REDUCED ARITHMETICALLY GORENSTEIN SCHEMES AND SIMPLICIAL POLYTOPES WITH MAXIMAL BETTI NUMBERS

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Abstract. An SI-sequence is a finite sequence of positive integers which is symmetric, unimodal and satisfies a certain growth condition. These are known to correspond precisely to the possible Hilbert functions of graded Artinian Gorenstein algebras with the Weak Lefschetz Property, a property shared by a nonempty open set of the family of all graded Artinian Gorenstein algebras having a fixed Hilbert function that is an SI sequence. Starting with an arbitrary SI-sequence, we construct a reduced, arithmetically Gorenstein configuration G of linear varieties of arbitrary dimension whose Artinian reduction has the given SI-sequence as Hilbert function and has the Weak Lefschetz Property. Furthermore, we show that G has maximal graded Betti numbers among all arithmetically Gorenstein subschemes of projective space whose Artinian reduction has the Weak Lefschetz Property and the given Hilbert function. As an application we show that over a field of characteristic zero every set of simplicial polytopes with fixed h-vector contains a polytope with maximal graded Betti numbers.

Contents

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1. Introduction

This paper addresses three fundamental questions about reduced arithmetically Gorenstein subschemes of projective space, over an arbitrary field K. First, we consider the question of possible Hilbert functions that can occur (in arbitrary codimension). Second, we consider the possible graded Betti numbers that can occur for the minimal free resolution of the homogeneous ideal of such a subscheme. In particular, we are interested in the problem of whether, among the arithmetically Gorenstein schemes with fixed Hilbert function, there is one with maximal graded Betti numbers.

Most importantly, we are interested in the question of liftability from the graded Artinian case: which properties of graded Artinian Gorenstein ideals lift to properties of the ideal of a *reduced*, arithmetically Gorenstein subscheme of projective space? (Of course without the "reduced" requirement all properties can be lifted by considering cones.) In particular, which Hilbert functions and which graded Betti numbers that occur at the graded Artinian level also occur for reduced arithmetically Gorenstein schemes? *Throughout this paper we will mean by "Artinian Gorenstein algebra" a graded Artinian Gorenstein algebra.* We also consider these questions for the even more special reduced, monomial, Gorenstein ideals occurring as Stanley-Reisner ideals of simplicial polytopes.

We will give complete answers to these questions for a large subset of the set of all arithmetically Gorenstein subschemes, namely for the ones whose Artinian reductions have the so-called *weak Lefschetz property*. The weak Lefschetz property is a very important notion for Artinian Gorenstein algebras; it says that the homomorphism induced between any two consecutive components by multiplication by a general linear form has maximal rank. J. Watanabe [58] had shown that an open dense subset of the compressed Artinian Gorenstein algebras with fixed socle degree and embedding dimension (i.e. the Artinian Gorenstein algebras having maximum Hilbert function given the socle degree and the embedding dimension) satisfy the Weak Lefschetz property.

Work of R. Stanley [56] (over fields of characteristic zero) and T. Harima (over arbitrary infinite fields) generalized Watanabe's result to show that a nonempty open set of graded Artinian Gorenstein algebras having a Hilbert function that is an SI-sequence (see below in this introduction) have the Weak Lefschetz property. *From now on we say that an arithmetically Gorenstein scheme X has the weak Lefschetz property if there is an Artinian reduction of X having the weak Lefschetz property.*

Codimension three Gorenstein ideals are quite well understood, thanks primarily to the structure theorem of Buchsbaum and Eisenbud [12]. Using this result, Diesel [14] gave a description of all possible sets of graded Betti numbers (hence all possible Hilbert functions) for graded *Artinian* Gorenstein ideals, leaving open the question of which of these could occur for reduced, arithmetically Gorenstein subschemes of projective space. Geramita and the first author solved this problem by showing that any set of graded Betti numbers allowed by Diesel in fact occurs for a reduced set of points in $\mathbb{P}^3$, a *stick figure* curve in $\mathbb{P}^4$, and more generally a "generalized stick figure" in $\mathbb{P}^n$. As a consequence, they showed that any codimension three arithmetically Gorenstein subscheme of $\mathbb{P}^n$ ($n \geq 3$) specializes to a good linear configuration with the same graded Betti numbers.

What can be said in higher codimension? In codimension ≥ 4 it is not even known precisely which Hilbert functions arise for Artinian Gorenstein ideals, and there is no analog to Diesel's work for graded Betti numbers. Still, for Hilbert functions there are some results in [7], [4] and [29], for example, and for resolutions there are important results in [19] which we will describe below. These results form the starting point for our work.

Although a complete classification of possible Hilbert functions for graded Artinian Gorenstein ideals of codimension ≥ 4 is not known, a very large class of such Hilbert functions has been shown to occur by Harima [29], namely those which are so-called SI-sequences. Roughly, this property says that the "first half" of the Hilbert function (which is a finite sequence of integers) is a differentiable O-sequence, in the sense of Macaulay [42] and Stanley [55]. It is known that not all Hilbert functions of Artinian Gorenstein ideals in codimension ≥ 5 have this property, but it is an open question whether they all do in codimension 4. Certainly they all do in codimension 3. Harima's approach was via liaison and sums of linked ideals, to which we will return shortly.

Prior to Harima's work, it was already shown by Billera and Lee [5] and Stanley [56] that an h-vector is the h-vector of a simplicial polytope if and only if it is an SI-sequence. This result is the so-called g-Theorem (cf. Theorem 9.5). As observed by Boij [7], this implies that for any given SI-sequence there exists a reduced Gorenstein algebra whose h-vector is that sequence. However, as Harima points out, Stanley's methods used in [56] involve hard results about toric varieties and topology and apply in characteristic zero only. Thus it is worth giving a different proof. The first part of this paper gives a new proof of this fact, providing a "lifting" of Harima's Artinian result (but not his proof). Our approach is similar to his, but with some important differences which will be described shortly. More precisely, our first main result is the following (see Theorem 8.13):

Theorem 1.1. *Let $\underline{h} = (1, c, h_2, \dots h_{s-2}, c, 1)$ be an SI-sequence and let K be an arbitrary field. Then, for every integer $d \geq 0$, there is a reduced arithmetically Gorenstein scheme $G \subset \mathbb{P}^{c+d}_K$ of dimension d, with the weak Lefschetz property, whose h-vector is $\underline{h}$, provided the field K contains sufficiently many elements.*

We were not able to mimic the approach of Harima to prove Theorem 1.1 (see Remark 2.7 and Remark 7.8). Such an approach would amount to adding the ideals of certain reduced arithmetically Cohen-Macaulay subschemes of projective space which are linked *by a complete intersection*, thus forming a reducible arithmetically Gorenstein subscheme of codimension one more with a "bigger" Hilbert function than the desired one, and then removing components to obtain an arithmetically Gorenstein subscheme with the desired Hilbert function. Instead, our approach is to link using arithmetically Gorenstein ideals (i.e. G-links, in the terminology of [40]) rather than complete intersections (i.e. CI-links). We refer to [45] for background on liaison. To the best of our knowledge this is one of the first occurrences in the literature of using Gorenstein liaison to construct interesting subschemes of projective space. In fact we show in Remark 10.2 that complete intersections do not suffice to obtain all the Hilbert functions.

The main features of our construction are the following.

- A simple calculation shows that the desired Hilbert function can be obtained (at least numerically) by linking an arithmetically Cohen-Macaulay scheme with the "obvious" Hilbert function (namely the one suggested by the "first half" of the desired h-vector) inside an arithmetically Gorenstein scheme with "maximal" Hilbert function. This calculation is given in section 6. The hard part is to verify that suitable schemes can actually be found to carry this out.

