] in the coming years to account for the progress made in the intervening years since its original publication, of which there is quite a bit.

Key words and phrases. liaison, basic double link, stick figure, hyperplane arrangement, graded Betti number, simplicial polytope, Gröbner basis, Ferrers ideal, Rees algebra, vertex decomposability, unprojection.

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What is less chronicled, though, are the many areas in which liaison techniques have been applied. In this paper we have selected a handful of examples of such applications. We begin, in section 2, by giving enough of a background to make the subsequent sections readable, and then we describe just a few of the many directions in which these tools have led to interesting, and perhaps surprising, contributions. The table of contents lists the topics that we will cover, and will not be repeated here.

One of the breakthroughs in liaison theory came in 1983 by Lazarsfeld and Rao [58], and in a sense it was intended as a warning that a classical idea for applying liaison was more limited than was previously known. They say: *Classically, linkage was seen as a method for producing interesting examples of space curves starting from simpler ones. ... A priori, one could hope – as some of the classical geometers apparently did – that techniques of liaison could be used to study space curves inductively, by linking a given curve to a (possibly very special) curve of lower degree or genus. Believing that at least for general curves such an approach is fundimentally flawed, Harris suggested that a general curve should in various senses be minimal in its liaison class. Our results may be seen, then, as giving additional support (if any is needed) to the philosophy that there is no easy way to get one's hands on a "general" curve.*

In a sense, the book of Martin-Deschamps and Perrin [61] showed that this result of Lazarsfeld and Rao was not restricted to general space curves, but in fact was just part of a beautiful and much larger picture for space curves. The paper [5], appearing at about the same time, showed that it was not even restricted to space curves.

The applications in this paper illustrate the fact that nevertheless, the classical ideas were not so far off. In many situations liaison *can* be used to study general objects, and in any case it can be used to produce examples of varieties or ideals with very nice properties, or to produce interesting results of other kinds.

For many applications, it is essential to use a more general concept of liaison. Classically and in the references above, complete intersections were used to link subschemes. However, in [83] it is already discussed that one could use, more generally, arithmetically Gorenstein subschemes to link. Several decades went by before a systematic investigation of Gorenstein liaison was initiated in [54]. It led to a flurry of new results whose power we illustrate in some of the following sections.

We end the paper with a short list of open questions from liaison theory, hoping that they supplement the descriptions of known applications as a motivation for further study in liaison theory, and that their eventual resolution will in turn lead to new applications. We also include a long list of references for the interested reader.

2. Background

Let $R = k[x_0, \dots, x_n]$, where k is at least an infinite field. In different parts of the paper we make different assumptions about k.

Definition 2.1. Let $C_1, C_2, X \subset \mathbb{P}^n$ be subschemes of the same dimension, with X arithmetically Gorenstein. Assume that $I_X \subset I_{C_1} \cap I_{C_2}$ and that $I_X : I_{C_1} = I_{C_2}$ and $I_X : I_{C_2} = I_{C_1}$. Then C_1 and C_2 are said to be *(directly) algebraically G-linked by X*, and we say that C_2 is *residual* to C_1 in X. We write $C_1 \overset{X}{\sim} C_2$. If X is a complete intersections, we say that C_1 and C_2 are *(directly) algebraically CI-linked*.

Suppose two subschemes C_1 and C_2 of $\mathbb{P}^n$ are directly G-linked by an arithmetically Gorenstein subscheme X, and assume that the last twist in the minimal free resolution of X is $-t$. Then it was shown in [83] that there is a short exact sequence of sheaves

$$0 \to \mathcal{I}_X \to \mathcal{I}_{C_1} \to \omega_{C_2}(n+1-t) \to 0. \tag{2.1}$$

Of course this gives a short exact sequence on global sections, since $\mathcal{I}_X$ has vanishing first cohomology. If furthermore C_1 and C_2 are arithmetically Cohen-Macaulay, and if we know a minimal free resolution for I_X and one for I_{C_1}, then a mapping cone gives a free resolution for the canonical module of C_2, and the dual of this resolution is a free resolution for I_{C_2}. (Something more general holds, but we will not need it here.) This construction for the free resolution of I_{C_2} is called the *mapping cone construction*.

In some sense, the theories of CI-linkage and G-linkage have moved in different directions, although in codimension two in projective space they coincide. Two important properties that these two kinds of linkage have in common are the invariance of the deficiency module under (even) liaison, and the formula for the Hilbert function of the residual scheme in the arithmetically Cohen-Macaulay case.

We will see in a moment that the invariance of the deficiency modules implies that the property of being arithmetically Cohen-Macaulay is preserved under liaison. Accepting this for now, we first mention the Hilbert function formula. Suppose V and V' are arithmetically Cohen-Macaulay schemes of codimension c directly linked by an arithmetically Gorenstein scheme X. It turns out that linkage is also preserved under general hyperplane sections, including Artinian reductions, so we can assume that J and J' are Artinian ideals directly linked by an Artinian Gorenstein ideal I in a polynomial ring R.

Let $\underline{c} = (1, c, c_2, \dots, c_{s-1}, c_s)$ be the Hilbert function of R/I (i.e. the h-vector of X). Note that $\underline{c}$ is symmetric. Let $\underline{a} = (1, a_1, \dots, a_t)$ be the Hilbert function of R/J and let $\underline{a}' = (1, a_1', \dots, a_u')$ be the Hilbert function of R/J'. Note that $a_1 \leq c$, with equality if and only if V is non-degenerate (and similarly for V'). By [21] Theorem 3 (see also [63] Corollary 5.2.19) we have the following result.

Theorem 2.2. *Under the above assumptions and notation, the h-vector of V_2 is given by*

$$a_i' = c_{s-i} - a_{s-i}$$

for $i \geq 0$.

If C is a subscheme of $\mathbb{P}^n$ of dimension r, then for $1 \leq i \leq r$ we will denote by $M^i(C)$ the *i-th deficiency module*, i.e. the graded R-module

$$M^i(C) = \bigoplus_{t \in \mathbb{Z}} H^i(\mathcal{I}_C(t)).$$

Recall that C is arithmetically Cohen-Macaulay (ACM) if and only if $M^i(C) = 0$ for all $1 \leq i \leq r$.

It was shown by Hartshorne and by Rao (cf. for instance [86]) that in any codimension (assuming dimension $r \geq 1$), up to shift $M^i(C)$ is an invariant of the even liaison class of C. There is also a result relating the modules under an odd number of links, involving dual modules. We omit this here, but note that it follows from this that the property of being arithmetically Cohen-Macaulay is thus an invariant of a liaison class.

In fact, the whole configuration of modules is invariant up to shift. However, except in one case (see below), they do not uniquely determine the even liaison class, and in codimension ≥ 3 we know very little about what invariant(s) uniquely determine an even liaison class.

In [10] Proposition 1.4, it was shown that in fact there is a left-most shift for this configuration of modules within an even liaison class:

Proposition 2.3. *Let $\mathcal{L}$ be an even liaison class of dimension r subschemes of $\mathbb{P}^n$ ($1 \leq r \leq n-2$). Then there exists $X \in \mathcal{L}$ such that for all $V \in \mathcal{L}$ and for all $1 \leq i \leq r$, we have*

$$M^i(V) \cong M^i(X)(-d) \ \ \textit{for some } d \geq 0.$$

Note that it is the same value of d for each of the modules. This motivates the following partition of a non-ACM even liaison class according to the shift of the modules:

Definition 2.4. Let $\mathcal{L}$ be an even liaison class of dimension r subschemes of $\mathbb{P}^n$. Then $\mathcal{L}^0$ is the set of subschemes whose associated modules attain the leftmost possible shift, and $\mathcal{L}^h$ is the set of subschemes whose associated modules are shifted h places to the right of the leftmost shift.

We now consider curves in $\mathbb{P}^3$. As a special case, if C is a curve in $\mathbb{P}^3$, we set

$$M(C) = \bigoplus_{t \in \mathbb{Z}} H^1(\mathcal{I}_C(t)).$$

This is the *Hartshorne-Rao module* of C. It serves (at least) two purposes in this paper:

- It is invariant for the even liaison class of C (Hartshorne-Rao), and in fact up to shift it determines the even liaison class (Rao – see Theorem 2.5);
- It measures the failure of C to be ACM In particular,

$$C \textit{ is ACM if and only if } \ M(C) = 0.$$

For codimension two subschemes of a smooth arithmetically Gorenstein variety (in particular, codimension two subschemes of $\mathbb{P}^n$), we have necessary and sufficient conditions for two subschemes to be in the same even liaison class (cf. [86], [76], [79]). However, for the purposes of this paper we focus on the necessary and sufficient condition found by Rao for curves in $\mathbb{P}^3$.

Theorem 2.5 (Rao [85])**.** (i) *Let C, C' be curves in $\mathbb{P}^3$ whose homogeneous ideals are unmixed (i.e. the curves are locally Cohen-Macaulay and equidimensional). Then C and C' are in the same even liaison class if and only if $M(C) \cong M(C')(\delta)$ for some $\delta \in \mathbb{Z}$.*

(ii) *Let M be a graded module of finite length over $k[x_0, x_1, x_2, x_3]$. Then there exists a curve $C \subset \mathbb{P}^3$ and a positive integer δ such that $M \cong M(C)(-\delta)$.*

Rao has also solved this question for locally Cohen-Macaulay, codimension two subschemes of projective space [86] in terms of stable equivalence classes of vector bundles. We omit the details here, except to remark that both Nagel [76] and Nollet [79] extended the result to codimension two subschemes that are equidimensional but not necessarily locally Cohen-Macaulay.

We also remark that among curves in $\mathbb{P}^3$ (and indeed, among codimension two subschemes of $\mathbb{P}^n$), the ACM subschemes *form* an even liaison class. In higher codimension, it is an interesting question to determine which ACM subschemes are *licci* (the CI-liaison class of a complete intersection), or *glicci* (the G-liaison class of a complete intersection). We do not deal with this question in this paper except to mention an important open question in the last section.

Next we recall the construction of Liaison Addition, which was discovered and first proved by Phil Schwartau in his Ph.D. thesis in 1982 [88]. We note that while Schwartau never published his thesis, the result has been generalized in the literature [10], [32], [47]. Below we state the version proved in his thesis, very slightly revised to agree with our terminology.

Theorem 2.6 (Schwartau, [88] Theorem 50)**.** *Let C, C' be codimension two subschemes of $\mathbb{P}^n$. Let $F \in I_C$ and $F' \in I_{C'}$ be homogeneous polynomials such that (F, F') forms a regular sequence, defining a complete intersection Y. Assume that $\deg F = d$, $\deg F' = d'$. Let $I = F' \cdot I_C + F \cdot I_{C'}$. Then*

(i) *I is a saturated ideal.*
(ii) *As sets, I defines $C \cup C' \cup Y$. This is also true as schemes if pairwise C, C' and Y have no common component.*
(iii) *Let X be the scheme defined by the saturated ideal I. Then for $1 \leq i \leq n-2$ we have*
$$M^i(X) = M^i(C)(-d') \oplus M^i(C')(-d).$$
(iv) *In particular, if C and C' are ACM then so is X.*
(v) *The Hilbert function of X satisfies*
$$h_X(t) = h_Y(t) + h_C(t-d') + h_{C'}(t-d).$$

The ideal I (or the subscheme X) is called the liaison addition *of C and C'.*

The construction of Basic Double Linkage was introduced by Lazarsfeld and Rao [58] in 1982 in the context of curves in $\mathbb{P}^3$. We first state it in the context of codimension two subschemes of $\mathbb{P}^n$.

Theorem 2.7 (Lazarsfeld-Rao [58])**.** *Let $C \subset \mathbb{P}^n$ be a codimension 2 subscheme. Let $F \in I_C$ and let A be a form such that (F, A) is a regular sequence. Let Y be the complete intersection subscheme defined by (F, A).*

Consider the ideal $J = A \cdot I_C + (F)$. Then:

(i) *$J = A \cdot I_C + (F)$ is a saturated ideal, defining a scheme X.*
(ii) *If I_C is unmixed then X is CI-linked to C in two steps.*
(iii) *In particular:*
  – *if C is ACM then so is X.*
  – *If $C \in \mathcal{L}^h$ and $\deg A = a$ then $X \in \mathcal{L}^{h+a}$.*
(iv) *If C and Y have no common component then $X = C \cup Y$ as schemes.*

The ideal J (resp. subscheme X) is called a Basic Double Link *of I_C (resp. of C).*

Notice that this version of basic double linkage can be viewed as a special case of Liaison Addition, by taking C' to be the empty set, with ideal R. This theorem has been generalized, and we next give a more general version. This version was discovered by the two authors with Kleppe, Miró-Roig and Peterson [54].

Theorem 2.8 ([54] Lemma 4.8, Proposition 5.10)**.** *Let $S \subset \mathbb{P}^n$ be a generically Gorenstein, ACM subscheme. Let $C \subset S$ be an equidimensional subscheme of codimension 1 in S, and let $A \in R$ be a homogeneous element of degree d such that $I_S : A = I_S$. Let*
$$J = A \cdot I_C + I_S.$$

(i) *J is unmixed (in particular saturated). Let Y be the scheme defined by J, so we have $J = I_Y$.*

(ii) $\deg(Y) = d \cdot \deg(S) + \deg(C)$.
(iii) *Y is ACM if and only if C is ACM.*
(iv) *Let C_A be the subscheme of S cut out by A. We have $I_{C_A} = I_S + (A)$. As sets, $Y = C \cup C_A$, and if A does not vanish on any component of C then this equality is also true as schemes.*
(v) *C and Y are evenly G-linked in two steps. When S is a complete intersection, C and Y are evenly CI-linked in two steps.*

The ideal J (resp. the subscheme Y) is called a Basic Double G-link *of I_C (resp. of C).*

Remark 2.9. We want to highlight the second half of item (v) of the above theorem. When S is a complete intersection, then Y is not only G-linked in two steps, but in fact CI-linked in two steps. In this case we call Y a *Basic Double CI-link* of C.

Example 2.10. As a first application, basic double linkage can be used in a simple way to show that powers of complete intersections are saturated and Cohen-Macaulay, and with a little more work, to find their graded Betti numbers. This was carried out in [37]. The main tool is Lemma 1.4 of that paper, which applies Theorem 2.8 above and says the following.

> *Let $F_1, \dots, F_r$ be a regular sequence in $R = k[x_0 \dots, x_n]$ with $\deg F_i = d_i$. Set $I = \langle F_1, \dots, F_r \rangle$ and $J = \langle F_2, \dots, F_r \rangle$. Then for each positive integer s,*
>
> $$I^s = J^s + F_1 \cdot I^{s-1}.$$
>
> *Furthermore, we have the following short exact sequence*
>
> $$0 \to J^s(-d_1) \to I^{s-1}(-d_1) \oplus J^s \to J^s + f_1 \cdot I^{s-1} = I^s \to 0.$$

Using this sequence and induction, it is not hard to see that R/I^s is Cohen-Macaulay (so in particular I^s is saturated), and using a mapping cone and an inductive argument, with a bit of work one gets the graded Betti numbers.

The original significance of Theorem 2.8 is that it showed how Gorenstein liaison can be thought of as a theory about divisors. Indeed, Corollary 5.14 of [54] gives that if S as above satisfies property G_1 and C is a divisor, then a divisor in the linear system $|C + tH|$ (where H is the hyperplane section class and $t \in \mathbb{Z}$) can be obtained from C in two Gorenstein links. The analogous statement for complete intersection liaison was already known (for instance [61]). Hartshorne has further explored the consequences of this point of view (e.g. [46]). See also [64].

As we saw above, in codimension two it is fairly well understood what determines an even liaison class, while in higher codimension it is wide open. Another natural question is whether the even liaison classes (say with fixed codimension) have a common structure. This question has a long history [58], [5], [61], [10], [12], [76], [79], [47], [45], [59]. The following was proposed in general in [10], although as we will see, the only positive result is in codimension two.

Definition 2.11 ([10] Definition 1.8)**.** Let $\mathcal{L}$ be an even liaison class of dimension r subschemes of $\mathbb{P}^n$. Then $\mathcal{L}$ has the *Lazarsfeld-Rao (LR)-Property* if the following conditions hold.

(a) If $V_1, V_2 \in \mathcal{L}^0$ then there is a deformation from one to the other through subschemes all in $\mathcal{L}^0$.

(b) Given $V_0 \in \mathcal{L}^0$ and $V \in \mathcal{L}^h$ ($h \geq 1$), there exists a sequence of subschemes $V_0, V_1, \dots, V_t$ such that for all i, $1 \leq i \leq t$, V_i is a basic double link of V_{i-1} and V is a deformation of V_t through subschemes all in $\mathcal{L}^h$.

Remark 2.12. This definition was motivated by the paper [58] of Lazarsfeld and Rao, who proved that this structure holds for a "general" curve in $\mathbb{P}^3$. Their motivation was not so much to prove a general structure theorem as it was to prove a conjecture of Harris that a "general" curve is the smallest in its (even) liaison class.

The first broad case not covered by [58] where this structure was proven was for arithmetically Buchsbaum curves in $\mathbb{P}^3$ ([9]; see also Remark 2.14). Much more generally, this structure (and more) was shown to hold for even liaison classes of unmixed curves in $\mathbb{P}^3$ by Martin-Deschamps and Perrin [61], and at about the same time for locally Cohen-Macaulay, equidimensional codimension two subschemes of $\mathbb{P}^n$ by Ballico, Bolondi and Migliore [5]. We quote the latter result since we will refer to it.

Theorem 2.13 ([5] Theorem 2.4)**.** *Every even liaison class of codimension two, locally Cohen-Macaulay, equidimensional subschemes of $\mathbb{P}^n$ has the Lazarsfeld-Rao property.*

Earlier, a very special case (but the first for dimension ≥ 2) was proven in [10]. It was generalized to unmixed codimension two subschemes by Nollet [79] and (separately) by Nagel [76].

It is known ([47], [59]) that a G-liaison class in codimension ≥ 3 does not have such a structure. It is an open question whether it holds for CI-liaison in codimension ≥ 3.

Remark 2.14. As already noted, for a curve C, $M(C)$ is a graded module over the polynomial ring, i.e. multiplication by a linear form L induces a homomorphism from any component $M(C)_t$ to the next. However, it sometimes happens that this multiplication is trivial for all L and all t. In this case, C is said to be a(n) *(arithmetically) Buchsbaum* curve.

Ignoring the shift, any finite sequence $(d_1, \dots, d_s)$ of non-negative integers (say $d_1, d_s \neq 0$) is the sequence of dimensions (up to shift) of the components of many possible graded modules of finite length, of which one, say M, is the one with trivial multiplication by linear forms. Ballico and Bolondi [4] have studied how these structures fit together, although we will not go into this here. By Rao's theorem (Theorem 2.5), there is a curve C so that $M(C)$ is some shift of M. By Basic Double Linkage, all rightward shifts of this module also exist for some curves in $\mathbb{P}^3$. Of course if $(d_1, \dots, d_s) \neq (e_1, \dots, e_t)$ then the modules, and hence the corresponding even liaison classes, are distinct. Thus each of these tuples represents a unique *Buchsbaum even liaison class*. A good deal of work has gone into studying Buchsbaum even liaison classes and Buchsbaum curves, some of which will be described here in passing. We also refer to work of Amasaki (e.g. [2]) on this subject.

Buchsbaum curves have provided a setting in which progress on several interesting questions was made, and liaison tools have played an important role. They arise in several ways in this paper.

Another liaison tool that has been very useful in the literature, in the construction of arithmetically Gorenstein subschemes of projective (and graded Artinian Gorenstein algebras) with desired properties, is often referred to as *sums of linked ideals*. We briefly describe the background. Although this method has been in existence for a long time, our exposition here is mostly from [33] and [68], and we will describe these applications in section 3.

It is well known that the sum of the ideals of two geometrically linked, arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^n$ is arithmetically Gorenstein of height one greater, whether they are CI-linked [83] or G-linked (cf. [63]). Harima ([39], Lemma 3.1) has computed the Hilbert function of the Gorenstein ideals so obtained in the case of CI-linkage under a special numerical assumption. Here we would like to record this result in a more general way, more in line with our needs.

Lemma 2.15. *Let $V_1 \stackrel{X}{\sim} V_2$, where X is arithmetically Gorenstein, V_1 and V_2 are arithmetically Cohen-Macaulay of codimension c with saturated ideals I_{V_1} and I_{V_2}, and the link is geometric (meaning that V_1 and V_2 have no common components). Then $I_{V_1} + I_{V_2}$ is the saturated ideal of an arithmetically Gorenstein scheme Y of codimension $c+1$. The Hilbert functions are related by Theorem 2.2. Then the sequence $d_i = (a_i + a'_i - c_i)$ is the* first difference *of the h-vector of Y.*

Example 2.16. A twisted cubic curve V_1 in $\mathbb{P}^3$ is linked to a line V_2 by the complete intersection of two quadrics. The intersection of these curves is the arithmetically Gorenstein zeroscheme Y consisting of two points. This is reflected in the following diagram of h-vectors:

$$\begin{array}{rccc} X: & 1 & 2 & 1 \\ V_1: & 1 & 2 & \\ V_2: & 1 & & \\ \Delta Y: & 1 & 0 & -1 \end{array}$$

adding the second and third rows and subtracting the first to obtain the fourth, and so the h-vector of Y is (1,1), obtained by "integrating" the vector $(1, 0-1)$. The notation ΔY serves as a reminder that the row is really the first difference of the h-vector of Y.

It is well known that the h-vector of an arithmetically Gorenstein subscheme of projective space is symmetric, and a lot of work has been done to try to describe the symmetric sequences $(1, h_1, \dots, h_e = 1)$ that arise as the h-vector of an arithmetically Gorenstein scheme (which from now on we will call *Gorenstein sequences*). Note that the h-vector of an arithmetically Cohen-Macaulay (e.g. arithmetically Gorenstein) scheme is the Hilbert function of its Artinian reduction. Some of this work involves restricting to certain classes of arithmetically Gorenstein schemes, for instance reduced ones. It is also interesting to understand how "non-unimodal" a Gorenstein sequence can be, and many papers have explored this.

An important special case of a Gorenstein sequence is a so-called *SI-sequence*:

Definition 2.17. A symmetric sequence of integers $\underline{h} = (1, h_1, \dots, h_1, 1)$ is an *SI-sequence* if

(i) $\underline{h}$ is an O-sequence (i.e. it satisfies Macaulay's growth condition, and hence is the Hilbert function of some Artinian algebra);
(ii) the positive part of the first difference, $(1, h_1 - 1, h_2 - h_2, \dots)$ is again an O-sequence.
In this case we say that $\underline{h}$ is a *differentiable O-sequence*.

Definition 2.18. An Artinian graded algebra R/I is said to have the *strong Lefschetz property (SLP)* if there exists a linear form ℓ such that for all i and all d the homomorphism $\times \ell^d : [R/I]_i \to [R/I]_{i+d}$ has maximal rank. It has the *weak Lefschetz property (WLP)* if the above holds just for the case $d = 1$.

Remark 2.19. Notice that SI-sequences are always unimodal. They are important for (at least) two reasons:

- When $h_1 = 3$, they are *exactly* the set of Hilbert functions of Artinian Gorenstein algebras [89], [94].

- In any codimension they are exactly the set of Hilbert functions of Artinian Gorenstein algebras with the *Weak Lefschetz Property* [39].

Interestingly, in codimension 3 it is not known if all Artinian Gorenstein algebras have the Weak Lefschetz Property, in spite of the two bullet points above.

3. Stick figures, Zeuthen's problem and configurations of linear subvarieties

The question of when an irreducible flat family of subschemes of projective space contains an element that is a union of linear varieties is a very classical one. In this case we say that any element of the family *specializes* to the union of linear varieties. Throughout this section we assume that our union of linear varieties is equidimensional.

The most desirable kind of union of linear varieties is one for which the singularities are as "nice" as possible. We will see that different terminology has been used in different situations. For curves, these are universally called *stick figures*," i.e. configurations of lines where at most two lines meet in a point. In higher dimension, we also want the components to meet "nicely," depending on the situation.

- In [11] the authors defined a *good linear configuration* of codimension two in $\mathbb{P}^n$ to be a union of codimension two linear varieties such that the intersection of any three has dimension at most $n-4$. This was mentioned also in [33], Remark 2.5.
- In [33] the authors defined a *good linear configuration* of codimension three in $\mathbb{P}^n$ "in the obvious way" without specifying it, but it is clear that it was meant that the intersection of any three of the codimension three linear components has dimension at most $n-5$.
- In [68] the authors defined a *generalized stick figure* to be a union of linear varieties of any dimension d such that the intersection of any three components has dimension at most $d-2$ (where the empty set is taken to have dimension -1).

The latter clearly includes all the previous special cases, so in this section from now on we will use the term "generalized stick figure," except for the case of curves where we simply use "stick figure."

3.1. **Stick figure curves in $\mathbb{P}^3$.** Returning to families, the most celebrated such problem is the *Zeuthen problem*. Indeed, quoting [44], "at the suggestion of H.G. Zeuthen, the Royal Danish Academy of Arts and Sciences proposed in 1901 a prize problem with a gold medal [80] p. 29:

> *To determine if every family of space curves – after the customary division – contains limit curves which are composed of lines. In the case of a negative answer there should also be an investigation of conditions for the existence of such limit curves, and possible restrictions on the results which have been found using such limit curves.*"

The first result in this direction was given by Gaeta [26], who showed that any arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$ specializes to a stick figure.

The Zeuthen problem was solved in full by Hartshorne [44] (his solution appeared in 1997), where he gives a careful discussion of the history of the problem, partial results, and the significance of a possible positive answer. He points out that "(F)rom the earliest work on this problem, it has been understood that a "curve composed of lines" should be taken to mean a stick figure, ...and that "limits" should preserve the arithmetic genus, so that we are dealing with flat families."

In this subsection we mention a partial result on space curves predating Hartshorne's complete solution and generalizing Gaeta's work. In the next subsection we will give some results on arithmetically Gorenstein subschemes of higher codimension. These results used different ideas from liaison.

In the spirit of Zeuthen's problem, we first give a result for space curves. We refer to Remark 2.14 for the definition of Buchsbaum curves and Buchsbaum even liaison classes.

Theorem 3.1 ([11] Proposition 3.4)**.** *Every Buchsbaum curve in $\mathbb{P}^3$ specializes to a stick figure.*

The proof is based on the following simple idea. Let C be a stick figure in $\mathbb{P}^3$ and assume that C lies on a surface S consisting of a union of planes, no three containing the same line. Assume further that no component of C lies in the singular locus of S. Let H be a general plane. Then the union of C and $S \cap H$ is again a stick figure. Notice that $C \cup (S \cap H)$ is a basic double link of C.

Let $\mathcal{L}$ be an even liaison class of Buchsbaum curves (i.e. an even liaison class all of whose elements are Buchsbaum with the same Hartshorne-Rao module). If we can show that $\mathcal{L}$ contains a minimal element $C_0 \in \mathcal{L}^0$ that is a stick figure, then the above idea, combined with the Lazarsfeld-Rao property (Theorem 2.13), shows that every $C \in \mathcal{L}$ (and hence every Buchsbaum curve in $\mathbb{P}^3$, since $\mathcal{L}$ is an arbitrary Buchsbaum liaison class in $\mathbb{P}^3$) specializes to a stick figure. (Actually, it is a little more subtle than this. The construction also needs the observation that the sequence of basic double links described in part (b) of Definition 2.11 can be chosen to be strictly increasing, in a sense that we omit here. See [63] Example 6.4.12 and [10] Corollary 5.3.)

So it remains to show that $\mathcal{L}^0$ contains a stick figure. This is done in [11] using Liaison Addition (Theorem 2.6) and induction (see in particular [11] Lemma 3.3). One first notes that a set of two skew lines is a stick figure, and its Hartshorne-Rao module is one-dimensional (occurring in degree 0). One then builds up any finite length module with trivial R-multiplication using Liaison Addition, using surfaces of suitably chosen degree (but as efficiently as possible). The fact that if the choices are as efficient as possible then the stick figure so constructed is minimal in its even liaison class is a consequence of the description of Buchsbaum curves in [8] and [31].

Example 3.2. Let us construct a Buchsbaum curve C that is a stick figure, and such that $M(C)$ is a module whose dimensions are $(1, 0, 2)$ (meaning 1-dimensional in some degree, 0 in the next degree, and 2-dimensional in the next). Let C_1 and C_2 be two sets of two skew lines, chosen generally. Let $F_i \in I_{C_i}$ $(i = 1, 2)$ be surfaces of degree 2, each a union of planes. Then by Theorem 2.6, $F_2 \cdot I_{C_1} + F_1 \cdot I_{C_2}$ is the saturated ideal of a stick figure Y of degree 8, with $\dim M(Y) = 2$, and the only non-zero component is 2-dimensional occurring in degree 2. (The fact that it is a stick figure of degree 8 is from the geometric interpretation of Liaison Addition and is left to the reader.)

Now let C_3 again be a sufficiently general choice of two skew lines, and let F_3 be a union of *four* planes containing C_3, chosen so that one plane contains one component of C_3, one contains the other component, and the other two are chosen generally. Then $F_1 F_2 \cdot I_{C_3} + F_3 \cdot I_Y$ defines a stick figure, C. Its module $M(C)$ is the direct sum of a shift of $M(Y)$ and a shift of $M(C_3)$. What are these shifts? $M(Y)$ is 2-dimensional in degree 2, and it gets shifted to the right by $\deg(F_3) = 4$ to degree 6. $M(C_3)$ is 1-dimensional in degree 0 and gets shifted by $\deg(F_1) + \deg(F_2) = 4$ to degree 4. Thus $M(C)$ has the desired dimensions, with the 1-dimensional component coming in degree 4. The fact that this is the minimal shift among modules in the even liaison class of C follows from [31] Corollary 3.10. See also [8] for relevant facts about the even liaison class of a Buchsbaum curve.