- We observe that if a Gorenstein scheme, G, is a generalized stick figure then any link which it provides is geometric, i.e. for any subscheme X of the same codimension as G, if Y is the residual to X in G then X and Y have no common components. Furthermore, we observe that the sum of the ideals of X and Y will define a reduced union of linear varieties of codimension one greater.
- We give a useful construction of arithmetically Cohen-Macaulay schemes: Given a nested sequence $V_1 \subset V_2 \subset \cdots \subset V_r$ of generically Gorenstein arithmetically Cohen-Macaulay subschemes of the same dimension, the union of generally chosen hypersurface sections of the V_i is again an arithmetically Cohen-Macaulay subscheme of $\mathbb{P}^n$, and its Hilbert function is readily computed. This construction is central to the argument, and is of independent interest (cf. for instance [47]).
- Using the construction just mentioned, we show in section 4 how to make arithmetically Cohen-Macaulay generalized stick figures with "maximal" Hilbert function in any codimension. We give a precise primary decomposition of these configurations. We also show how a generalized stick figure with arbitrary Hilbert function can be found as a subconfiguration of such a "maximal" one.
- We show how to construct arithmetically Gorenstein generalized stick figures with "maximal" Hilbert function which contain the "maximal" generalized stick figures just mentioned. Note that the Artinian reductions of our arithmetically Cohen-Macaulay schemes with maximal Hilbert functions are compressed Cohen-Macaulay algebras in the sense of Iarrobino [36], [37]. We give a precise primary decomposition of these configurations.
- Tying things together, we show that this set-up allows us to carry out the program described in the first step above.

As a result, we have a purely combinatorial way of getting arithmetically Gorenstein schemes with the desired Hilbert functions. As a consequence, we have the interesting fact that our links are obtained by beginning with the "linking" arithmetically Gorenstein scheme and then finding *inside it* the subschemes which are to be linked. This is the opposite of the usual application of liaison.

A more precise statement of Harima's result is that *a sequence is an SI-sequence if and only if it is the Hilbert function of an Artinian Gorenstein algebra with the weak Lefschetz property.* In order to check that the arithmetically Gorenstein schemes that we construct have the weak Lefschetz property we need a higher-dimensional analogue of the weak Lefschetz property which does not refer to an Artinian reduction. To this end we introduce in Section 7 the *subspace property.* This property is even defined for arithmetically Cohen-Macaulay schemes. We show that this property is preserved under hyperplane sections (including Artinian reductions) and that, in the case of an Artinian Gorenstein algebra, it is equivalent to the weak Lefschetz property. Moreover, we prove that the arithmetically Gorenstein schemes constructed above have the subspace property, finishing the argument for the proof of Theorem 1.1.

Furthermore, the subspace property allows us to address a question posed to us by Tony Geramita: Harima showed that every SI-sequence is the Hilbert function of some Artinian Gorenstein algebra with the weak Lefschetz property. However, the question remained whether an Artinian reduction of a reduced arithmetically Gorenstein subscheme of projective space can possess the weak Lefschetz property. Since we do not know which Artinian Gorenstein algebras lift to reduced subschemes, it was just possible that for some SI-sequences there are no reduced arithmetically Gorenstein algebras whose Artinian reduction has that SI-sequence and has the Weak Lefschetz property. However, it follows immediately from Theorem 1.1 that this is not the case. This also follows from work of Stanley in case the ground field has characteristic zero. He has shown in [56] that in characteristic zero the Stanley-Reisner ring of a simplicial polytope has the weak Lefschetz property.

The last part of the paper addresses the question of resolutions. An important result in this direction was obtained in [19]. There the authors showed that for many, but not all, SI-sequences $\underline{h}$ they can produce an Artinian Gorenstein algebra A with Hilbert function $\underline{h}$ possessing the weak Lefschetz property, and such that among Artinian Gorenstein algebras with the weak Lefschetz property and Hilbert function $\underline{h}$, A has maximal graded Betti numbers. Our second main result of this paper is a generalization of this theorem for reduced arithmetically Gorenstein schemes of any dimension (since the Artinian reduction has the weak Lefschetz property and *every* SI-sequence can be produced) (see Theorem 8.13):

Theorem 1.2. *Let $\underline{h} = (1, c, h_2, \ldots, h_t, \ldots, h_s)$ be an SI-sequence. Then the scheme G described in Theorem 1.1 has maximal graded Betti numbers among arithmetically Gorenstein subschemes of $\mathbb{P}^n$ with the weak Lefschetz property and h-vector $\underline{h}$.*

If the embedding dimension satisfies $n + 1 \geq s + c$, we can choose the Gorenstein scheme G described in Theorem 1.1 as one defined by a reduced monomial ideal. Thus G corresponds to a simplicial complex. If P is a simplicial polytope then the Stanley-Reisner ring $K[\Delta(P)]$ of the boundary complex $\Delta(P)$ of P is a reduced Gorenstein K-algebra. The h-vector and the graded Betti numbers of P are the corresponding numbers of $K[\Delta(P)]$. The famous g-theorem mentioned above (see Theorem 9.5) characterizes the h-vectors of simplicial polytopes as SI-sequences. For the Betti numbers we have the following result analogous to Theorem 1.2 (see Theorem 9.6).

Theorem 1.3. *Let K be a field of characteristic zero and let $\underline{h}$ be an SI-sequence. Then there is a simplicial polytope with h-vector $\underline{h}$ having maximal graded Betti numbers among all simplicial polytopes with h-vector $\underline{h}$.*

For the proof we have to relate Theorem 1.2 and the results obtained in Sections 6 and 7 to the work of Billera and Lee in [5]. It uses the combinatorial description of the irreducible components of the arithmetically Gorenstein schemes occurring in Theorem 1.1 (cf. Theorem 6.3). This description allows an interpretation of a construction for simplicial polytopes with the help of Gorenstein linked ideals. In the course of the proof of Theorem 1.3 we also show that the upper bounds on the Betti numbers for arbitrary graded Cohen-Macaulay K-algebras with given Hilbert function in [3] and [35] are best possible even for shellable simplicial complexes.

The maximal graded Betti numbers mentioned in Theorems 1.2 and 1.3 can be computed effectively (cf. Theorems 8.13 and 9.6).

In spite of our results above and the conjectured generalization of the g-theorem to simplicial spheres, we would like to pose the following question:

Question 1.4. *Does* every *reduced, arithmetically Gorenstein subscheme of projective space possess the weak Lefschetz property?*

In case this question has an affirmative answer, this would imply that we have characterized all Hilbert functions of reduced arithmetically Gorenstein schemes and that among the reduced arithmetically Gorenstein schemes with fixed Hilbert function we always have the existence of a scheme with maximal Betti numbers.

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2. Background

We begin by recording the following notation and conventions.

- Let K be any field (of arbitrary characteristic). We set
  - $R = K[x_0, \ldots, x_n]$ and $\mathbb{P}^n = \operatorname{Proj} R$;
  - $T = K[z_1, \ldots, z_c]$ and $\bar{T} = K[z_2, \ldots, z_c]$, where $c < n$ (this will be used primarily in section 5).
  - $S = K[y_1, \ldots, y_c]$ (this will be used primarily in section 5).
- Following [40] we say that two homogeneous ideals I_1 and I_2 are CI-linked (resp. G-linked) if they are linked using a complete intersection (resp. a Gorenstein ideal which is not necessarily a complete intersection). That is, we require

$$J \subset I_1 \cap I_2, \quad J : I_1 = I_2, \quad J : I_2 = I_1$$

  where J is a complete intersection (resp. a Gorenstein ideal which is not necessarily a complete intersection).
- Let V_1 and V_2 respectively be the linked subschemes of $\mathbb{P}^n$ defined by the ideals I_1 and I_2 above. Then V_1 and V_2 are also said to be CI-linked (resp. G-linked). If X is the Gorenstein ideal defined by J, we write $V_1 \overset{X}{\sim} V_2$. The property of being linked forces V_1 and V_2 to be equidimensional of the same dimension (as X). The link is *geometric* if V_1 and V_2 have no common component.
- If X is a subscheme of $\mathbb{P}^n$ with saturated ideal I_X, and if $t \in \mathbb{Z}$ then the Hilbert function of X is denoted by

$$h_X(t) = h_{R/I_X}(t) = \dim_K [R/I_X]_t.$$

- If X is arithmetically Cohen-Macaulay of dimension d then $A = R/I_X$ has Krull dimension $d+1$ and a general set of $d+1$ linear forms forms a regular sequence for A. Taking the quotient of A by such a regular sequence gives a zero-dimensional Cohen-Macaulay ring called the *Artinian reduction* of A (or of X – cf. [45]). The

Hilbert function of the Artinian reduction of R/I_X is called the *h-vector* of R/I_X (or of X). This is a finite sequence of integers.
- If X is arithmetically Gorenstein with h-vector $(1, c, \dots, h_s)$ then this h-vector is symmetric ($h_s = 1$, $h_{s-1} = c$, etc.) and s is called the *socle degree* of X.