3.2. **Arithmetically Gorenstein generalized stick figures of codimension three.** It is known from work of Stanley [89] and Buchsbaum and Eisenbud [15] exactly what Hilbert functions can occur for codimension three arithmetically Gorenstein subschemes, and in fact from their work we also know what sets of graded Betti numbers can occur. (From now on we will refer to *Betti diagrams* as a way of collecting the graded Betti numbers for a given graded module, in what is now a standard way. For Gorenstein algebras we will refer to *Gorenstein Betti diagrams.*) Concerning the Hilbert functions, the corresponding Gorenstein sequences are the SI-sequences (Remark 2.19). Diesel [22] described an algorithm to find all possible Betti diagrams given the SI-sequence. She also showed that the Gorenstein algebras for such a Hilbert function form an irreducible family.

Although the possible Betti diagrams for a given Hilbert function were known, it was not known "how nice" the arithmetically Gorenstein subschemes are for any such Betti diagram. In particular, does each such irreducible family contain a reduced set of points in the case of arithmetically Gorenstein zero-dimensional schemes in $\mathbb{P}^3$, a stick figure in the case of curves in $\mathbb{P}^4$, or a generalized stick figure in the case of codimension three subschemes in $\mathbb{P}^n$? This was shown in the affirmative in [33], and in fact not only for each family but indeed for each possible Betti diagram.

Theorem 3.3 ([33] Theorem 2.1, Corollary 2.4, Remark 2.5)**.** *For any Gorenstein Betti diagram for codimension three subschemes of $\mathbb{P}^n$, there is a arithmetically Gorenstein generalized stick figure having that Betti diagram.*

The idea of the proof is as follows. We begin with a possible Gorenstein Betti diagram. From this diagram one finds the Betti diagram of a suitable arithmetically Cohen-Macaulay codimension two subscheme V_1, a suitable complete intersection X containing it, and the Betti diagram of the residual scheme V_2 (using the mapping cone construction mentioned above), so that the sum of the linked ideals is arithmetically Gorenstein and has the desired Gorenstein Betti diagram. This is done using a certain mapping cone, building off of the resolutions for the linked curves. (So far this is all numerical.) Then we use Gaeta's result mentioned above (generalized to $\mathbb{P}^n$) to arrange that V_1, X and V_2 are all generalized stick figures. This gives that $I_{V_1} + I_{V_2}$ defines an arithmetically Gorenstein generalized stick figure of codimension three with the desired Betti diagram.

With this result for codimension three Gorenstein subschemes, it is natural to wonder what we can say about higher codimension.

3.3. **Arithmetically Gorenstein (generalized) stick figures of any codimension.** Recall from Remark 2.19 that in codimension three, the h-vector of a (codimension three) arithmetically Gorenstein subscheme is always an SI-sequence. We also noted that in higher codimension, the SI-sequences are exactly the Hilbert functions of Artinian Gorenstein algebras with the Weak Lefschetz Property. In this setting, though, the Artinian Gorenstein algebras with the same Hilbert function do not in general form an irreducible family.

Remark 3.4. It has also been asked (and we conjecture) whether a general Artinian reduction of a reduced, arithmetically Gorenstein subscheme of projective space necessarily has the Weak Lefschetz Property in characteristic zero ([69] Question 3.8). Of course since Artinian Gorenstein algebras exist with non-unimodal Hilbert function, the extension of such an ideal in a larger polynomial ring defines a non-reduced arithmetically Gorenstein subscheme whose general Artinian reducton does not have the Weak Lefschetz Property. Also, Mats Boij has shown us an example of a reduced, arithmetically Gorenstein set of points in projective space such that a *special* Artinian reduction fails to have the Weak Lefschetz property. Thus the assumptions *general* and *reduced* are important in this question. In any case, the problem of classifying all possible Hilbert functions of Artinian Gorenstein algebras is probably intractable, and the problem of classifying the Hilbert functions of reduced, arithmetically Gorenstein subschemes of projective space is still open. Thus the fact that we at least do know precisely the Hilbert functions of Artinian Gorenstein algebras with the Weak Lefschetz Property is a very welcome result.

In view of Remark 3.4, it is natural to wonder whether every SI-sequence also occurs as the h-vector of a reduced, arithmetically Gorenstein subscheme, and then it is worth asking if it also occurs for a generalized stick figure. In this subsection we are also interested in the question of whether there is a set of maximal graded Betti numbers among arithmetically Gorenstein subschemes with the given Hilbert function which have general Artinian reduction with the Weak Lefschetz Property.

The story starts with the paper [29] of A.V. Geramita, T. Harima and Y.S. Shin. In that paper they used CI-liaison to construct certain Artinian Gorenstein algebras with the Weak Lefschetz Property (and as such, whose Hilbert function is an SI-sequence). They were not interested in generalized stick figures or reduced arithmetically Gorenstein algebras, and most importantly they did not produce an example for every possible SI-sequence. They did, however, prove the extremality of the graded Betti numbers for the class of algebras that they constructed. The paper [68] goes beyond these results (and in fact Remark 10.2 of [68] shows that CI-links are not enough to produce all SI-sequences).

In fact, [68] was innovative in the application of liaison in two ways. First, it was one of the first papers that applied G-liaison (rather than CI-liaison) to construct interesting objects. Second, and more surprisingly, the approach was in some sense the reverse of the usual one: instead of starting with a scheme and producing a Gorenstein scheme containing it to produce a desired link, the approach was to first produce a totally reducible (i.e. union of linear varieties) arithmetically Gorenstein scheme and find *within it* a suitable arithmetically Cohen-Macaulay subscheme, and then perform the desired G-link. The scheme with the desired Hilbert function is then obtained as a sum of G-linked ideals.

The first main result of [68] is the following. We will return to this paper when we discuss simplicial polytopes.

Theorem 3.5 ([68] Theorem 1.1)**.** *Let $\underline{h} = (1, c, h_2, \dots, h_{s-2}, c, 1)$ be an SI-sequence and let K be an arbitrary field containing sufficiently many elements. Then for every integer $d \geq 0$ there is a reduced arithmetically Gorenstein union of linear varieties, $G \subset \mathbb{P}^{c+d}$, of dimension d, whose general Artinian reduction has the Weak Lefschetz Property, and whose h-vector is $\underline{h}$.*

Remark 3.6. 1. Of course the assumption that K has sufficiently many elements depends on the choice of $\underline{h}$.

2. The theorem does not, unfortunately, guarantee that the arithmetically Gorenstein scheme produced is a generalized stick figure, only that it is reduced. However, the "large" Gorenstein scheme referred to before the statement of the theorem *is* a generalized stick figure, and this is what guarantees that when it is used to produced a sum of linked ideals, the result will again be reduced. The authors conjecture that the schemes are, in fact, again generalized stick figures ([68] Remark 6.5).

As mentioned, the arithmetically Gorenstein union of linear varieties produced in Theorem 3.5 also has an extremality property. We remark that the schemes produced in [29] also have such a property.

Theorem 3.7. *Fix an SI-sequence $\underline{h}$. The scheme produced in Theorem 3.5 with h-vector $\underline{h}$ has maximal graded Betti numbers among arithmetically Gorenstein subschemes of $\mathbb{P}^n$ whose general Artinian reductions have the Weak Lefschetz Property and Hilbert function $\underline{h}$.*

In a way analogous to the approach of [29], the idea is to arrange that the linked varieties are not only arithmetically Cohen-Macaulay, but in fact have extremal Betti numbers for their (prescribed) Hilbert functions.

4. The singular locus of a hyperplane arrangement

In this section we will assume that k has characteristic zero.

If $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{P}^n$, it is defined by a product, F, of linear forms such that none is a scalar multiple of another. Let

$$J = \langle F_{x_0}, F_{x_1}, \dots, F_{x_n} \rangle$$

be the Jacobian ideal, generated by the partial derivatives of F. Note that J is not necessarily saturated, and that the saturation J^{sat} is not necessarily unmixed. It does, however, have height two. Consider a primary decomposition of J,

$$J = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s \cap \dots$$

where $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ are all the primary components of height 2. For each such primary ideal, let $\mathfrak{p}_i$ be the associated prime. Then set

$$J^{top} = \bigcap_{1 \le i \le s} q_i \quad \text{and} \quad \sqrt{J} = \bigcap_{1 \le i \le s} p_i.$$

Both J^{top} and $\sqrt{J}$ are unmixed ideals of height 2. We denote by X^{top} the scheme defined by J^{top} and by X^{red} the scheme defined by $\sqrt{J}$.

Remark 4.1. Although the saturation of J can still have embedded components, it is interesting to study the unmixed singular locus of $\mathcal{A}$, and this could refer to either X^{top} (which is not necessarily reduced) or to X^{red}. The results described here are a contribution to this.

As a first step to studying the schemes X^{top} and X^{red}, one asks if they are necessarily ACM. If they are not, are there conditions that guarantee that they are ACM? And in terms of the invariants $M^i(C)$ (which we saw in section 2 is a measure of the failure to be ACM), how far can these schemes be from being ACM? In this section we address all of these questions.

We first give a version of Liaison Addition (Theorem 2.6) in the language of arrangements, that we will use in the rest of this section.

Theorem 4.2 (Arrangement version of Liaison Addition in $\mathbb{P}^3$)**.** *Let $\mathcal{A}_1 = \bigcup_{i=1}^{s_1} H_i$ and $\mathcal{A}_2 = \bigcup_{i=1}^{s_2} H'_i$ be plane arrangements in $\mathbb{P}^n$ with corresponding schemes X_i^{top}, X_i^{red} $(i = 1, 2)$.*

$(*)$ Assume that no plane of $\mathcal{A}_1$ contains a component of X_2^{red} and vice versa.

Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, with schemes X^{top} and X^{red}. Then for each $1 \le i \le n-2$,

$$M^i(X^{top}) \cong M^i(X_1^{top})(-s_2) \oplus M^i(X_2^{top})(-s_1).$$

Similarly,

$$M^i(X^{red}) \cong M^i(X_1^{red})(-s_2) \oplus M^i(X_2^{red})(-s_1).$$

In particular, if X_1^{top} and X_2^{top} (resp. X_1^{red} and X_2^{red}) are ACM then also X^{top} (resp. X^{red}) is ACM.

We also give a version of Basic Double Linkage (Theorem 2.7) in the language of arrangements.

Theorem 4.3 (Arrangement version of Basic Double Linkage in $\mathbb{P}^n$)**.** *Let $\mathcal{A}$ be an arbitrary hyperplane arrangement in $\mathbb{P}^n$ with corresponding schemes X^{top} and X^{red}. Let H be a plane* not containing any component of *X^{red}. Let $\mathcal{A}' = \mathcal{A} \cup H$, with corresponding schemes Y^{top} and Y^{red}. Then X^{top} and Y^{top} are linked in two steps, as are X^{red} and Y^{red}. In particular:*

(i) *We have isomorphisms*

$$M^i(Y^{top}) \cong M^i(X^{top})(-1) \quad \text{and} \quad M^i(Y^{red}) \cong M^i(X^{red})(-1)$$

for $1 \le i \le \dim X = \dim Y$.

(ii) *X^{top} (resp. X^{red}) is ACM if and only if Y^{top} (resp. Y^{red}) is ACM.*

A very special case of Theorem 4.3 is the following.

Corollary 4.4 ([27])**.** *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^n$ and assume that no three hyperplanes contain the same codimension two linear subvariety. Then $X^{top} = X^{red}$ is ACM.*

Proof. One simply notes that the intersection of two planes is ACM, and then applies Theorem 4.3 successively for the remaining planes. □

Remark 4.5. The schemes described in Corollary 4.4 are called *codimension two star configurations*. Theorem 2.8 can be used to extend this result. Assume that the intersection of any j of the hyperplanes of $\mathcal{A}$ is either empty or of codimension j. Then for any $1 \le c \le \min(s, n)$ let $V_c(\mathcal{A})$ be the union of the codimension c linear varieties defined by the intersections of these hyperplanes, taken c at a time. In [27] these were called *codimension c star configurations*. Then it was shown in [27] (among other results) that $V_c(\mathcal{A})$ is also ACM. The machinery of Theorem 2.8 can also be used to give Hilbert functions and Betti numbers of $V_c(\mathcal{A})$, which we omit here.

Our first main result is obtained by combining Liaison Addition (Theorem 4.2) and Basic Double Linkage (Theorem 4.3). It says that under a condition on the hyperplanes, X^{top} and X^{red} are both ACM.

Theorem 4.6 ([71])**.** *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^n$. Assume that*

$(*)$ $\left\{ \begin{array}{l} \text{no linear factor of } F \text{ is in the associated prime of any two } \underline{\text{non-reduced}} \text{ components} \\ \text{of } J^{top}. \end{array} \right.$

Then both R/J^{top} and $R/\sqrt{J}$ are Cohen-Macaulay (i.e. X^{top} and X^{red} are ACM).

If $(*)$ *fails then both* X^{top} *and* X^{red} may *fail to be ACM.*

Remark 4.7. In 1983, Hiroaki Terao [92] conjectured that the freeness of a hyperplane arrangement is a combinatorial property, i.e. whether it is determined from its intersection lattice. The above result does not address freeness. Nevertheless, notice that $(*)$ is a combinatorial property of the intersection lattice of $\mathcal{A}$. Thus one can also ask whether the property that X^{top} (resp. X^{red}) is Cohen-Macaulay is a combinatorial property of the intersection lattice of $\mathcal{A}$. Theorem 4.6 says that the answer is yes provided that condition $(*)$ holds, so the issue is whether it is true without condition $(*)$.

We now focus on arrangements in $\mathbb{P}^3$. As indicated in Theorem 4.6, if condition $(*)$ fails then it is possible for either X^{top} or X^{red} or both to fail to be ACM. The first example, where X^{top} fails, was given in [75] Example 4.5 for hyperplane arrangements in $\mathbb{P}^3$, and in their example X^{red} is ACM. Experimenting with subarrangements of their example, we were able to find examples for X^{red} to fail to be ACM while X^{top} is ACM, and examples where both fail to be ACM. In all cases, the dimension of the Hartshorne-Rao module was 1. As such, these curves are automatically Buchsbaum.