The underlying idea of [25] was to produce the desired arithmetically Gorenstein scheme as a sum of CI-linked, arithmetically Cohen-Macaulay codimension two schemes, chosen so that the complete intersection linking them is already a union of linear subvarieties with nice intersection properties. This guarantees that the intersection of the linked schemes will also be a good linear configuration. The deformation result arose as a consequence of work of Diesel [14] about the irreducibility of the Hilbert scheme of Gorenstein Artinian k-algebras of codimension three with given Hilbert function and degrees of generating sets.

Diesel's methods do not hold in higher codimension. However, as a first step in the above direction in higher codimension, we conjecture that every Hilbert function which arises as the Hilbert function of an arithmetically Gorenstein scheme in fact occurs for some arithmetically Gorenstein scheme which is a generalized stick figure. It is not yet known which Hilbert functions in fact arise in this way, but a large class is known thanks to [29] (as recalled below), and we will prove our conjecture for this class.

Our approach is to combine ideas of [29] and [25], with several new twists. We recall some facts and definitions from the former.

Definition 2.1. Let $\underline{h} = (h_0, \dots, h_s)$ be a sequence of positive integers. $\underline{h}$ is called a *Gorenstein sequence* if it is the Hilbert function of some Gorenstein Artinian k-algebra. $\underline{h}$ is *unimodal* if $h_0 \le h_1 \le \cdots \le h_j \ge h_{j+1} \ge \cdots \ge h_s$ for some j. $\underline{h}$ is called an SI-sequence (for Stanley-Iarrobino) if it satisfies the following two conditions:

(i) $\underline{h}$ is symmetric, i.e. $h_{s-i} = h_i$ for all $i = 0, \dots, \lfloor \frac{s}{2} \rfloor$.
(ii) $(h_0, h_1 - h_0, h_2 - h_1, \dots, h_j - h_{j-1})$ is an O-sequence, where $j = \lfloor \frac{s}{2} \rfloor$; i.e. the "first half" of $\underline{h}$ is a *differentiable* O-sequence.

Remark 2.2. From the above definition of an SI-sequence it follows that

$$(h_0, h_1 - h_0, h_2 - h_1, \dots, h_t - h_{t-1})$$

is an O-sequence, where $t = \min\{i | h_i \ge h_{i+1}\}$. However, we cannot replace the given condition by this one. For instance, the sequence

$$(1, 3, 6, 6, 7, 6, 6, 3, 1)$$

satisfies this condition (with $t = 2$) and is unimodal and symmetric, but the last condition of the definition is not satisfied, and we do not want to allow this kind of behavior.

Definition 2.3. A graded Artinian algebra $A = \bigoplus_{i-1}^{s} A_i$, with $A_s \neq 0$, is said to have the *weak Lefschetz property* (WLP for short) if A satisfies the following two conditions.

(i) The Hilbert function of A is unimodal.
(ii) There exists $g \in A_1$ such that the k-vector space homomorphism $g : A_i \to A_{i+1}$ defined by $f \mapsto gf$ is either injective or surjective for all $i = 0, 1, \dots, s-1$.

Since we are interested in the question of lifting, we will say that *an arithmetically Cohen-Macaulay subscheme X of $\mathbb{P}^n$ has the weak Lefschetz property* if there is an Artinian reduction of X having the weak Lefschetz property.

In [29], this property is also called the *weak Stanley property.* Harima proved the following very interesting result ([29] Theorem 1.2).

Theorem 2.4. *Let $\underline{h} = (h_0, h_1, \dots, h_s)$ be a sequence of positive integers. Then $\underline{h}$ is the Hilbert function of some Gorenstein Artinian k-algebra with the WLP if and only if $\underline{h}$ is an SI-sequence.*

This provides a huge class of Hilbert functions for which it is known that there is an Artinian Gorenstein ideal. In codimension 3 it is known that all Artinian Gorenstein ideals have Hilbert functions which are SI-sequences, and in codimension ≥ 5 it is known that not all do. (Codimension 4 is open.) Our goal is to provide, for each such SI-sequence in any codimension, a reduced set of points (or more generally a reduced union of linear varieties) which is arithmetically Gorenstein and whose h-vector is the given SI-sequence. See [45] for more on h-vectors.

It is well known that the sum of the ideals of two geometrically linked, arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^n$ is arithmetically Gorenstein of height one greater, whether they are CI-linked [50] or G-linked (cf. [45]). Harima ([29], Lemma 3.1) has computed the Hilbert function of the Gorenstein ideals so obtained in the case of CI-linkage; the proof is the same for G-linkage. However, we would like to record this result in a different (but equivalent) way, more in line with our needs.

Lemma 2.5. *Let $V_1 \overset{X}{\sim} V_2$, where X is arithmetically Gorenstein, V_1 and V_2 are arithmetically Cohen-Macaulay of codimension c with saturated ideals I_{V_1} and I_{V_2}, and the link is geometric. Then $I_{V_1} + I_{V_2}$ is the saturated ideal of an arithmetically Gorenstein scheme Y of codimension $c+1$. The Hilbert functions are related as follows.*

Let $\underline{c} = (1, c, c_2, \dots, c_{s-1}, c_s)$ be the h-vector of X; note that $c_{s-1} = c$ and $c_s = 1$. Let $\underline{g} = (1, g_1, \dots, g_r)$ be the h-vector of V_1 (so we have $g_1 \leq c$, with equality if and only if V_1 is non-degenerate, and $r \leq s$). Let $\underline{g}' = (1, g_1', \dots)$ be the h-vector of V_2. By [13] *Theorem 3 (see also* [45] *Corollary 5.2.19) we have the h-vector of V_2 given by*

$$g_i' = c_{s-i} - g_{s-i}$$

for $i \geq 0$. Then the sequence $d_i = (g_i + g_i' - c_i)$ is the first difference *of the h-vector of Y.*

Example 2.6. A twisted cubic curve V_1 in $\mathbb{P}^3$ is linked to a line V_2 by the complete intersection of two quadrics. The intersection of these curves is the arithmetically Gorenstein zeroscheme Y consisting of two points. This is reflected in the diagram

$$\begin{array}{rccc} X: & 1 & 2 & 1 \\ V_1: & 1 & 2 & \\ V_2: & 1 & & \\ \Delta Y: & 1 & 0 & -1 \end{array}$$

adding the second and third rows and subtracting the first to obtain the fourth, and so the h-vector of Y is (1,1), obtained by "integrating" the vector $(1, 0-1)$. The notation

ΔY serves as a reminder that the row is really the first difference of the h-vector of Y, and it will be used in the remaining sections.