This leads to the question of by how much the Cohen-Macaulay property can fail to hold, and one answer is provided by another application of Liaison Addition:

Theorem 4.8. *Let* $r \geq 1$ *be a positive integer. Then:*

(i) *There exists a positive integer* N *and an arrangement* $\mathcal{A}$ *in* $\mathbb{P}^3$*, such that*

$$\dim M(X^{top})_i = \begin{cases} r & \text{if } i = N; \\ 0 & \text{if } i \neq N \end{cases}$$

(ii) *The same result holds for* X^{red}*, although the value of* N *is not necessarily the same.*
(iii) *For each* $h \geq 1$ *we can replace* N *by* $N + h$ *and obtain the same result.*
(iv) *The curves obtained above are all in the same even liaison class.*

The proof involves making sufficiently general "copies" of the curves described before the statement of the theorem, and applying Liaison Addition $r - 1$ times, keeping careful track of the degrees. Then part (iii) is obtained by basic double linkage using general linear forms (i.e. adding general hyperplanes to $\mathcal{A}$). The conclusion (iv) is a direct application of Rao's theorem (Theorem 2.5). Notice that the curves described here are again automatically Buchsbaum, having only one non-zero component in the Hartshorne-Rao module.

Remark 4.9. It is natural to ask which other even liaison classes arise among the schemes X^{top} or X^{red} for arrangements in $\mathbb{P}^3$. We have found several examples to show that the curves in Theorem 4.8 are not the only non-ACM examples. However, a classification remains out of reach. We would be very interested to know, for example, whether any arithmetically Buchsbaum curve whose Hartshorne-Rao module is non-zero in more than one degree can arise in this way.

5. The Eisenbud-Green-Harris conjecture and Cayley-Bacharach

It is hard to imagine a more appropriate topic, in a paper on applications of liaison theory, than a description of the paper "An application of liaison theory to the Eisenbud-Green-Harris conjecture" [17] by Ernest Chong.

The classical Cayley-Bacharach theorem (see [23] Exercise 21.24) says the following. Let Z be a reduced complete intersection of two plane cubics in $\mathbb{P}^2$. Let $P \in Z$ be any point and let $Y = Z\backslash P$. If F is any cubic vanishing at the eight points of Y, then F must also vanish at P. Another way to say this is that the Hilbert function of Z and the Hilbert function of Y agree in degrees ≤ 3. More precisely, the Hilbert function of Z is $(1, 3, 6, 8, 9, 9, ...)$ and the Hilbert function of *any* eight points of Z must be $(1, 3, 6, 8, 8, \dots)$.

Remark 5.1. It is beyond the scope of this paper to discuss all the different directions beyond this result that have been studied, but we make a few comments.

The classical Cayley-Bacharach theorem has led to the notion of the *Cayley-Bacharach property* for a set of points, defined as follows. It is not hard to show that given *any* reduced subset of degree d in any projective space $\mathbb{P}^n$ (not only $\mathbb{P}^2$) with Hilbert function

$$(1, n+1, h_2, \dots, h_{t-1}, h_t = d, d, \dots),$$

with $h_{t-1} < d$, there must be at least one subset of $d-1$ points with the truncated Hilbert function

$$(1, n+1, h_2, \dots, h_{t-1}, h_t - 1 = d-1, d-1, \dots).$$

(See for instance [34].) The set Z has the Cayley-Bacharach property if this is true for *every* choice of $d-1$ points. A standard fact, which follows easily from liaison (specifically from Theorem 2.2) is that any arithmetically Gorenstein set of points in any projective space has the Cayley-Bacharach property. One generalization has been the notion of the *uniform position property*, which has been important in the study of the genus of space curves (see for instance [41]). A set of points Z in $\mathbb{P}^n$ has the uniform position property if, for any fixed cardinality p, all subsets of p points have the same Hilbert function (which must be the truncation at level p of the Hilbert function of Z).

The Cayley-Bacharach property has been studied in many papers, for example [16] and [55] (both of which use liaison as a tool).

The fact that the Cayley-Bacharach property is closely related to liaison has been studied for many years (see for instance [21]), but in 1996 David Eisenbud, Mark Green and Joe Harris wrote the beautiful paper [24], starting with historical versions of this result and developing the theory until they arrived at several versions of what is now called the Eisenbud-Green-Harris (EGH) conjecture. We refer the reader to [24] for all of the beautiful intricacies and interrelations between the different ideas, and here we will focus on the version addressed by Chong.

Let $S = k[x_1, \dots, x_n]$, where k is an infinite field. The version of the conjecture quoted by Chong is the following.

Conjecture 5.2 (Eisenbud-Green-Harris Conjecture [24])**.** *Let $2 \leq e_1 \leq \cdots \leq e_n$ be integers. If $I \subsetneq S$ is a homogeneous ideal that minimally contains an $(e_1, \dots, e_n)$-regular sequence of forms, then there exists a homogeneous ideal $J \subsetneq S$ containing $x_1^{e_1}, \dots, x_n^{e_n}$, such that I and J have the same Hilbert function.*

Some special cases of this conjecture have been proven (we refer to [17] for a partial list). Chong's idea is to prove it for a new special case using a result of the current authors in [70]. His main theorem proves it for height three Gorenstein ideals:

Theorem 5.3 ([17] Theorem 1)**.** *Let $2 \leq e_1 \leq e_2 \leq e_3$ be integers. If $I \subsetneq k[x_1, x_2, x_3]$ is a homogeneous Gorenstein ideal that minimally contains an (e_1, e_2, e_3)-regular sequence of forms, then there exists a monomial ideal J in $k[x_1, x_2, x_3]$ containing $x_1^{e_1}, x_2^{e_2}, x_3^{e_3}$, such that I and J have the same Hilbert function.*

As noted, Chong's idea was to use liaison to prove this result. First, it follows immediately from Theorem 2.2 that if I_1 and I_2 are two Cohen-Macaulay ideals (of any codimension) with the same Hilbert function, and if $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are complete intersections in I_1 and I_2, respectively, of the same type, then the linked ideals $\mathfrak{c}_1 : I_1$ and $\mathfrak{c}_2 : I_2$ have the same Hilbert function.

The key ingredient of Chong's proof is the notion of minimal linkage, which we now describe. Given an ideal I, there is certainly an initial degree d_1 in which I is non-zero (i.e. the initial degree of I). Then there is a smallest degree $d_2 \geq d_1$ such that I contains a regular sequence of type (d_1, d_2). Continuing in this way, there is a smallest c-tuple (where c is the codimension of I) $(d_1, d_2, \ldots, d_c)$ for which I contains a regular sequence of those degrees. Certainly such a regular sequence can be used to perform a link of $I := I_1$, obtaining a residual ideal I_2.

Next, one can apply the same construction to the residual I_2. The resulting tuple is lexicographically smaller than or equal to the original one. One sequentially applies this construction (in [62] it was called the *minimal link procedure*) until one of two things happens. Either the smallest tuple is no smaller than the one just before, or else one arrives at a complete intersection ideal.

In the former case, if R/I is not Cohen-Macaulay one can hope that the final ideal is minimal in its even liaison class, in the sense of Definition 2.11. It was shown in [70] that among curves in $\mathbb{P}^3$, in some even liaison classes this is true, and in others it is not true. In the latter case, i is said to be *licci* (i.e. in the linkage class of a complete intersection) in particular, and hence R/I is Cohen-Macaulay. We now focus on the Cohen-Macaulay situation.

In codimension two, Gaeta proved that this procedure always leads to a complete intersection. In codimension 3, an example was given in [28] of a licci ideal that could not be minimally linked to a complete intersection, but no proof was given. This was remedied in [49], where it was even shown that for the Hilbert function $(1, 3, 6, 8, 7, 6, 2)$ there exist three ideals I_1, I_2, I_3 such that all three quotients have this Hilbert function, but one is not licci, one is licci but cannot be linked to a complete intersection by minimal links, and one is licci and can be linked to a complete intersection by minimal links. The latter two even have the same graded Betti numbers.

This all goes to show that it is interesting to study what properties of an ideal give that it can be minimally linked to a complete intersection. (The example just mentioned shows that the Hilbert function and the graded Betti numbers are not enough, in general.) One such property is that of being Gorenstein of codimension three.

J. Watanabe showed in [93] that any such ideal is licci, but he did not consider minimal links. Watanabe's result is extended to minimal links:

Theorem 5.4 ([70] Theorem 6.3)**.** *If $I \subset k[x_1, x_2, \ldots, x_n]$ $(n \geq 3)$ is a homogeneous Gorenstein ideal then I can be minimally linked to a complete intersection.*

We now return to Chong's nice idea to use this result to prove Theorem 5.3. He first weakens the notion of minimal links down to a complete intersection.

Definition 5.5. Let I be a licci ideal of height r. Suppose there exists a sequence of CI-links

$$I = I_0 \overset{J_1}{\sim} I_1 \overset{J_2}{\sim} \cdots \overset{J_s}{\sim} I_s$$

where I_s is a complete intersection. Say the type of J_i is $\mathbf{a}^{(i)} \in \mathbb{Z}_+^r$, and assume

$$\mathbf{a}^{(1)} \geq \cdots \geq \mathbf{a}^{(s)}$$

in the lexicographic order. Then I is said to be a *sequentially bounded* licci ideal. If, furthermore, J_1 is a minimal link, we say that I is a sequentially bounded licci ideal *that admits a minimal first link*.

Chong then proves the following important theorem.

Theorem 5.6. *Let $2 \leq e_1 \leq \cdots \leq e_n$ be integers. If $I \subsetneq S = k[x_1, \dots, x_n]$ is a sequentially bounded licci ideal that admits a minimal first link and minimally contains an $(e_1, \dots, e_n)$-regular sequence of forms, then there exists a monomial ideal $J \subsetneq S$ containing $x_1^{e_1}, \dots, x_n^{e_n}$ such that I and J have the same Hilbert function.*

The proof is a bit technical, but essentially the existence of the specified sequence of links starting with I down to a complete intersection allows one to construct a numerically equivalent sequence of links using monomial ideals.

Once this is established, Theorem 5.3 follows immediately from Theorem 5.4.

6. The genus of space curves

A very classical problem, going back well over a century, is to classify the smooth curves in $\mathbb{P}^3$ (also called *space curves*). In particular, one can ask which pairs (d, g) occur for a smooth space curves, and what role is played by the least degree of a surface containing the curve. Furthermore, what Hilbert functions can arise for such curves, or for their general hyperplane sections? This can also be extended from smooth curves to locally Cohen-Macaulay, equidimensional curves (see for example [43]), but we consider here only the smooth case.

As of about 1980, two outstanding references for much of what was known at the time were Hartshorne's book [42] (Chapter IV, section 6) and Harris's Montreal Notes [40]. And of course one can extend all this to curves in $\mathbb{P}^n$, where open questions remain. It is very far beyond the scope of this paper to describe this rich history, but two results in fact are connected (and use) liaison, and we will describe these. The first was written by J. Harris [41] and appeared in 1980, and the second was written by R. Maggioni and A. Ragusa and appeared in 1988.

The first complete answer to the question of which pairs (d, g) occur for smooth, non-degenerate space curves is due to Gruson and Peskine [36], and we omit the slightly technical statement. A weaker question is to ask for a *bound* on the genus of a smooth curve of degree d, and this was settled many years ago:

$$g(C) \leq \begin{cases} (\frac{d}{2} - 1)^2, & d \text{ even} \\ \\ \left(\frac{d-1}{2}\right)\left(\frac{d-3}{2}\right), & d \text{ odd} \end{cases}$$

with equality if and only if C lies on a quadric surface and is either a complete intersection (in the even case) or residual to a line (in the odd case) in a suitable complete intersection with a quadric. See [41] or [45]. The latter attributes this result to Castelnuovo (1893).

Among curves not lying on a quadric but lying on a cubic surface there is a new bound:

$$g(C) \leq \begin{cases} \frac{d^2}{6} - \frac{d}{2} + 1, & d \equiv 0 \pmod 3 \\ \\ \frac{d^2}{6} - \frac{d}{2} + \frac{1}{3}, & d \equiv 1, 2 \pmod 3 \end{cases}$$

and Harris notes that these three values represent (respectively) (i) the genus of the complete intersection of a cubic and a surface of degree $\frac{d}{3}$, (ii) the genus of the residual to a conic in a complete intersection of a cubic and a surface of degree $\frac{d+2}{3}$, or (iii) the genus of the residual to a line inside a complete intersection of a cubic and a surface of degree $\frac{d+1}{3}$ (see [41]).

Notice that in some sense what matters is not so much that the extremal curves in the latter case fail to lie on a quadric surface as that they *do* lie on an *irreducible* cubic surface. So, for instance, if $d = 10$ then such a curve may lie on a quadric surface, but then (by Bezout's theorem) it has no hope of lying on an irreducible cubic surface. If $d = 5$, on the other hand, then the argument is a little bit more subtle: Bezout's theorem does not rule out the possibility that C lie on both a quadric surface and an irreducible cubic surface, but the residual must be a line, which is arithmetically Cohen-Macaulay, so C also must be arithmetically Cohen-Macaulay and we must have that C has genus 2, which is extremal using either formula.

Harris's idea was to extend this to higher degree surfaces. He produces a formula (which we will not repeat here, but which is analogous to the two formulas above) which gives an upper bound for the genus of a smooth curve lying on an irreducible surface of degree k. His argument breaks into two cases, namely $d > k(k-1)$ and $d \leq k(k-1)$, as one might guess from the preceding paragraph. Furthermore, he shows that the extremal curves are always residual, in a suitable complete intersection, to a plane curve of degree $n = \lceil \frac{d-1}{k} \rceil + 1$.

Central to Harris's argument was an approach using hyperplane sections, and studying the Hilbert function (or more precisely the h-vector) of the corresponding points in $H = \mathbb{P}^2$. It was in this paper that he introduced the crucial notion of points being in *uniform position*, meaning that all subsets of m points (for any m) have the same Hilbert function. One also says that the points have the *Uniform Position Property*. Harris showed that the general hyperplane section of a reduced, irreducible curve $C \subset \mathbb{P}^3$ has this property. Furthermore, he shows that the h-vector of the general hyperplane section of C is of (what has come to be called) *decreasing type*. This means that the beginning of the h-vector agrees with the polynomial ring, i.e. `(1,2,3,4,...)`, then is possibly flat at the highest point, and after that is *strictly decreasing*. So for example `(1,2,3,4,5,5,5,3,2)` is of decreasing type, while `(1,2,3,4,5,5,5,3,2,2)` is not.

A more general bound for the genus (now for curves in $\mathbb{P}^r$) was given in [40], by Eisenbud and Harris. In this book the authors ask what may be the Hilbert function of the general hyperplane section of a reduced, irreducible curve in $\mathbb{P}^r$. Furthermore, what may be the Hilbert function of a set of points in $\mathbb{P}^{r-1}$ with the Uniform Position Property? And are the two answers the same?