Remark 2.7. Many papers on Hilbert functions or minimal free resolutions of Gorenstein ideals (for instance [19], [20] [21], [25], [26], [27], [28]) use as the method of constructing Gorenstein ideals this notion of adding the ideals of geometrically CI-linked Cohen-Macaulay ideals. In codimension three this was enough, as shown in [25]. However, in higher codimension this is not enough. For instance, in [21] Remark 3.5 the authors say that with their construction method they cannot obtain the h-vector

$$1 \quad 4 \quad 10 \quad n \quad 10 \quad 4 \quad 1$$

where $14 \leq n \leq 19$. Indeed, one can check that taking $n = 19$, it is impossible to obtain such a Hilbert function as the sum of geometrically CI-linked Cohen-Macaulay ideals (see Example 10.2).

Of course such an h-vector is a special case of Harima's theorem. Harima begins with the same complete intersection trick, but he adds a very nice extra ingredient which we summarize briefly in Lemma 2.9. We would like to remark that to extend this approach to the non-Artinian case, one would have to prove that the sum of CI-linked ideals can be done in such a way that a very precisely determined subset of the resulting Gorenstein set of points can always be found lying on a hyperplane, and that the residual is again Gorenstein, arriving at the desired h-vector. (Note, however, that Harima's trick works for a general linear form, while in higher dimension the hyperplane is very special since it contains a large number of the points.) In this paper we do not take this approach. However, we show in Section 7 that the configurations that we obtain have what we call the "subspace property," and this corresponds precisely to the higher dimensional analog of Harima's trick. See Remark 7.8. This is a by-product of a completely different approach using Gorenstein liaison.

Definition 2.8. Let X be a finite set of points in $\mathbb{P}^n$ with Hilbert function $h_X(i)$. Then $\sigma(X) = \min\{i | \Delta h_X(i) = 0\}$.

Lemma 2.9 ([29], Lemma 3.3)**.** *Let X and Y be two finite sets of points in $\mathbb{P}^n$ such that $X \cap Y = \emptyset$ and $X \cup Y$ is a complete intersection, and put $A = K[x_0, \dots, x_n]/(I(X)+I(Y))$. Furthermore put $a = \sigma(X) - 1$, $b = \sigma(X \cup Y) - \sigma(X) - 1$, $c = \sigma(X \cup Y) - 1$. Assume that $2\sigma(X) \leq \sigma(X \cup Y)$ and $|X| \geq 2$. Let $L \subset \mathbb{P}^n$ be a hyperplane defined by a polynomial $G \in R_1$ and let $g \in A_1$ be the image of G. Assume that $X \cap L = \emptyset$. Let d be an integer such that $1 \leq d \leq \sigma(X \cup Y) - 2\sigma(X)$ and let $[0 : g^d]$ denote the homogeneous ideal generated by homogeneous elements $f \in A$ such that $g^d f = 0$. Then the Hilbert function $h_{A/[0:g^d]}(i)$ is a Gorenstein SI-sequence as follows:*

$$h_{A/[0:g^d]}(i) = \begin{cases} h_X(i), & \text{if } i = 0, 1, \dots, a-1, \\ |X|, & \text{if } i = a, \dots, b-d, \\ h_X(c-1-i-d), & \text{if } i = b+1-d, \dots, c-1-d,, \end{cases} \tag{2.1}$$

and $\sigma(A/[0 : g^d]) = c - d$.

Remark 2.10. Let $X \subset \mathbb{P}^n$ be a projective subscheme. Then its Castelnuovo-Mumford regularity is

$$\operatorname{reg}(X) := \operatorname{reg}(I_X) = \min\{j \mid h^i(\mathbb{P}^n, \mathcal{I}_X(j-i) = 0 \text{ for all } i \text{ with } 1 \leq i \leq \dim X + 1 \}.$$

If X is arithmetically Cohen-Macaulay and its h-vector is $(1, h_1, \dots, h_s)$ with $h_s > 0$ then $\operatorname{reg}(X) = s + 1$. Thus, we observe that $\sigma(X) = \operatorname{reg}(X)$ if X is a zeroscheme.

3. Generalized stick figures and a useful construction

In this section we give a construction which is useful for producing arithmetically Cohen-Macaulay subschemes of projective space, especially unions of linear varieties with "nice" singularities. It is an application of a result in [40]. As a consequence, in the next section we show how to construct such unions of linear varieties with "maximal" Hilbert function among the arithmetically Cohen-Macaulay schemes with fixed regularity and initial degree.

The notion of a *stick figure* curve was introduced classically, and it has culminated with the recent solution by Hartshorne of the so-called *Zeuthen problem* [31]. A stick figure is simply a union of lines, no three of which meet in a point. We will make use of the following type of configuration, which was introduced for codimension two in [10] and in this generality in [46].

Definition 3.1. Let V be a union of linear subvarieties of $\mathbb{P}^m$ of the same dimension d. Then V is a *generalized stick figure* if the intersection of any three components of V has dimension at most $d-2$ (where the empty set is taken to have dimension -1). In particular, if $d = 1$ then V is a stick figure.

Remark 3.2. The property of being a generalized stick figure has a useful consequence in liaison. As mentioned in Lemma 2.5, if V_1 and V_2 are arithmetically Cohen-Macaulay subschemes of projective space which are geometrically linked by an arithmetically Gorenstein scheme X_c (c is the codimension) then $I_{V_1} + I_{V_2}$ is the saturated ideal of an arithmetically Gorenstein subscheme X_{c+1} of codimension $c+1$. In this paper we want to use this fact to construct many reduced arithmetically Gorenstein subschemes of projective space. A key observation for us is thus the following. If our "linking scheme" $X_c \subset \mathbb{P}^n$ is a generalized stick figure, with $c < n$, then the sum of the linked ideals defines the arithmetically Gorenstein scheme X_{c+1} which is again a reduced union of linear varieties. Indeed, the components of X_{c+1} are generically the intersection of two linear varieties.

Remark 3.3. Let $\{L_1, \dots, L_p\}$ be a set of generically chosen linear forms in the ring $K[x_0, \dots, x_N]$ (i.e. hyperplanes in $\mathbb{P}^N$). Let

$$\begin{array}{rcl} A_1 & = & (L_{i_1}, \dots, L_{i_c}) \\ A_2 & = & (L_{j_1}, \dots, L_{j_c}) \\ A_3 & = & (L_{k_1}, \dots, L_{k_c}) \end{array}$$

be three *different* ideals generated by subsets of $\{L_1, \dots, L_p\}$ defining codimension c linear varieties. Then these three varieties meet in codimension $c+1$ if and only if there are exactly $c+1$ different linear forms among the $3c$ generators. That is, the union of these three varieties violate the condition of being a generalized stick figure if and only if there are exactly $c+1$ different linear forms among the $3c$ generators of the ideals A_1, A_2 and A_3.

We now give our construction for arithmetically Cohen-Macaulay schemes. We begin by recalling the notion of *Basic Double G-linkage* introduced in [40], so called because of part (iv) and the notion of Basic Double Linkage ([41], [9], [24]).

Lemma 3.4 ([40] Lemma 4.8, Remark 4.9 and Proposition 5.10)**.** *Let $J \supset I$ be homogeneous ideals of $R = K[x_0, \dots, x_n]$, defining schemes $W \subset V \subset \mathbb{P}^n$ such that* $\operatorname{codim} V + 1 = \operatorname{codim} W$. *Let $f \in R$ be an element of degree d such that $I : f = I$. Then we have*

(i) $\deg(I + f \cdot J) = d \cdot \deg I + \deg J$.
(ii) *If I is perfect and J is unmixed then $I + f \cdot J$ is unmixed.*
(iii) $J/I \cong [(I + f \cdot J)/I](d)$.
(iv) *If V is arithmetically Cohen-Macaulay with property G_0 and J is unmixed then J and $I + f \cdot J$ are linked in two steps using Gorenstein ideals.*
(v) *The Hilbert functions are related by*

$$\begin{aligned} h_{R/(I+f\cdot J)}(t) &= h_{R/(I+(f))}(t) + h_{R/J}(t-d) \\ &= h_{R/I}(t) - h_{R/I}(t-d) + h_{R/J}(t-d) \end{aligned}$$

Lemma 3.4 should be interpreted as viewing the scheme W defined by J as a divisor on the scheme V defined by I, and adding to it a hypersurface section H_f of V defined by the polynomial f. Note that $I_{H_f} = I_V + (f)$. If V and W are arithmetically Cohen-Macaulay then the divisor $W + H_f$ is again arithmetically Cohen-Macaulay (by step 4). As an immediate application we have the following by successively applying Lemma 3.4.

Corollary 3.5. *Let $V_1 \subset V_2 \subset \dots \subset V_r \subset \mathbb{P}^n$ be arithmetically Cohen-Macaulay schemes of the same dimension, each generically Gorenstein. Let $H_1, \dots, H_r$ be hypersurfaces, defined by forms $F_1, \dots, F_r$, such that for each i, H_i contains no component of V_j for any $j \le i$. Let W_i be the arithmetically Cohen-Macaulay schemes defined by the corresponding hypersurface sections: $I_{W_i} = I_{V_i} + (F_i)$. Then*

(i) *viewed as divisors on V_r, the sum Z of the W_i (which is just the union if the hypersurfaces are general enough) is in the same Gorenstein liaison class as W_1.*
(ii) *In particular, Z is arithmetically Cohen-Macaulay.*
(iii) *As ideals we have*

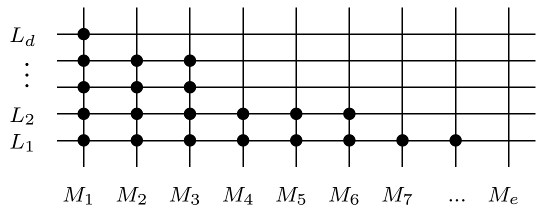
$$I_Z = I_{V_r} + F_r \cdot I_{V_{r-1}} + F_r F_{r-1} I_{V_{r-2}} + \dots + F_r F_{r-1} \cdots F_2 I_{V_1} + (F_r F_{r-1} \cdots F_1).$$

(iv) *Let $d_i = \deg F_i$. The Hilbert functions are related by the formula*

$$\begin{aligned} h_Z(t) &= h_{W_r}(t) + h_{W_{r-1}}(t-d_r) + h_{W_{r-2}}(t-d_r-d_{r-1}) + \dots \\ &\quad + h_{W_1}(t-d_r-d_{r-1}-\dots-d_2). \end{aligned}$$

Remark 3.6. Parts (ii), (iii) and (iv) of Corollary 3.5 have been proved independently by Ragusa and Zappalà ([52] Lemma 1.5).

Corollary 3.7. *Let $R = K[x_0, \dots, x_n]$. Consider the complete intersection (A, B) in R, where $A = \prod_{i=1}^d L_i$ and $B = \prod_{i=1}^e M_i$. Thinking of the L_i and M_i as hyperplanes which are pairwise linearly independent, the intersection of any L_i with M_k is a codimension* *two linear variety, $P_{i,k}$. Consider a union Z of such varieties subject to the condition that if $P_{i,k} \subset Z$ then $P_{j,\ell} \subset Z$ for all (j,ℓ) satisfying $j \leq i$ and $\ell \leq k$:*



Then Z is arithmetically Cohen-Macaulay.

Proof. Apply Corollary 3.5, using $V_1 = M_1$, $V_2 = M_1 \cup M_2$, etc. and taking the hypersurface sections obtained by suitable multiples of the L_i (starting with L_d and working backwards), working our way down the picture. This corollary can also be obtained using the lifting results of [46]. □

Remark 3.8. In the case $n = 2$ these are not necessarily the k-configurations of [28], since consecutive lines L_i are allowed to have the same number of points. This situation has been studied in [26], and such a configuration in $\mathbb{P}^2$ is a special case of a *weak k-configuration*. (In general a weak configuration does not require the existence of the "vertical" M_i.) It was shown in [26] that the Hilbert function of a weak k-configuration in $\mathbb{P}^2$ can be immediately read from the number of points on each line. This type of special case was also extended to higher codimension in [46], section 4.

In the case of lines in $\mathbb{P}^3$, Corollary 3.7 is essentially contained in [1].

4. A construction of arithmetically Cohen-Macaulay and Gorenstein ideals with "maximal" Hilbert function

In this section we show how to construct certain Gorenstein subschemes of projective space with two properties: they are generalized stick figures and they have very special Hilbert function (maximal until degree t, then constant until degree $s-t$; note that the "flat" part in the middle can be arbitrarily long). The Hilbert function that we seek, for codimension c, corresponds to the h-vector

$$\left(1, c, \binom{c+1}{c-1}, \dots, \binom{c-1+t}{c-1}, \dots, \binom{c-1+t}{c-1}, \dots, \binom{c+1}{c-1}, c, 1\right) \tag{4.1}$$

where the terms in the middle are all equal to $\binom{c-1+t}{c-1}$ and the last non-zero entry is in degree s. Then we have that the first occurrence of the value $\binom{c-1+t}{c-1}$ is in degree t and the last in degree $s-t$. Thus, the h-vector is the maximum bounded above by $\binom{c-1+t}{c-1}$, of socle degree (the degree of the last non-zero term) s.

In order to construct such an ideal, we will first give a construction of arithmetically Cohen-Macaulay ideals with maximal Hilbert function (expressed as an h-vector)

$$\left(1, c, \binom{c+1}{2}, \dots, \binom{c-1+t}{t}\right). \tag{4.2}$$

The procedure is inductive, producing first a suitable answer in codimension one, then codimension two, then codimension three, etc. We will construct, in codimension c, an arithmetically Cohen-Macaulay generalized stick figure $Z_{c,t}$ with "maximal" h-vector given by (4.2).

Theorem 4.1. *Let R be a polynomial ring of dimension $n+1 > c$. Choose a set of $2t+c$ linear forms*

$$\mathcal{M}_{c,t} = \left\{ M_0, \dots, M_{t+\lfloor \frac{c-1}{2} \rfloor}, L_0, \dots, L_{t+\lfloor \frac{c-2}{2} \rfloor} \right\}$$

in R. Define

$$\begin{aligned} I_{Z_{c,t}} &= \bigcap_{0 \le i_1 \le i_2 < i_3 \le i_4 < \dots < i_{c-1} \le i_c \le t + \frac{c-2}{2}} (M_{i_1}, L_{i_2}, M_{i_3}, L_{i_4}, \dots, L_{i_c}) \quad \text{if } c \text{ is even,} \\ I_{Z_{c,t}} &= \bigcap_{0 \le i_1 \le i_2 < i_3 \le i_4 < \dots \le i_{c-1} < i_c \le t + \frac{c-1}{2}} (M_{i_1}, L_{i_2}, M_{i_3}, L_{i_4}, \dots, M_{i_c}) \quad \text{if } c \text{ is odd.} \end{aligned} \tag{4.3}$$

(If $c = 1$ the above range is understood to be $0 \le i_1 \le t$.) If each subset consisting of $c+1$ of the $2t+c$ linear forms generates an ideal of codimension $c+1$ then $I_{Z_{c,t}}$ is a reduced Cohen-Macaulay ideal with h-vector

$$\left(1, c, \binom{c+1}{2}, \dots, \binom{c+t-1}{t}\right).$$

If each subset consisting of $c+2$ of the $2t+c$ linear forms generates a complete intersection of codimension $c+2$ then $I_{Z_{c,t}}$ defines a generalized stick figure.

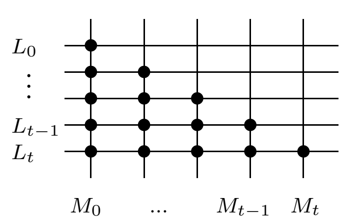
Proof. Consider the complete intersection (A, B) in codimension 2, where

$$A = \prod_{i=0}^{t+\lfloor \frac{c-1}{2} \rfloor} M_i \quad \text{and} \quad B = \prod_{i=0}^{t+\lfloor \frac{c-2}{2} \rfloor} L_i$$

(note that both products start with $i = 0$). Let us denote the scheme defined by (A, B) by G_2 (subscripts here will refer to the codimension).