This question was the launching point for the second liaison-related result that we describe here, namely the paper [60] of R. Maggioni and A. Ragusa. In this paper the authors show that when $r = 3$, the answers are indeed the same and the possible h-vectors are exactly those of decreasing type. Part of this of course was done by Harris, and the task remaining for the authors was to show that given an h-vector of decreasing type, there exists a smooth curve (in fact an arithmetically Cohen-Macaulay smooth curve) whose general (in fact arbitrary) hyperplane section has the given h-vector.

As mentioned, the proof uses liaison. One starts with an h-vector $\underline{h}$ of decreasing type. From $\underline{h}$ one can read the least degree, a_1, of a minimal generator of the ideal I_C of any arithmetically Cohen-Macaulay curve C with this h-vector. One can also read the degree, a_2, where such curve would have its second minimal generator. In general it is not necessarily true that I_C contains a regular sequence of type (a_1, a_2), but it is true if C is irreducible, and it is true for the general hyperplane section of C. One then formally produces the "residual" h-vector to $\underline{h}$, by a complete intersection of type (a_1, a_2), using Theorem 2.2. Call this sequence $\underline{h}'$.

Next, the authors construct a reduced, ACM union of lines, C', in $\mathbb{P}^3$ whose h-vector is $\underline{h}'$. They show that C' lies on a smooth surface, S, of degree a_1. They do this with a variation of Bertini's theorem. Finally, they look at the general residual to C' in a complete intersection of S and a surface of degree a_2; that is, they look at the linear system $|a_2H - C'|$ on S and show that the general element is smooth. This general element is the desired smooth arithmetically Cohen-Macaulay curve C.

Remark 6.1. In the above argument we have ignored the issue of why we need decreasing type. As the authors remark, in this case it can never happen that C (with h-vector $\underline{h}$) lies in a complete intersection of type (a_1, a_2).

7. Liaison and Graded Betti Numbers

Liaison theory has been used in a number of contexts in order to achieve information on minimal free resolutions. We will highlight a few instances.

It is useful to recall a module version of the exact sequence of sheaves (2.1). Let I, J be ideals of a polynomial ring $R = k[x_0, \dots, x_n]$ that are directly linked by a Gorenstein ideal $\mathfrak{c} \subset R$, that is, $\mathfrak{c} : I = J$ and $\mathfrak{c} : J = I$. Then there is a short exact sequence (see, e.g., [76, Lemma 3.5])

$$0 \to \mathfrak{c} \hookrightarrow I \to \omega_{R/J}(-t) \to 0, \tag{7.1}$$

where t is the integer such that $\omega_{R/\mathfrak{c}} \cong R/\mathfrak{c}(t)$ and $\omega_{R/J} \cong \operatorname{Ext}^c_R(M, R)(-t)$ is the canonical module of R/J with $c = n + 1 - \dim R/J$, the codimension of J. If R/J is Cohen-Macaulay then the Betti numbers of R/J and those of its canonical module determine each other. Explicitly, if

$$0 \to F_c \to \cdots \to F_1 \to R \to R/J \tag{7.2}$$

is a graded minimal free resolution of R/J over R, then dualizing with respect to R gives a graded minimal free resolution of $\operatorname{Ext}^c_R(M, R)$,

$$0 \to R \to F_1^* \to \cdots \to F_c^* \to \operatorname{Ext}^c_R(M, R) \to 0.$$

Consider an ideal I of R that is minimally generated by s homogeneous polynomials of degrees $d_1, \dots, d_s$. This is not enough information to determine the Hilbert function of R/I. However, if the generators of I are sufficiently general polynomials with the specified degrees, then the Hilbert function of R/I depends only on the integers $n, d_1, \dots, d_s$. In fact, Fröberg's conjecture (see [25]) predicts this Hilbert function. This conjecture is known in a number of cases, but open in general.

The graded Betti numbers of such ideals are much less understood. In fact, one can show that the graded Betti numbers of an ideal I generated by sufficiently general forms are determined by n and the generator degrees $d_1, \dots, d_s$. However, even in interesting special cases there is not even a conjecture that gives a precise description of the minimal graded free resolution of I. Note that the ideal I is resolved by a Koszul complex if $s \le n+1$. Thus the first interesting case is $s = n+2$, that is, I is an almost complete intersection. Although even in this case the graded Betti numbers are not known in general, many partial results have been established by the first author and Miro-Roig in [65] (see also [66]). Most notably, if $n = 2$ and so $s = 4$ the Betti numbers have been determined in [65, Theorems 4.2]. In any case, the authors obtain very good upper bounds on the Betti numbers. The basic strategy is to use induction on the number of variables.

Consider a complete intersection $\mathfrak{c}$ and an almost complete intersection $I = (\mathfrak{c}, f)$. Then I is linked by c to a Gorenstein ideal $J = \mathfrak{c} : I = \mathfrak{c} : f$. Using duality, the exact sequence (7.2) becomes

$$0 \to R/J(-\deg f) \to R/\mathfrak{c} \to R/I \to 0. \tag{7.3}$$

Now assume that f and the generators of $\mathfrak{c}$ are sufficiently general. Then $R/\mathfrak{c}$ has the Strong Lefschetz Property, and thus so does $R/(\mathfrak{c} : f) = R/J$ because f is generic. Moreover, one gets $[R/J]_j = [R/\mathfrak{c}]_j$ if $j \leq \frac{1}{2}(d_0 + \cdots + d_n - n - 1 - \deg f)$, where $d_0, \ldots, d_n$ are the degrees of the generators of $\mathfrak{c}$. Let $\ell \in R$ be a generic linear form. The last equality implies $[R/(J, \ell)]_j = [R/(\mathfrak{c}, \ell)]_j$ if $j \leq \frac{1}{2}(d_0 + \cdots + d_n - n - 1 - \deg f)$. Using that R/J has the Weak Lefschetz Property, one knows the Hilbert function of $R/(J, \ell)$. Moreover, $R/(\mathfrak{c}, \ell)$ is isomorphic to a quotient of $R/\ell R \cong k[x_0, \ldots, x_{n-1}]$ by an almost complete intersection whose generators have degrees $d_0, \ldots, d_n$. Thus, by induction on n we have information on the graded Betti numbers of $R/(\mathfrak{c}, \ell)$, and so on the graded Betti numbers of $R/(J, \ell)$ in low degrees. Invoking [68, Proposition 8.7], this gives upper bounds on the graded Betti numbers of R/J in low degrees. Since the resolution of R/J is self-dual as R/J is Gorenstein, one obtains upper bounds on all graded Betti numbers of R/J. Finally, using Sequence (7.3), this gives information on the resolution of R/I. The base case for the induction is $n = 2$, where R/J is a Gorenstein ideal of codimension three. Thus, its minimal free resolution is known by work of Diesel [22].

It turns out that the obtained bounds are optimal in several situations, once cancellations in the mapping cone procedure are taken into account. In general, it is a difficult problem to establish if a cancellation occurs or not. Since I is a generic complete intersection, one may hope that its minimal free resolution has few if any ghost terms, that is, free summands that appear in consecutive homological degrees. Note that ghost terms cannot be entirely avoided. For example, if I has two generators of degree 5 and one generator of degree 10, there is a Koszul syzygy of degree 10 producing a ghost summand $R(-10)$. A natural conjecture, due to Iarrobino [21], predicted that these Koszul syzygies are the only source of ghost terms. However, this is too optimistic. Consider, for example, a generic almost complete intersection I in three variables with generator degrees $4, 4, 4, 8$. Its minimal free resolution has the form (see [65, Example 4.3]):

$$0 \to \begin{matrix} R(-10) \\ \oplus \\ R(-11)^2 \end{matrix} \to \begin{matrix} R(-8)^3 \\ \oplus \\ R(-9)^2 \\ \oplus \\ R(-10) \end{matrix} \to \begin{matrix} R(-4)^3 \\ \oplus \\ R(-8) \end{matrix} \to I \to 0.$$

It has a ghost term $R(-10)$, which is not a consequence of a Koszul syzygy. The presence of ghost terms makes it challenging to predict the minimal free resolution when the number of variables is large.

More recently, liaison theory has been used with regards to a conjecture of Mustaţă [74] on the minimal free resolution of a general set of points X on an irreducible subvariety $S \subset \mathbb{P}^n$. Essentially, the conjecture posits that the top part of the Betti diagram of R/I_X is the Betti diagram of R/I_S and the bottom part has only two rows with no ghost terms. Although it is not true in general, this conjecture has motivated several investigations.

Assume X is a general set of points on a surface S of $\mathbb{P}^3$. In this case, Mustaţă's conjecture has been established if S is a smooth quadric [34], a smooth cubic (see [72] and [73]) or a general quartic surface [6]. The last two results use liaison theory in order to prove the conjecture by induction on the number of points on X. Once the result is shown for a certain number of points this set is linked by a suitable Gorenstein set of points to a larger set of points. Sequence (7.1) is used to guarantee that the new set of points satisfies the conjecture as well. Establishing the existence of suitable Gorenstein sets of points is rather subtle. Thus, in [6] the conjecture is first shown for sets of points on a carefully constructed quartic surface. Semicontinuity implies then the desired result on a general quartic surface.

As indicated in Theorem 3.5 above, a different set of tools from liaison theory has been used in order to construct reduced Gorenstein schemes with prescribed properties. In fact, the methods also provide information on their graded Betti numbers.

A key is to use geometric linkage. Suppose ideals I and J are geometrically linked, that is, I and J do not have associated prime ideals in common and $I \cap J$ is a Gorenstein ideal of codimension, say, c. Then Sequence (7.1) implies

$$\omega_{R/J}(-t) \cong I/I \cap J \cong (I+J)/J.$$

It follows that $I+J$ is a Gorenstein ideal of codimension $c+1$ that fits into an exact sequence

$$0 \to \omega_{R/J}(-t) \to R/J \to R/(I+J) \to 0. \tag{7.4}$$

As pointed out above, if R/J is Cohen-Macaulay, then a resolution of R/J determines a resolution of its canonical module $\omega_{R/J}$. Using Sequence (7.4), one obtains upper bounds on the graded Betti numbers of $I+J$. If the Castelnuovo-Munford regularity of J is large enough compared to the regularity of $I \cap J$, then these bounds are sharp by [68, Corollary 8.2]. This is an important ingredient of the following result.

Theorem 7.1 ([68, Theorem 8.13]). *Let A be a graded Gorenstein k-algebra whose Artinian reduction has the Weak Lefschetz Property. Then for any integers i, j, there is an upper bound on $\dim_k[\operatorname{Tor}_i^R(A,k)]_j$ depending on the Hilbert function of A.*

Moreover, given any Hilbert function of a graded Gorenstein k-algebra of positive dimension whose Artinian reduction has the Weak Lefschetz Property, there is a reduced *Gorenstein algebra with these properties such that the above bounds are equalities for every i and j, provided the field k has sufficiently many elements,*

The Gorenstein algebras proving sharpness of the bounds are constructed using sums of geometrically G-linked ideals. The bounds are a consequence of [68, Proposition 8.7] that compares the graded Betti numbers of an Artinian Gorenstein algebra with those of $A/\ell A$, where $\ell \in [A]_1$ is sufficiently general. Since $A/\ell A$ is Cohen-Macaulay its graded Betti numbers are bounded above by whose of R/L where L is a lexicographic ideal such that R/L has the same Hilbert function as A. The graded Betti numbers of a lexicographic ideal were explicitly computed in [77, Proposition 3.8].

The above result has consequences for the theory of simplicial polytopes. Consider a d-dimensional simplicial polytope P. Let $\Delta(P)$ its boundary complex. The Stanley-Reisner ring $k[P] = R/I_{\Delta(P)}$ is a Gorenstein ring of dimension d. The so-called g-theorem classifies face vectors of simplicial polytopes, equivalently, Hilbert functions of $k[P]$. In particular, Stanley showed in [90] that a general Artinian reduction of $k[P]$ has the Weak Lefschetz Property if k has characteristic zero.

Theorem 7.2 ([68, Theorem 9.5])**.** *Suppose k has characteristic zero. If P is a d-dimensional simplicial polytope then there is an upper bound for any graded Betti number of the Stanley-Reisner ring $k[P]$ that depends only on the face vector of P.*

Moreover, for every face vector of a simplicial polytope, there is a simplicial polytope with the given face vector such that the above bounds are all simultaneously sharp.

As pointed out in [77], any empty simplex of a simplicial polytope P corresponds to a minimal generator of the monomial ideal $I_{\Delta(P)}$. Hence, Theorem 7.2 implies a conjecture by Kalai, Kleinschmidt and Lee on the number of empty simplices of a simplicial polytope (see [77, Theorem 2.3]). Additional work is needed to establish the following result:

Theorem 7.3 ([77, Corollary 4.16])**.** *Let P be a d-dimensional simplicial polytope with n vertices, which is not a simplex. Then P has at most $\binom{g+k}{g-1} + \binom{g+k-1}{g-1}$ empty simplices of dimension $\leq k$, where $g = n - d - 1$.*

8. Gröbner bases and Rees algebras

Suppose $I \subset R$ is a homogeneous ideal with a generating set G whose initial monomials (with respect to some monomial ordering on R) generate a monomial ideal I'. Using Buchberger's classical criterion in order to decide whether G is a Gröbner basis of I is often not feasible. In interesting cases, liaison theory offers an alternate approach. In fact, if I and I' can be linked to complete intersections of the same type and both chains of links have the same pattern, then this implies that both, R/I and R/I', are Cohen-Macaulay ideals with the same Hilbert function. Hence the inclusion $I' \subset I$ must be an equality, and G is indeed a Gröbner basis of I.

In order to implement this basic idea it is often enough to leave out every other ideal in a chain of direct links by using suitable generalizations of basic double links as discussed in Theorems 2.7 and 2.8. Here we give a more algebraic version.

Definition 8.1.
(i) Let $\mathfrak{a} \subset I \subset R$ be homogeneous ideals such that $\operatorname{codim} \mathfrak{a} + 1 = \operatorname{codim} I$ and $R/\mathfrak{a}$ is Cohen-Macaulay. If $f \in R$ is homogeneous with $\mathfrak{a} : f = \mathfrak{a}$, then the ideal $fI + \mathfrak{a}$ is called a *basic double link* of degree $\deg f$ on $\mathfrak{a}$.
(ii) Let $\mathfrak{a}, I, J$ be unmixed homogeneous ideals of R such that $\mathfrak{a} \subset I \cap J$, $\operatorname{codim} \mathfrak{a} + 1 = \operatorname{codim} I = \operatorname{codim} J$ and $R/\mathfrak{a}$ is Cohen-Macaulay. If there is an isomorphism of graded R-modules $J/\mathfrak{a} \cong (I/\mathfrak{a})(-t)$, then it is said that J is obtained from I by an *elementary biliaison* of height t on $\mathfrak{a}$.

The above names are motivated by the following result.