When $c = 2$, the scheme $Z_{2,t}$ has the following form (where the components of $Z_{2,t}$ are represented by dots, and the set of all intersection points is G_2):

(4.4)



Using Corollary 3.7, it follows immediately that the h-vector of $Z_{2,t}$ is

$$(1, 2, 3, 4, \dots, t, t+1)$$

as claimed.

Now consider the case $c = 3$. Let V_1 be the subset of $Z_{2,t}$ consisting of all "dots" lying on L_0 (there is just one). Let V_2 be the union of the components of X lying on *either* L_0 *or* L_1, and so on. Clearly we have

$$V_1 \subset V_2 \subset \cdots \subset V_{t+1}$$

and all V_i are arithmetically Cohen-Macaulay. We will apply Corollary 3.5, so to that end let us set $F_i = M_i$ $(1 \leq i \leq t+1)$ and let W_i the hyperplane section of V_i with F_i. Then we have

$$\begin{aligned} \deg W_1 &= 1 \\ \deg W_2 &= 1+2=3 \\ &\vdots \\ \deg W_{t+1} &= 1+2+\cdots+(t+1) = \tbinom{t+2}{2} \end{aligned}$$

and by Corollary 3.5 the union of the W_i is arithmetically Cohen-Macaulay of codimension 3 with the h-vector

$$\left(1, 3, \ldots, \binom{t+2}{2}\right)$$

(use Lemma 3.4). Clearly we also have

$$W_1 \subset W_1 \cup W_2 \subset W_1 \cup W_2 \cup W_3 \cdots .$$

Remark 3.3 quickly shows that each of these is a generalized stick figure. But clearly the union of the W_i has the form described in (4.3), so this is $Z_{3,t}$ and we have completed the case $c = 3$.

To pass to codimension 4 using Corollary 3.5 again, we now take $V_1, \ldots, V_t$ to be the codimension 3 schemes just produced (i.e. set V_1 now to be the W_1 just produced, V_2 to be the $W_1 \cup W_2$ just produced, etc.), and we take $F_i = L_i$ $(1 \leq i \leq t+1)$. As before we have a generalized stick figure in codimension 4 which is arithmetically Cohen-Macaulay and has maximal Hilbert function. Note that the components of the arithmetically Cohen-Macaulay scheme $Z_{4,t}$ we have produced have precisely the form described in (4.3).

We continue by induction to finally produce the desired arithmetically Cohen-Macaulay scheme of codimension c with h-vector given by (4.2). (Note that the range for F_i changes: for instance, in codimension 5 we have $F_i = M_i$ $(2 \leq i \leq t+2)$. It is a generalized stick figure by Remark 3.3. $\square$

Remark 4.2. We will now turn to the construction of the arithmetically Gorenstein generalized stick figure with h-vector given by (4.1). First we check numerically what is needed, using Lemma 2.5 (see also Example 2.6). In the table below, the values occurring between degrees $t+1$ and $s-t$ on any row are constant. G_{c-1} is the h-vector of a codimension $c-1$ arithmetically Gorenstein scheme which links arithmetically Cohen-Macaulay schemes Z_{c-1} to Y_{c-1}, and the sum of linked ideals gives an arithmetically Gorenstein scheme G_c with the h-vector claimed in (4.1).

	degree:												
	0	1	2	...	t	$t+1$	...	$s-t$	$s-t+1$	...	s	$s+1$
G_{c-1}	1	$c-1$	$\binom{c}{c-2}$	...	$\binom{c-2+t}{c-2}$	$\binom{c-2+t}{c-2}$	...	$\binom{c-2+t}{c-2}$	$\binom{c-2+t}{c-2}$	...	$c-1$	1
Z_{c-1}	1	$c-1$	$\binom{c}{c-2}$	...	$\binom{c-2+t}{c-2}$							
Y_{c-1}	1	$c-1$	$\binom{c}{c-2}$	...	$\binom{c-2+t}{c-2}$	$\binom{c-1+t}{c-2}$	...	$\binom{c-1+t}{c-2}$				
ΔG_c	1	$c-1$	$\binom{c}{c-2}$	...	$\binom{c-2+t}{c-2}$	0	...	0	$-\binom{c-2+t}{c-2}$	...	$-(c-1)$	-1
G_c	1	c	$\binom{c+1}{c-1}$	...	$\binom{c-1+t}{c-1}$	$\binom{c-1+t}{c-1}$	...	$\binom{c-1+t}{c-1}$	$\binom{c-2+t}{c-1}$	...	1	0

Note that in the above table, if we take as G_{c-1} a Gorenstein scheme which "levels off" in degree $t+1$ rather than degree t, then the residual Hilbert function changes, but the Hilbert function of the sum of the linked ideals, G_c, does not change.

The basic idea of our proof will be to assume by induction that a generalized stick figure G_{c-1} exists with the desired Hilbert function, and that it contains $Z_{c-1,t}$, hence giving us a geometric link. Then we have to show that the scheme G_c obtained by adding the linked ideals is again a generalized stick figure and contains $Z_{c,t}$ (which we have already described). This allows for the construction in the next codimension, hence completing the induction. In fact, we are able to give the components of G_c very explicitly! We will refine the notation for G_c in the proof, to account for more data.

Theorem 4.3. *Let R be a polynomial ring of dimension $n+1 > c \geq 1$. Let $s \geq 2t$. Let*

$$\mathcal{N}_{c,s,t} = \left\{ M_0, \ldots, M_{t+\lfloor \frac{c-1}{2} \rfloor}, L_0, \ldots, L_{s-t+\lfloor \frac{c-2}{2} \rfloor} \right\}$$

be a subset of $s+c$ linear forms in R. Define an ideal $I_{G_{c,s,t}}$ as follows. If c is even then $I_{G_{c,s,t}} = A_{c,s,t} \cap B_{c,s,t} \cap C_{c,s,t}$ where

$$\begin{aligned}
A_{c,s,t} &= \bigcap_{0 \leq i_1 \leq i_2 < i_3 \leq \cdots < i_{c-1} \leq i_c \leq t+\frac{c-2}{2}} (M_{i_1}, L_{i_2}, \ldots, M_{i_{c-1}}, L_{i_c}), \\
B_{c,s,t} &= \bigcap_{0 \leq i_1 < i_2 \leq i_3 < \cdots \leq i_{c-1} < i_c \leq t+\frac{c-2}{2}} (L_{i_1}, M_{i_2}, \ldots, L_{i_{c-1}}, M_{i_c}), \\
C_{c,s,t} &= \bigcap_{\substack{0 \leq i_1 \leq i_2 < i_3 \leq \cdots < i_{c-1} \leq t+\frac{c-2}{2} \\ t+\frac{c}{2} \leq i_c \leq s-t+\frac{c-2}{2}}} (M_{i_1}, L_{i_2}, \ldots, M_{i_{c-1}}, L_{i_c}).
\end{aligned}$$

If c is odd then $I_{G_{c,s,t}} = A'_{c,s,t} \cap B'_{c,s,t} \cap C'_{c,s,t}$ *where*

$$\begin{aligned} A'_{c,s,t} &= \bigcap_{0 \le i_1 \le i_2 < i_3 \le \cdots \le i_{c-1} < i_c \le t + \frac{c-1}{2}} (M_{i_1}, L_{i_2}, \ldots, L_{i_{c-1}}, M_{i_c}), \\ B'_{c,s,t} &= \bigcap_{0 \le i_1 < i_2 \le i_3 < \cdots < i_{c-1} \le i_c \le t + \frac{c-3}{2}} (L_{i_1}, M_{i_2}, \ldots, M_{i_{c-1}}, L_{i_c}), \\ C'_{c,s,t} &= \bigcap_{\substack{0 \le i_1 \le i_2 < i_3 \le \cdots \le i_{c-1} \le t + \frac{c-3}{2} \\ t + \frac{c-1}{2} \le i_c \le s - t + \frac{c-3}{2}}} (M_{i_1}, L_{i_2}, \ldots, L_{i_{c-1}}, L_{i_c}). \end{aligned}$$

If each subset of $\mathcal{N}_{c,s,t}$ consisting of $c+1$ elements generates a complete intersection then $I_{G_{c,s,t}}$ is a reduced Gorenstein ideal contained in $I_{Z_{c,t}}$ with h-vector

$$\left(1, c, \binom{c+1}{2}, \ldots, \underbrace{\binom{c+t-1}{t}, \ldots, \binom{c+t-1}{t}}_{\text{flat part}}, \ldots, \binom{c+1}{2}, c, 1\right)$$

where the final "1" occurs in degree s, and $t+1$ is the initial degree of the ideal if $c \geq 2$. If each subset of $\mathcal{N}_{c,s,t}$ consisting of $c+2$ elements generates a complete intersection of codimension $c+2$ then $G_{c,s,t}$ is a generalized stick figure.

Proof. It is clear from the description that $I_{G_{c,s,t}}$ is a reduced ideal.

Next we verify that if $\mathcal{N}_{c,s,t}$ has the property that each subset of $c+2$ elements generates a complete intersection of codimension $c+2$ then $G_{c,s,t}$ is a generalized stick figure. To prove this, consider first the case c even. Suppose we have three components

$$\begin{aligned} P &= (p_1, p_2, \ldots, p_c) \\ Q &= (q_1, q_2, \ldots, q_c) \\ R &= (r_1, r_2, \ldots, r_c) \end{aligned}$$

(each taken from $A_{c,s,t}$, $B_{c,s,t}$ or $C_{c,s,t}$). In order for P and Q to have $c-1$ entries in common, we must be able to take out an entry from P and replace it with a new entry, giving Q. Putting both of these entries in and removing a third one must give R. Because of the rigid form of the components given in the statement of the theorem, one can just check that this is impossible.

Next suppose c is odd and $s = 2t$. Note that in this case $C'_{c,s,t}$ is empty. Then the argument is similar to the one above.

The case where c is odd and $s > 2t$ is handled similarly. Here is it slightly tricky to prove that we get a generalized stick figure because of the fact that condition $C'_{c,s,t}$ allows two consecutive $L's$ at the end. But the subscript of the second one is bounded below by $t + \frac{c-1}{2}$, and this fact is needed to complete the proof. For example, the linear forms

$$\begin{aligned} &(M_1, L_3, M_5, L_7, M_8) \\ &(M_1, L_3, M_5, L_7, L_9) \\ &(L_3, M_5, L_7, M_8, L_9) \end{aligned}$$

seem at first glance to be a counter-example, but the "9" in L_9 cannot simultaneously be bounded above by $t+\frac{c-3}{2}$ and below by $t+\frac{c-1}{2}$.

The fact that $Z_{c,t} \subset G_{c,s,t}$ follows from the observation that $A_{c,s,t}$ (resp. $A'_{c,s,t}$) is $I_{Z_{c,t}}$, thanks to Theorem 4.1.

It remains to show that $I_{G_{c,s,t}}$ is a Gorenstein ideal with the correct h-vector. For this we proceed by induction on $c \geq 1$. If $c = 1$ then $I_{G_{1,s,t}} = (M_0 \cdot \ldots \cdot M_t \cdot L_0 \cdot \ldots \cdot L_{s-t-1})$ is a principal ideal of degree $s+1$, thus having the claimed properties.

If $c = 2$ then $I_{G_{2,s,t}} = (M_0 \cdot \ldots \cdot M_t, L_0 \cdot \ldots \cdot L_{s-t})$ is a complete intersection by assumption on $\mathcal{N}_{c,s,t}$. It is easy to check its h-vector.

Now suppose that $c \geq 3$. We distinguish two cases.

Case 1: Assume that c is odd. We have at our disposal the set

$$\mathcal{N}_{c,s,t} = \left\{ M_0, \ldots, M_{t+\frac{c-1}{2}}, L_0, \ldots, L_{s-t+\frac{c-3}{2}} \right\}$$

where each subset of $c+1$ elements is linearly independent (over K). Let us temporarily re-name the linear form $M_{t+\frac{c-1}{2}}$ with the new name $L_{s-t+\frac{c-1}{2}}$. Hence we now have the set

$$\mathcal{N}_{c-1,s+1,t} = \left\{ M_0, \ldots, M_{t+\frac{c-3}{2}}, L_0, \ldots, L_{s-t+\frac{c-1}{2}} \right\}.$$

By induction we can use $\mathcal{N}_{c-1,s+1,t}$ to get the Gorenstein ideal $I_{G_{c-1,s+1,t}}$ which defines a generalized stick figure. The configuration $Z_{c-1,t}$ is formed using the set $\mathcal{M}_{c-1,t} = \{M_0, \ldots, M_{t+\frac{c-3}{2}} L_0, \ldots, L_{t+\frac{c-3}{2}}\}$. Since $s \geq 2t$, we observe that

$$\mathcal{M}_{c-1,t} \subset \mathcal{N}_{c-1,s+1,t} \backslash \{L_{s-t+\frac{c-1}{2}}\}. \tag{4.5}$$

Now we re-name $L_{s-t+\frac{c-1}{2}}$ back to $M_{t+\frac{c-1}{2}}$. The configuration $G_{c-1,s+1,t}$ then becomes (with the re-naming) a configuration, which we now call $G'_{c-1,s+1,t}$, whose ideal is

$$I_{G'_{c-1,s+1,t}} = A_{c-1,s+1,t} \cap B_{c-1,s+1,t} \cap \tilde{C}_{c-1,s+1,t} \cap \tilde{D}_{c-1,s+1,t} \tag{4.6}$$

where

$$\begin{aligned} \tilde{C}_{c-1,s+1,t} &= \bigcap_{\substack{0 \leq i_1 \leq i_2 < i_3 \leq \cdots < i_{c-2} \leq t+\frac{c-3}{2} \\ t+\frac{c-1}{2} \leq i_{c-1} \leq s-t+\frac{c-3}{2}}} (M_{i_1}, L_{i_2}, \ldots, M_{i_{c-2}}, L_{i_{c-1}}) \\ \tilde{D}_{c-1,s+1,t} &= \bigcap_{0 \leq i_1 \leq i_2 < i_3 \leq \cdots < i_{c-2} \leq t+\frac{c-3}{2}} (M_{i_1}, L_{i_2}, \ldots, M_{i_{c-2}}, M_{t+\frac{c-1}{2}}), \end{aligned}$$

where it is understood that $\tilde{C}_{c-1,s+1,t} = R$ if $s = 2t$.

By the observation (4.5), this re-naming does not affect any component of $Z_{c-1,t}$, which is defined by $A_{c-1,s+1,t}$. Of course the naming of the linear forms does not affect the properties of the configurations, and so all together we have that $G'_{c-1,s+1,t}$ is an arithmetically Gorenstein generalized stick figure with the same h-vector as $G_{c-1,s+1,t}$ and containing $Z_{c-1,t}$.

Since $G'_{c-1,s+1,t}$ is a generalized stick figure containing $Z_{c-1,t}$, it provides a geometric link to the residual

$$Y = G'_{c-1,s+1,t} \backslash Z_{c-1,t}.$$

By Remark 4.2, the ideal $I_{G_{c,s,t}}$ given in the statement of the theorem will have the desired properties if we can show that $I_{G_{c,s,t}} = I_{Z_{c-1,t}} + I_Y$.