Theorem 8.2. *(a) Suppose $J = fI + \mathfrak{a}$ is a basic double link of I height $t = \deg f$.*
(i) If I is a perfect ideal, then so is J. Moreover, their Hilbert functions are related by

$$h_{R/J}(j) = h_{R/I}(j-t) + h_{R/\mathfrak{a}}(j) - h_{R/\mathfrak{a}}(j-t) \quad \textit{for all } j \in \mathbb{Z}.$$

In particular, I and J have the same codimension.
(ii) If I is unmixed and $R/\mathfrak{a}$ is generically Gorenstein then J is unmixed and Gorenstein linked to I in two steps.
(b) Suppose J is obtained from I by an elementary biliaison of height t.

(i) *The Hilbert functions are related by*

$$h_{R/J}(j) = h_{R/I}(j-t) + h_{R/\mathfrak{a}}(j) - h_{R/\mathfrak{a}}(j-t) \quad \textit{for all } j \in \mathbb{Z}.$$

(ii) *If If I and J are unmixed and $R/\mathfrak{a}$ is generically Gorenstein then I and J are Gorenstein linked to I in two steps.*

Proof. (a) Claim (i) is part of [54, Lemma 4.8] and (ii) is shown in [54, Proposition 5.10]

(b) The first assertion is an immediate consequence of the definition. The second assertion is shown in [46]. □

The concepts of basic double links and elementary biliaison are closely related.

Remark 8.3. (i) If $fI + \mathfrak{a}$ is a basic double link of I, then there is a graded isomorphism $(I/\mathfrak{a})(-\deg f) \cong J/\mathfrak{a}$. Thus, basic double linkage is a special case of elementary biliaison.

(ii) If J is obtained from I by an elementary biliaison of height t, then there are homogeneous polynomials $f, g \in R$ with $\deg f = t + \deg g$, $\mathfrak{a} : f = \mathfrak{a} = \mathfrak{a} : g$ and $fI + \mathfrak{a} = gJ + \mathfrak{a}$. Thus, I and J are related via two basic double links, and, by Theorem 8.2(a), J can be obtained (not optimally) from I by four Gorenstein links.

The above result implies a sufficient condition for a set of polynomials to be a Gröbner basis (with respect to a given term order) for the ideal that they generate. We denote by $\operatorname{in}(I)$ the initial ideal of I with respect to the chosen term order.

Lemma 8.4 ([35, Lemma 1.12])**.** *Fix a monomial order on R. Consider an ideal J that is obtained from I by an elementary biliaison of height t on $\mathfrak{a}$. If the initial ideals $\operatorname{in}(I)$ and $\operatorname{in}(\mathfrak{a})$ are perfect and there is a monomial ideal $J' \subset J$ that is obtained from $\operatorname{in}(I)$ by an elementary biliaison of height t on $\operatorname{in}(\mathfrak{a})$, then $J' = \operatorname{in}(J)$.*

Following [35], we illustrate the use of this Gröbner basis criterion in a simple well-known case.

Theorem 8.5. *Let $X = (x_{i,j})$ be an $m \times n$ matrix with $m \leq n$ whose entries are distinct variables. Then the set of maximal minors of X forms a Gröbner basis of the ideal $I_m(X)$, generated by the maximal minors of X.*

Sketch of Proof. Fix a monomial order such that the product of the variables on the main diagonal of any maximal minor is its initial monomial, and denote by J' the ideal generated by these monomials. We want to show that $\operatorname{in}(I_m(X)) = J'$.

We use induction on $|X| = mn$. If $m = 1$, then $J' = I_1(X)$ is generated by variables.

Let $m \geq 2$. If $n = m$, then $I_m(X)$ is a principal ideal. Let $n \geq m + 1$. Denote by Z the $m \times (n-1)$ matrix obtained from X by deleting the last column, and let Y be the $(m-1) \times (n-1)$ matrix obtained from Z by deleting the last row. The induction hypothesis implies that the maximal minors of Y and of Z are Gröbner bases of $I_{m-1}(Y)$ and $I_m(Z)$. It follows by inspection that

$$J' = \operatorname{in}(I_{m-1}(Y)) + x_{m,n} \operatorname{in}(I_m(Z)).$$

Since $x_{m,n}$ does not appear in Y, we get $\operatorname{in}(I_{m-1}(Y)) : x_{m,n} = \operatorname{in}(I_{m-1}(Y))$. Moreover, we have $\operatorname{codim} I_{m-1}(Y) = m - n + 1 = 1 + \operatorname{codim} I_m(Z)$. Hence J' is a basic double link of $\operatorname{in}(I_{m-1}(Y))$ of height one on $\operatorname{in}(I_m(Z))$. The proof of [54, Theorem 3.6] shows that $I_m(X)$ is obtained from $I_{m-1}(Y)$ by an elementary biliaison of height one on $I_m(Z)$. Hence Lemma 8.4 gives $J' = \operatorname{in} I_m(X)$. □

Even if a linkage pattern of an ideal is not known, variations of the above aproach can be productive. We illustrate the idea by explicitly determining equations of some blow-up algebras. If I is an ideal of R, then its *Rees algebra* is the ring $R[It] = \bigoplus_{j \geq 0} I^j t^j \subset R[t]$, where t is a new variable. The *special fiber ring* of $I \subset R$ is the algebra

$$\mathcal{F}(I) = \bigoplus_{j \geq 0} I^j / \mathfrak{m} I^j \cong R[It] \otimes_R R/\mathfrak{m},$$

where $\mathfrak{m} = (x_0, \ldots, x_n)$ is the unique maximal homogeneous ideal of R. Both rings are finitely generated k-algebras, and so they are quotients of polynomial rings by suitable ideals. One often refers to generators of these ideals as equations of the Rees algebra and the special fiber ring, respectively. Determining these equations is typically a challenging problem. We describe a solution for some important classes of monomial ideals.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ be a vector such that $0 \leq \mu_1 \leq \cdots \leq \mu_n < \lambda_n$ and $\mu_i \geq i - 1$ for $i = 1, \ldots, n$. Set $m = \lambda_1$. Following [19], define a *generalized Ferrers ideal* $I_{\lambda-\mu}$ as

$$I_{\lambda-\mu} := (x_i y_j \mid 1 \leq i \leq n, \mu_i < j \leq \lambda_i) \subset k[x_1, \ldots, x_n, y_1, \ldots, y_m].$$

It is isomorphic to a Ferrers ideal as considered in [18]. Substituting $y_j \mapsto x_j$ gives the *specialized Ferrers ideal*

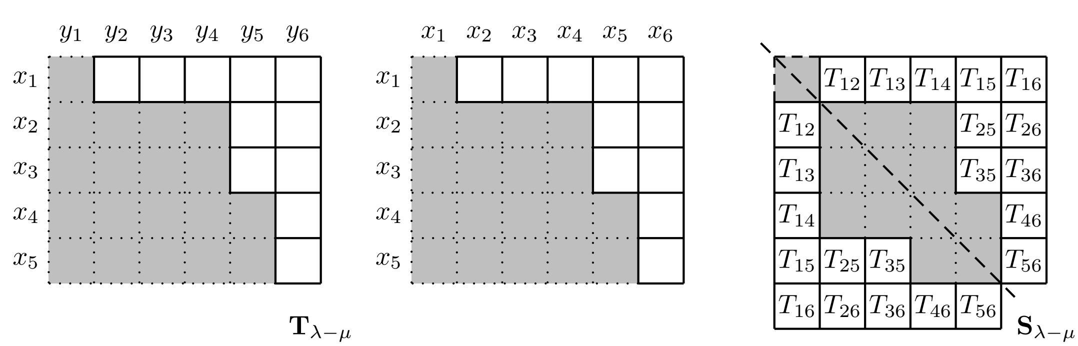
$$\overline{I}_{\lambda-\mu} := (x_i x_j \mid 1 \leq i \leq n, \mu_i < j \leq \lambda_i) \subset K[x_1, \ldots, x_{\max\{n,m\}}].$$

Note that any squarefree strongly stable monomial ideal corresponds to a unique ideal $\overline{I}_{\lambda-\mu}$ with $\mu = (1, 2, \ldots, n)$.

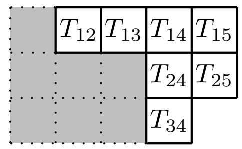
The above ideals can be visualized using a suitable tableau. Form a skew shape $\mathbf{T}_{\lambda-\mu}$, obtained from the Ferrers diagram $\mathbf{T}_\lambda$ by removing the leftmost μ_i boxes in row i. Then the generators of $I_{\lambda-\mu}$ and $\overline{I}_{\lambda-\mu}$ correspond to the boxes of the $\mathbf{T}_{\lambda-\mu}$, where the rows are labelled by $x_1, \ldots, x_n$ and the columns by $y_1, \ldots, y_m$ and $x_1, \ldots, x_m$, respectively. We may also label a box in position (i, j) of $\mathbf{T}_{\lambda-\mu}$ by a variable $T_{i,j}$. Thus, it corresponds to a polynomial ring

$$k[\mathbf{T}_{\lambda-\mu}] := K[T_{ij} \mid 1 \leq i \leq n, \mu_i < j \leq \lambda_i].$$

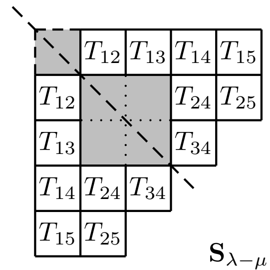
The *symmetrized tableau* $\mathbf{S}_{\lambda-\mu}$ is obtained by reflecting $\mathbf{T}_{\lambda-\mu}$ along the main diagonal. It may have holes along the main diagonal. For example, if $\lambda = (6, 6, 6, 6, 6)$ and $\mu = (1, 4, 4, 5, 5)$, one gets



Observe that in general neither $\mathbf{T}_{\lambda-\mu}$ nor $\mathbf{S}_{\lambda-\mu}$ is a ladder. Denote by $I_2(\mathbf{T}_{\lambda-\mu})$ and $I_2(\mathbf{S}_{\lambda-\mu})$ the ideals in $K[\mathbf{T}_{\lambda-\mu}]$ generated by the determinants of 2×2 submatrices of $\mathbf{T}_{\lambda-\mu}$ and $\mathbf{S}_{\lambda-\mu}$, respectively. For instance, if $\lambda = (5, 5, 4)$ and $\mu = (1, 3, 3)$ we obtain



$\mathbf{T}_{\lambda-\mu}$



$\mathbf{S}_{\lambda-\mu}$

and so

$$I_2(\mathbf{T}_{\lambda-\mu}) = (T_{14}T_{25} - T_{15}T_{24})$$

and

$$I_2(\mathbf{S}_{\lambda-\mu}) = (T_{14}T_{25} - T_{15}T_{24}, T_{12}T_{34} - T_{13}T_{24}).$$

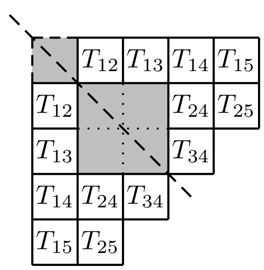
We need one further construction. Given vectors $\lambda, \mu \in \mathbb{Z}^n$ as above, set

$$\lambda' = (\lambda_1 + 1, \lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1) \in \mathbb{Z}^{n+1}$$

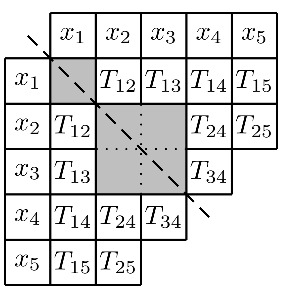
and

$$\mu' = (1, \mu_1 + 1, \mu_2 + 1, \mu_n + 1) \in \mathbb{Z}^{n+1}.$$

Augment the tableau $\mathbf{S}_{\lambda-\mu}$ with a new top row and a new leftmost column. Leave the new northwest corner empty and fill the new top row with the variables $x_1, \dots, x_m$ from left to right and the leftmost column with $x_1, \dots, x_m$ from top to bottom. Up to the names of the variables, the augmented tableau is the same as $\mathbf{S}_{\lambda'-\mu'}$.



$\mathbf{S}_{\lambda-\mu}$



The augmented tableau $\mathbf{S}_{\lambda'-\mu'}$

Theorem 8.6 ([20, Theorem 4.2 and Corollary 4.6])**.** *The special fiber ring and the Rees algebra of $\overline{I}_{\lambda-\mu}$ are determinantal rings.*

More precisely, there are graded isomorphisms

$$\mathcal{F}(\overline{I}_{\lambda-\mu}) \cong k[\mathbf{T}_{\lambda-\mu}]/I_2(\mathbf{S}_{\lambda-\mu})$$

and, if $\mu_1 \leq n$,

$$R[\overline{I}_{\lambda-\mu}t] \cong \mathcal{F}(\overline{I}_{\lambda'-\mu'}) \cong k[\mathbf{T}_{\lambda'-\mu'}]/I_2(\mathbf{S}_{\lambda'-\mu'}).$$

By [20, Remark 4.5], the above result also gives a description of the special fiber ring and the Rees algebra of a generalized Ferrers ideal $I_{\lambda-\mu}$ as established first in [18]. In particular, one has $\mathcal{F}(I_{\lambda-\mu}) \cong K[\mathbf{T}_{\lambda-\mu}]/I_2(\mathbf{T}_{\lambda-\mu})$.

As explained in [20, Remark 4.5], the assumption $\mu_1 \leq n$ for the second isomorphism is harmless. Its proof is similar to that of the first isomorphism. The latter is shown as follows.

By [20, Theorem 2.4] the 2-minors of $\mathbf{T}_{\lambda-\mu}$ and $\mathbf{S}_{\lambda-\mu}$ form a Gröbner basis of $I_2(\mathbf{T}_{\lambda-\mu})$ and $I_2(\mathbf{S}_{\lambda-\mu})$, respectively. Their initial ideals can be obtained from ideals generated by variables via sequences of basic double links, which, in particular, allows one to determine the codimension of these ideals (see [20, Theorem 3.3]). Consider now the algebra epimorphism

$$\pi\colon k[\mathbf{T}_{\lambda-\mu}] \twoheadrightarrow k[x_i x_j \mid x_i x_j \in \overline{I}_{\lambda-\mu}] \cong \mathcal{F}(\overline{I}_{\lambda-\mu}),$$

induced by $\pi(T_{ij}) = x_i x_j$. Since π maps all 2-minors of $\mathbf{S}_{\lambda-\mu}$ to zero we get $I_2(\mathbf{S}_{\lambda-\mu}) \subset \ker \pi$. Both ideals are prime ideals (see [20, Proposition 3.5]). Thus, the desired equality follows if the two ideals have the same codimension. This is indeed true as a comparison of the codimension of $I_2(\mathbf{S}_{\lambda-\mu})$ and $\dim \mathcal{F}(\overline{I}_{\lambda-\mu})$ reveals.

9. Vertex decomposability

The use of liaison-theoretic methods to study simplicial complexes has been pioneered in [78]. The starting point is a well-known bijection between squarefree monomial ideals and simplicial complexes.

Recall that a *simplicial complex* Δ on n vertices is a collection of subsets of $[n] = \{1, \ldots, n\}$ that is closed under inclusion. The elements of Δ are called the *faces* of Δ. The dimension of a face F is $|F| - 1$. The *Stanley-Reisner ideal* of Δ is $I_\Delta = (\prod_{i \in F} x_i \mid F \subseteq [n],\ F \notin \Delta) \subset R = k[x_1, \ldots, x_n]$, and the corresponding *Stanley-Reisner ring* is $k[\Delta] = R/I_\Delta$. Note that the dimensions of Δ and $k[\Delta]$ determine each other because $\dim \Delta = \dim k[\Delta] - 1$. We say that Δ has an algebraic property such as Cohen-Macaulayness if $K[\Delta]$ has this property. For more details on simplicial complexes, Stanley-Reisner rings and their algebraic properties we refer to the books of Bruns-Herzog [14] and Stanley [91].

Following [78, Definition 2.2], a squarefree monomial I is said to be *squarefree glicci* if I can be linked in an even number of steps to a complete intersection I' generated by variables such that every other ideal in the chain linking I to I' is a squarefree monomial ideal. In other words the simplicial complex Δ corresponding to I can be "linked" to a simplex in an even number of steps, where every other step corresponds to a simplicial complex.

Example 9.1. Denote by Δ the simplicial complex on [4] consisting of 4 vertices. Its Stanley-Reisner ideal is $I_\Delta = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4)$. It is squarefree glicci because

$$I_\Delta = x_4 \cdot (x_1, x_2, x_3) + (x_1x_2, x_1x_3, x_2x_3)$$

implies that I_Δ is a basic double link of (x_1, x_2, x_3).

Provan and Billera introduced in [84] an important property of a simplicial complex. To state it recall that, given a vertex j of a simplicial complex Δ, the *link* of j is

$$\operatorname{lk}_\Delta(j) = \{G \in \Delta \mid \{j\} \cup G \in \Delta, \{j\} \cap G = \emptyset\},$$

and the *deletion* with respect to j is

$$\Delta_{-j} = \{G \in \Delta \mid \{j\} \cap G = \emptyset\}.$$

A pure simplicial complex Δ is said to be *vertex decomposable* if Δ is a simplex or equal to $\{\emptyset\}$, or there exists a vertex j such that $\operatorname{lk}_\Delta j$ and Δ_{-j} are both pure and vertex-decomposable and $\dim \Delta = \dim \Delta_{-j} = \dim \operatorname{lk}_\Delta j + 1$.

Every vertex decomposable simplicial complex is shellable, and so Cohen-Macaulay. Thus, the following concept, introduced in [78, Definition 3.1], is less restrictive. A pure simplicial simplex $\Delta \neq \emptyset$ on $[n]$ is said to be *weakly vertex decomposable* if there is some $j \in [n]$ such that Δ is a cone over the weakly vertex-decomposable deletion Δ_{-j} or there is some $j \in [n]$ such that $\mathrm{lk}_\Delta(j)$ is weakly vertex decomposable and Δ_{-j} is Cohen-Macaulay of the same dimension as Δ.

We now relate these combinatorial concepts via liaison theory. For a simplicial complex Δ on $[n]$, consider any vertex $j \in [n]$. Then the cone over the link $\mathrm{lk}_\Delta(j)$ with apex j considered as complex on $[n]$ has as Stanley-Reisner ideal $J_{\mathrm{lk}_\Delta(j)} = I_\Delta : x_j$. Denote by $J_{\Delta_{-j}} \subset R$ the extension ideal of the Stanley-Reisner ideal of Δ_{-j} considered as a complex on $[n] \setminus \{j\}$. Note that x_j does not divide any of the minimal generators of $J_{\Delta_{-j}}$, thus $J_{\Delta_{-j}} : x_j = J_{\Delta_{-j}}$. Furthermore, it follows that

$$I_\Delta = x_j J_{\mathrm{lk}_\Delta(j)} + J_{\Delta_{-j}}. \tag{9.1}$$

Comparing with Definition 8.1 and Theorem 8.2, this equation implies that Δ is a basic double link of the cone over its link $\mathrm{lk}_\Delta(j)$ and Gorenstein linked to it in two steps if Δ is pure and if the deletion Δ_{-j} is Cohen-Macaulay and has the same dimension as Δ when both are considered as complexes on $[n]$. These observations lead to the following result.

Theorem 9.2 ([78, Theorem 3.3])**.** *If Δ is a weakly vertex decomposable simplicial complex, then Δ is squarefree glicci. In particular, Δ is Cohen-Macaulay.*

This result applies to a number of well-studied classes of simplicial complexes. In fact, it is known that any pure shifted complex, any matroid complex, any Gorenstein complex and any 2-Cohen-Macaulay complex (see [3] for the definition) is weakly vertex decomposable.

It has been observed in [78] that in general both of the properties considered in the above theorem depend on the characteristic of the ground field.

Example 9.3 ([78, Example 5.5])**.**
(i) Consider a tringulation Δ of the real projective plane $\mathbb{P}^2$ with six vertices. Using the notation from [14, p. 236], its Stanley-Reisner ideal in $k[x_1, \ldots, x_6]$ is

$$I_\Delta = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6).$$

If $\operatorname{char} k \neq 2$ this is a 2-dimensional Cohen-Macaulay complex, whereas Δ is *not* Cohen-Macaulay if $\operatorname{char} k = 2$.

(ii) Let $R = k[x_1, \ldots, x_7]$ and denote by $\mathfrak{a}$ the extension ideal of I_Δ in R. Set $J = (x_1, \ldots, x_4) \subset R$. Consider the squarefree monomial ideal

$$I = x_7 J + \mathfrak{a}.$$

Since $R/\mathfrak{a}$ is Cohen-Macaulay if and only if $\operatorname{char} k \neq 2$, I is a basic double link of the complete intersection J if $\operatorname{char} k \neq 2$. It follows that in this case I is squarefree glicci and that the induced simplicial complex Δ' is weakly vertex-decomposable. However, if $\operatorname{char} k = 2$ then Δ' is not Cohen-Macaulay and so neither (squarefree) glicci nor weakly vertex decomposable.

Example 9.3(i) also gives rise to a challenging problem. One of the main open questions in liaison theory is whether every Cohen-Macaulay ideal is glicci. In view of the above dependence of the Cohen-Macaulayness of $k[\Delta]$ on the characteristic, the following problem was proposed in [78, Problem 5.3]:

Problem 9.4. *Decide whether the Stanley-Reisner ideal of the above triangulation of $\mathbb{P}^2_{\mathbb{R}}$ is glicci if* $\operatorname{char} k \neq 2$.

Recently, in [52] Klein and Rajchgot established a vast generalization of Theorem 9.2. To discuss it, it is useful to rewrite Equation (9.1) as

$$I_\Delta = J_{\mathrm{lk}_\Delta(j)} \cap (x_j, J_{\Delta_{-j}}). \tag{9.2}$$

Note that $(x_j, J_{\Delta_{-j}}) = (x_j, I_\Delta)$ is the Stanley-Reisner ideal of the deletion Δ_{-j} when it is considered as a simplicial complex on $[n]$. We also observed that $J_{\mathrm{lk}_\Delta(j)} = I_\Delta : x_j$. Knutson, Miller, Yong introduced in [51] geometric vertex decomposition as an analog of the decomposition in Equation (9.2) for an ideal $I \subset R = k[x_1, \ldots, x_n]$ that is not necessarily homogeneous. We need some notation.

Let y be any variable of R. Any nonzero polynomial $f \in R$ can be uniquely written as $f = y^d q + r$ with polynomials $q, r \in R$ and $d \in \mathbb{N}_0$ such that no monomial in $q \neq 0$ is divisible by y, no monomial in r is divisible by y^d if $d > 0$ and $r = 0$ if $d = 0$. Set $\mathrm{in}_y f = y^d q$ and define the initial ideal of I with respect to y as

$$\mathrm{in}_y I = (\mathrm{in}_y f \mid f \in I) \subset R.$$

A monomial order $<$ on R is said to be *y-compatible* if $\mathrm{in}_< f = \mathrm{in}_<(\mathrm{in}_f f)$ for every $f \neq 0$ in R. Consider now a Gröbner basis G of an ideal $I \subset R$ with respect to a y-compatible order. Write each element of G as above, that is,

$$G = \{y^{d_i} q_i + r_i \mid 1 \leq i \leq s\},$$

and so in particular $\mathrm{in}_y(y^{d_i} q_i + r_i) = y^{d_i} q_i$. Define the following ideals of R:

$$\mathfrak{b}_{y,I} = (q_i \mid 1 \leq i \leq s) \quad \text{and} \quad \mathfrak{a}_{y,I} = (q_i \mid d_i = 0).$$

Note that these definitions do not depend on the choice of Gröbner basis G because one has by [51, Theorem 2.1],

$$\mathfrak{b}_{y,I} = \bigcup_{i \geq 1} (\mathrm{in}_y I : y^i) \quad \text{and} \quad (y, \mathfrak{a}_{y,I}) = (y, \mathrm{in}_y I).$$

Definition 9.5 ([51])**.** If

$$\mathrm{in}_y I = \mathfrak{b}_{y,I} \cap (y, \mathfrak{a}_{y,I}) \tag{9.3}$$

then this is called a *geometric vertex decomposition of I with respect to y.*

Comparing with Equation (9.2) and the discussion below it, it follows that Equation (9.2) is a geometric vertex decomposition of I_Δ with respect to x_j. Using the definition of a vertex decomposable simplicial complex as a role model, one says (see [52, Definition 2.6]) that an unmixed ideal I of R is *geometrically vertex decomposable* if (i) $I = R$ or I is generated by variables, or (ii) for some variable y of R, $\mathrm{in}_y I = \mathfrak{b}_{y,I} \cap (y, \mathfrak{a}_{y,I})$ is a geometric vertex decomposition and the contractions of $\mathfrak{b}_{y,I}$ and $\mathfrak{a}_{y,I}$ to $k[x_1, \ldots, \hat{y}, \ldots, x_n]$ are geometrically vertex decomposable. By induction, it follows that every geometrically vertex decomposable ideal is radical.

Similarly to weakly vertex decomposable simplicial complexes, there are also *weakly geometrically vertex decomposable ideals*, see [52, Definition 4.6]. Analogously to Theorem 9.2, one has:

Theorem 9.6 ([52, Corollary 4.8])**.** *Any weakly geometrically vertex decomposable ideal $I \subset R$ is both radical and glicci. In particular, R/I is Cohen-Macaulay.*

This result applies, for example, to Schubert determinantal ideals and homogeneous ideals of lower bound Cluster algebras [52, Propositions 5.2 and 5.3]. The key observation for establishing Theorem 9.6 is that a geometric vertex decomposition often gives rise to an elementary biliaison (see Definition 8.1(ii)).

Theorem 9.7 ([52, Theorem 4.1])**.** *Suppose an unmixed ideal $I \subset R$ has a geometric vertex decomposition with respect to some variable y of R such that neither $\mathfrak{b}_{y,I} = \mathfrak{a}_{y,I}$ nor $\mathfrak{b}_{y,I} = R$. If $I, \mathfrak{a}_{y,I}$ and $I, \mathfrak{a}_{y,I}$ are homogeneous then there is a graded isomorphism $I/\mathfrak{a}_{y,I} \cong (\mathfrak{b}_{y,I}/\mathfrak{a}_{y,I})(-1)$.*

Remarkably, some form of converse to this result is true as well.

Theorem 9.8 ([52, Theorem 6.1])**.** *Fix a y-compatible monomial order and consider ideals $I, \mathfrak{b}, \mathfrak{a}$ with $\mathfrak{a} \subset I \cap \mathfrak{b}$. Suppose that y^2 does not divide any term of any element of the reduced Gröbner basis of I and that no term of any element of the reduced Gröbner basis of $\mathfrak{a}$ is divisible by y. If there is an isomorphism of $R/\mathfrak{a}$ modules $I/\mathfrak{a} \to \mathfrak{b}/\mathfrak{a}$ induced by multiplication with $\frac{f}{g}$ with $\frac{\mathrm{in}_y f}{g} = y$ then $\mathrm{in}_y I = \mathfrak{b} \cap (y, \mathfrak{a})$ is a geometric vertex decomposition.*

Combining these results with Lemma 8.4, allows one to determine Gröbner bases of further classes of ideals (see [52, Corollary 4.13] and [53]).

10. Unprojections

In 1983 Kustin and Miller introduced a construction of Gorenstein ideals in local Gorenstein rings, starting from smaller such ideals. More precisely, given Gorenstein ideals $\mathfrak{b} \subset \mathfrak{a}$ with grades g and $g-1$, respectively, in a Gorenstein local ring R, in [56] they construct a new Gorenstein ideal I of grade g in a larger Gorenstein ring $R[v]$. Here v is a new indeterminate. In [57] they give an interpretation for their construction via liaison theory. The Kustin-Miller construction has been used to produce many interesting classes of Gorenstein ideals. In birational geometry it is known as *unprojection* (see, e.g., [81, 82, 13]). Following [38], we discuss a modification of the Kustin-Miller construction in the case of graded rings within the framework of Gorenstein liaison theory.

Let R be a graded Gorenstein k-algebra. Let $\mathfrak{a}$ and $\mathfrak{b} \subset \mathfrak{a}$ be homogeneous Gorenstein ideals in R of codimension g and $g-1$, respectively. The embedding $\mathfrak{b} \hookrightarrow \mathfrak{a}$ induces the following commutative diagram, where the rows are minimal free resolutions of $R/\mathfrak{b}$ and $R/\mathfrak{a}$, respectively:

$$\begin{array}{ccccccccccccc} & & 0 & \longrightarrow & B_{g-1} = R(-u) & \xrightarrow{b_{g-1}} & \cdots & \longrightarrow & B_1 & \xrightarrow{b_1} & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \alpha_{g-1}} & & & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0 = \mathrm{id}} & & \\ 0 & \longrightarrow & A_g = R(-v) & \xrightarrow{a_g} & A_{g-1} & \xrightarrow{a_{g-1}} & \cdots & \longrightarrow & A_1 & \xrightarrow{a_1} & R & \longrightarrow & 0 \end{array} \tag{10.1}$$

Fixing bases for all the free modules, we identify the maps with their coordinate matrices. As above, we denote by ω_M the canonical module of a graded R-module M. It is isomorphic to the k-dual of the local cohomology module $H^{\dim M}_{\mathfrak{m}}(M)$.

Theorem 10.1 ([38, Theorem 3.1])**.** *Assume $d = u - v \geq 0$. Let $y \in \mathfrak{a}$ be a homogeneous element such that $\mathfrak{b} : y = \mathfrak{b}$. The embedding $\mu : (\mathfrak{b}, y) \hookrightarrow \mathfrak{a}$ induces an R-module homomorphism $\omega_{R/\mathfrak{a}} \to \omega_{R/(\mathfrak{b},y)}$ that is multiplication by some homogeneous element $\omega \in R$. Its degree is $d + \deg y$.*

Assume there is a homogeneous element $f \in R$ of degree d such that $\mathfrak{b} : (\omega + fy) = \mathfrak{b}$. Consider the ideal I obtained from $\mathfrak{a}$ by the two Gorenstein links

$$\mathfrak{a} \sim_{(\mathfrak{b},y)} \sim_{(\mathfrak{b},\omega+fy)} I,$$

that is, $I = (\mathfrak{b}, \omega + fy) : [(\mathfrak{b}, y) : \mathfrak{a}]$. Then I is a Gorenstein ideal with the same codimension as $\mathfrak{a}$. It can be written as

$$I = \mathfrak{b} + (\alpha^*_{g-1} + (-1)^g f a^*_g) = (\mathfrak{b}, \alpha^*_{g-1} + (-1)^g f a^*_g),$$

where α^*_{g-1} and a^*_g are interpreted as row vectors and "+" indicates their component-wise sum whose entries, together with generators of $\mathfrak{b}$, generate I.

Observe that a sufficiently general choice of the element f always gives a desired element $\omega + fy$ in Theorem 10.1, at least if the field k is infinite.

We illustrate the result by a simple example.

Example 10.2. Consider the complete intersections $\mathfrak{a} = (x, y, z)$ and $\mathfrak{b} = (x^2 - z^2, y^2 - z^2)$ in the polynomial ring $k[x, y, z]$, where k is a field of characteristic zero. Linking $\mathfrak{a}$ by $\mathfrak{b} + (z^2)$, we get as residual $J = \mathfrak{b} + (z^2, xyz)$. Choosing $f = 5z$, we link J by $\mathfrak{b} + (xyz + fz^2)$ to

$$I = \mathfrak{b} + (xf + yz, yf + xz, zf + xy) = (x^2 - z^2, y^2 - z^2, xz, yz, xy + 5z^2).$$

Notice that for the second link we cannot take $f = z$ because $xyz + z^3$ is a zero divisor modulo $\mathfrak{b}$.

Given a minimal free resolution of $\mathfrak{b}$, it is easy to determine minimal free resolutions of the ideals $(\mathfrak{b}, y)$ and $(\mathfrak{b}, \omega + fy)$ that are used for the links in Theorem 10.1. Combined with the mapping cone procedure applied twice to sequences as in (7.1), one obtains a free resolution of I. However, this resolution is not minimal if $g \geq 3$. In fact, by identifying the construction in Theorem 10.1 as an elementary biliaison one gets a smaller free resolution.

Theorem 10.3 ([38, Theorem 4.1])**.** *Adopt the notation and assumptions of Theorem 10.1. Then there is a short exact sequence of graded R-modules*

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{b})(-d) \longrightarrow R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0.$$

Moreover, the ideal I has a graded free resolution of the form

$$0 \to B_{g-1}(-d) \to \begin{matrix} A_{g-1}(-d) \\ \oplus \\ B_{g-2}(-d) \end{matrix} \to \begin{matrix} B_{g-2} \\ \oplus \\ A_{g-2}(-d) \\ \oplus \\ B_{g-3}(-d) \end{matrix} \to \dots \to \begin{matrix} B_2 \\ \oplus \\ A_2(-d) \\ \oplus \\ B_1(-d) \end{matrix} \to \begin{matrix} B_1 \\ \oplus \\ A_1(-d) \end{matrix} \to I \to 0.$$

Notice that the maps in the constructed free resolution of I are described in the proof of the statement.

Corollary 10.4 ([38, Proposition 4.3])**.** *The homogeneous Gorenstein ideal $I = (\mathfrak{b}, \alpha^*_{g-1} + (-1)^g f a^*_g)$ in Theorem 10.1 is obtained from $\mathfrak{a}$ by an elementary biliaison on $\mathfrak{b}$.*

Proof. The short exact sequence in Theorem 10.3 gives a graded isomorphism $\mathfrak{a}/\mathfrak{b}(-d) \cong I/\mathfrak{b}$. Since $\mathfrak{b}$ is Gorenstein the claim follows directly from the definition of an elementary biliaison. ∎

The free resolution constructed in Theorem 10.3 is often minimal. In fact, if the polynomial f is not a unit and each map α_i in Diagram (10.1) is minimal whenever $1 \le i \le g-1$, that is, $\operatorname{Im}\alpha_i \subset \mathfrak{m}A_i$, then the resolution of I described in Theorem 10.3 is a graded minimal free resolution of I (see [38, Corollary 4.2]).

We illustrate the versatility of the above construction by some examples. Even if one starts with complete intersections the resulting Gorenstein ideal is more complicated.

Example 10.5 ([38, Example 51])**.** Let $R = k[x_1, \dots, x_n$ be a polynomial ring. For an integer g with $2 \le g \le n$, consider two ideals that are generated by regular sequences

$$\mathfrak{b} = (x_1^{m_1}, x_2^{m_2}, \cdots, x_{g-1}^{m_{g-1}}) \subset (x_1^{n_1}, x_2^{n_2}, \cdots, x_g^{n_g}) = \mathfrak{a}.$$

If $d := \sum_{i=1}^{g-1} m_i - \sum_{i=1}^{g} n_i \ge 0$ then, for a sufficiently general polynomial $f \in R$ of degree d,

$$I = (x_1^{m_1}, \cdots, x_{g-1}^{m_{g-1}}, fx_1^{n_1}, \cdots, fx_{g-1}^{n_{g-1}}, fx_g^{n_g} + \prod_{j=1}^{g-1} x_j^{m_j - n_j})$$

is a Gorenstein ideal. Moreover, if $m_j > n_j$ for each $j = 1, \dots, g-1$, then the resolution in Theorem 10.3 is a minimal free resolution of I.

The next example shows that every Artinian Gorenstein ideal whose Castelnuovo-Mumford regularity is three can be obtained by one elementary biliaison from a complete intersection.

Example 10.6. Consider an ideal $I \subset R = k[x_1, \dots, x_n]$ such that R/I is a graded compressed Gorenstein algebra with h-vector $(1, n, 1)$. According to Sally [87, Theorem 1.1], after a suitable change of coordinates any such ideal is of the form

$$I = (x_i x_j \mid 1 \le i < j \le n) + (x_1^2 - c_1 x_n^2, \dots, x_{n-1}^2 - c_{n-1} x_n^2),$$

where $c_1, \dots, c_{n-1} \in k$ are suitable units. It can be obtained by an elementary biliaison as in Theorem 10.1 from $\mathfrak{a} = (x_1, \dots, x_n)$ on $\mathfrak{b}R$, where $\mathfrak{b}$ is a Sally ideal in $n-1$ variables, namely

$$\mathfrak{b} = (x_i x_j \mid 1 \le i < j \le n-1) + (x_1^2 - \frac{c_1}{c_{n-1}} x_{n-1}^2, \dots, x_{n-2}^2 - \frac{c_{n-2}}{c_{n-1}} x_{n-1}^2).$$

More precisely, there are the following links

$$\mathfrak{a} \sim_{(\mathfrak{b}, x_n)} (\mathfrak{b}, x_n, x_{n-1}^2) \sim_{(\mathfrak{b}, x_{n-1}^2 - c_{n-1} x_n^2)} I.$$

Note that $(\mathfrak{b}, x_n, x_{n-1}^2) = (x_1, \dots, x_{n-1})^2 + (x_n)$.

We now consider some codimension four Gorenstein ideals with 9 generators and 16 syzygies. Such Gorenstein ideals are investigated in depth from the point of view of unprojections in [13].

Example 10.7. Let $R = k[a, b, c, d, e, f, x, y, z]$ be a polynomial ring in 9 variables over a field k. Consider a generic 3×3 symmetric matrix A and a generic skew-symmetric matrix B:

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}.$$

For $\lambda \neq 0$ in k, define a 6×6 skew-symmetric matrix $N = \begin{bmatrix} B & A \\ -A & \lambda B \end{bmatrix}$. The ideal $\mathfrak{a}$ generated by the 4×4 Pfaffians of N is a homogeneous Gorenstein ideal of grade 4:

$$\begin{aligned}\mathfrak{a} =&(b^2 - ad + \lambda x^2, bc - ae + \lambda xy, c^2 - af + \lambda y^2, cd - be + \lambda xz, ce - bf + \lambda yz,\\ &e^2 - df + \lambda z^2, cx - by + az, ex - dy + bz, fx - ey + cz).\end{aligned}$$

It is the defining ideal of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ into $\mathbb{P}^8$ and a typical case of a Tom unprojection (see [13, 82]). In particular, $\mathfrak{a}$ is equal to the ideal generated by the 2×2 minors of a 3×3 generic matrix $A+\sqrt{-\lambda}B$. Hence, the Gulliksen and Negård complex gives its minimal free resolution:

$$0 \longrightarrow R(-6) \xrightarrow{a_4} R^9(-4) \xrightarrow{a_3} R^{16}(-3) \xrightarrow{a_2} R^9(-2) \xrightarrow{a_1} \mathfrak{a} \longrightarrow 0.$$

In order to perform the construction of Theorem 10.1, we choose the first three listed generators of $\mathfrak{a}$ to define a complete intersection

$$\mathfrak{b} = (b^2 - ad + \lambda x^2, bc - ae + \lambda xy, c^2 - af + \lambda y^2)$$

inside $\mathfrak{a}$. Then we link as follows:

$$\mathfrak{a} \sim_{(\mathfrak{b}, cd-be+\lambda xz)} (\mathfrak{b}, cd - be + \lambda xz, ax) \sim_{(\mathfrak{b}, ax+(cd-be+\lambda xz))} I.$$

Explicitly, the resulting ideal I is

$$\begin{aligned}I =&(e^2 - df - cx + by + az + \lambda z^2, ce - bf + ay + \lambda yz, cd - be + ax + \lambda xz,\\ &c^2 - af + \lambda y^2, bc - ae + \lambda xy, ac + \lambda fx - \lambda ey + \lambda cz, b^2 - ad + \lambda x^2,\\ &ab + \lambda ex - \lambda dy + \lambda bz, a^2 + \lambda cx - \lambda by + \lambda az).\end{aligned}$$

It has the same Betti table as $\mathfrak{a}$. In fact, I is generated by the 4×4 Pfaffians of the matrix

$$M = \begin{bmatrix} 0 & x & y & a & b & c \\ -x & 0 & \frac{1}{\lambda}a + z & b & d & e \\ -y & -\frac{1}{\lambda}a - z & 0 & c & e & f \\ -a & -b & -c & 0 & \lambda x & \lambda y \\ -b & -d & -e & -\lambda x & 0 & a + \lambda z \\ -c & -e & -f & -\lambda y & -a - \lambda z & 0 \end{bmatrix}.$$

Using the description of the minimal free resolution in Theorem 10.3, one can compare the Castelnuovo-Mumford regularities of the ideals involved in Theorem 10.1. In fact, one has (see the proof of [38, Corollary 4.4])

$$\operatorname{reg} I - \operatorname{reg} \mathfrak{a} = 2d.$$

In particular, we get $\operatorname{reg} I \geq \operatorname{reg} \mathfrak{a}$, which is also expressed by saying that I has been obtained from $\mathfrak{a}$ by an *ascending* elementary biliaison. The last equation also leads to an explicit example of a Gorenstein ideal that cannot be obtained using the construction of Theorem 10.1 with a strictly ascending biliaison.

Example 10.8 ([38, Example 5.5])**.** Let I be a generic Artinian Gorenstein ideal in $R = k[x_1, \dots, x_5]$ with h-vector $(1, 5, 5, 1)$, where k is an infinite field. It has the least possible Betti numbers. Its graded minimal free resolution has the form

$$0 \to R(-8) \to R^{10}(-6) \to R^{16}(-5) \to R^{16}(-3) \to R^{10}(-2) \to I \to 0. \tag{10.2}$$

This is the key to showing that there are no Gorenstein ideals $\mathfrak{a}$ and $\mathfrak{b}$ to produce I using a biliaison as in Theorem 10.1 that is strictly ascending, i.e., $d > 0$ or, equivalently, $\mathfrak{a}$ has smaller regularity than I.

11. Open questions

We end with a short list of open questions from liaison theory. Besides being important in and of themselves from a theoretical perspective, it is to be hoped that their resolution will lead to further examples of beautiful and unexpected applications.

1. It is well-known that if a homogeneous ideal I is glicci then it is Cohen-Macaulay. What about the converse: is every Cohen-Macaulay ideal glicci? The first result in this direction is still arguably the cleanest in that it is a direct generalization of Gaeta's theorem ([54] Theorem 3.6): if I is the ideal of maximal minors of a homogeneous $t \times (t + c)$ matrix, and if I has the expected height $c + 1$, then I is glicci. As mentioned on page 28, this converse is one of the main open questions in liaison theory and was first proposed in [54], page 18. See Problem 9.4 above for a particular example. One can also ask, more generally, whether for curves in $\mathbb{P}^n$, $n \geq 4$, the Hartshorne-Rao module determines the even Gorenstein liaison class.
2. We have seen several applications of the LR property (see Definition 2.11 and Theorem 2.13), and as noted above, this property is only known to hold in codimension two. It was studied in the context of Gorenstein liaison in higher codimension in [47], and the general conclusion was that there is no hope of getting an analogous result in that setting. However, it seems to us to be quite reasonable to hope that for CI-liaison in higher codimension, the analogous property *does* hold. And since it had so many applications in codimension two, one can furthermore expect many consequences in higher codimension.
3. We have seen above that questions about the genus of curves in $\mathbb{P}^n$, and about possible Hilbert functions of sets of points in uniform position, have used liaison theory to make advances. One kind of measure of uniformity is given by the Cayley-Bacharach property, and we saw above that Chong used liaison to say something also here. It seems almost certain that Gorenstein liaison will open still further doors for us in this direction. Is there in fact an approach via Gorenstein liaison?
4. We saw above in sections 3 and 7 that liaison theory has been used to produce a broad family of arithmetically Gorenstein unions of linear varieties in any codimension, with important properties. Predominant among these are the fact that the general Artinian reduction has the WLP, and the fact that the graded Betti numbers are maximal in a precise sense. In the paper [69] is a discussion of how this relates to the so-called g-conjecture and, perhaps, an even stronger result as a consequence of a positive answer to the following open question: Does the general Artinian reduction of an arithmetically Gorenstein set of points have the WLP? SLP?
5. In several papers (see, e.g., [45, 46]) Hartshorne has studied aspects of the following open question: Can every Gorenstein ideal be produced by an ascending elementary biliaison from another Gorenstein ideal? This is interesting in its own right, but it

would also give further applications along the lines of unprojection, as described in section 10.

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