We have to show

(a) For every choice of a component from $A_{c-1,s+1,t}$ (i.e. from $Z_{c-1,t}$) and a component from either $B_{c-1,s+1,t}$, $\tilde{C}_{c-1,s+1,t}$ or $\tilde{D}_{c-1,s+1,t}$ *which meet in codimension* c, their intersection occurs in either $A'_{c,s,t}$, $B'_{c,s,t}$ or $C'_{c,s,t}$.

(b) Every component of $A'_{c,s,t}$, $B'_{c,s,t}$, or $C'_{c,s,t}$ is the intersection of a component from $A_{c-1,s+1,t}$ and a component from either $B_{c-1,s+1,t}$, $\tilde{C}_{c-1,s+1,t}$ or $\tilde{D}_{c-1,s+1,t}$.

All of these involve an analysis of how many linear forms can be common to two (or three) of the components of our configurations. For (a), in order for the two components in codimension $c-1$ to meet in codimension c, their intersection must be defined by only c linear forms, so they must have exactly $c-2$ linear forms in common. We have to determine all the ways that this can happen and show that we always get a component of $G_{c,s,t}$. This analysis then works backwards to show just how to produce any component of $G_{c,s,t}$ as the intersection of two components of $G'_{c-1,s+1,t}$, which answers (b).

To answer these two questions, we leave it to the reader to verify that in order to meet in codimension c,

- a component from $A_{c-1,s+1,t}$ (i.e. $Z_{c-1,t}$) meets a component from $B_{c-1,s+1,t}$ either in the form $B'_{c,s,t}$ (and *all* components of $B'_{c,s,t}$ arise in this way) or in the form $A'_{c,s,t}$ (and all components of $A'_{c,s,t}$ arise in this way *except* those where $M_{i_c} = M_{t+\frac{c-1}{2}}$).
- a component from $A_{c-1,s+1,t}$ meets a component from $\tilde{C}_{c-1,s+1,t}$ in the form $C'_{c,s,t}$ (and *all* components of $C'_{c,s,t}$ arise in this way).
- a component from $A_{c-1,s+1,t}$ meets a component from $\tilde{D}_{c-1,s+1,t}$ in the form $A'_{c,s,t}$ where $M_{i_c} = M_{t+\frac{c-1}{2}}$, taking care of those components which were "missing" from the first set above.

To do this one checks how it is possible to remove one entry from the first component and one entry from the second and have the remaining entries equal. (There are very few possibilities.)

Case 2: Assume that c is even. Then $\mathcal{N}_{c,s,t} = \mathcal{N}_{c-1,s+1,t}$. By induction, the arithmetically Gorenstein scheme $G_{c-1,s+1,t}$ contains $Z_{c-1,t}$. We put $Y := G_{c-1,s+1,t} \backslash Z_{c-1,t}$, the residual scheme. The assertion follows because $I_{G_{c,s,t}} = I_{Z_{c-1,t}} + I_Y$, which can be shown as in Case 1 (and is easier).

This completes the construction of the arithmetically Gorenstein generalized stick figure with "maximal" h-vector. $\square$

Example 4.4. To construct the h-vector $(1,4,4,1)$ we take $s=3$, $t=1$, $c=4$, and we get the components (M_0, L_0, M_1, L_1), (M_0, L_0, M_1, L_2), (M_0, L_0, M_2, L_2), (M_0, L_1, M_2, L_2), (M_1, L_1, M_2, L_2), (L_0, M_1, L_1, M_2), (M_0, L_0, M_1, L_3), (M_0, L_0, M_2, L_3), (M_0, L_1, M_2, L_3), (M_1, L_1, M_2, L_3).

Remark 4.5. As was the case with Theorem 4.1, the construction of the arithmetically Gorenstein scheme with "maximal Hilbert function" described in Theorem 4.3 can be viewed in a very concrete, geometrical way, especially for low codimension. For example, let us produce $G_{3,s,t}$. We start with the set

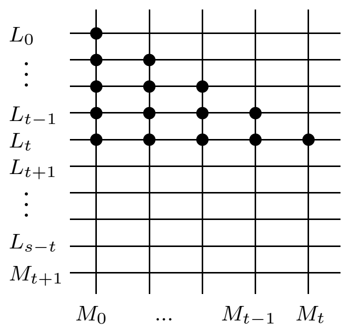
$$\mathcal{N}_{3,s,t} = \{M_0, \ldots, M_{t+1}, L_0, \ldots, L_{s-t}\}$$

The result of renaming M_{t+1} to L_{s-t+1}, considering $G_{2,s+1,t}$ and renaming back is that we have the complete intersection (A, B) where

$$A = \prod_{i=0}^{t} M_i, \quad B = \left(\prod_{i=0}^{s-t} L_i \right) \cdot M_{t+1}.$$

The scheme $Z_{2,t}$ is a subconfiguration. Let Y be the residual to $Z_{2,t}$ in this complete intersection. In the following diagram, $Z_{2,t}$ is given by the dots and Y by the intersection points without dots:

(4.7)



Clearly this is a geometric link. Let $G_{3,s,t}$ be the Gorenstein scheme obtained by $I_{G_{3,s,t}} = I_{Z_{2,t}} + I_{Y_{2,s+1,t+1}}$. One can check geometrically that the components of $G_{3,s,t}$ are of the form described in the statement of the theorem. The simplest way to see this is to use the description of $G_{2,s,t}$ and $Z_{2,t}$. Since the codimension of $G_{3,s,t}$ is 3 and it is the intersection of $Z_{2,t}$ and $Y_{2,s+1,t+1}$, each component corresponds to a *pair* of intersection points in (4.7), where one intersection point comes from $Z_{2,t}$ (dots) and one from $Y_{2,s+1,t+1}$ (non-dots), provided these intersection points lie on the same vertical or horizontal line (L_i or M_i) so that the codimension will be 3. Then it is a simple matter to verify that the components have the form claimed in the statement of the theorem.

Remark 4.6. In Theorems 4.1 and 4.3 we assumed the existence of sufficiently general linear forms. This can be guaranteed if, for example, the field K contains sufficiently many elements or the polynomial ring has dimension $n+1 \geq 2t+c$ (in order to construct $Z_{c,t}$) or $n+1 \geq s+c$ (in order to construct $G_{c,s,t}$).

Throughout the rest of the paper it is understood that, whenever one of the schemes $Z_{c,t}$ or $G_{c,s,t}$ is mentioned, it indeed exists and is a reduced scheme.

5. A construction of arithmetically Cohen-Macaulay generalized stick figures with arbitrary Hilbert function

In this section we give a construction of an arithmetically Cohen-Macaulay generalized stick figure of codimension c in $\mathbb{P}^n$, having an arbitrary possible Hilbert function H: that is, the h-vector $\Delta^{n-c+1}H$ is an arbitrary O-sequence. The key goal is to see that they can be viewed as subconfigurations of the generalized stick figures $Z_{c,t}$ constructed in the previous section, and hence are automatically contained in $G_{c,s,t}$. Moreover, we derive a combinatorial description of the components of the constructed generalized stick figures.

Definition 5.1. Let $>$ denote the degree-lexicographic order on monomials in the ring $T = K[z_1, \ldots, z_c]$, i.e.

$$z_1^{a_1} \cdots z_c^{a_c} > z_1^{b_1} \cdots z_c^{b_c}$$

if the first nonzero coordinate of the vector

$$\left(\sum_{i=1}^{c} (a_i - b_i), a_1 - b_1, \ldots, a_c - b_c \right)$$

is positive. Let J be a monomial ideal. Let m_1, m_2 be monomials in T of the same degree such that $m_1 > m_2$. Then J is a *lex-segment ideal* if $m_2 \in J$ implies $m_1 \in J$.

Notation 5.2. For a graded module M we denote by $a(M)$ the initial degree:

$$a(M) := \inf\{t \in \mathbb{Z} | [M]_t \neq 0\}.$$

Lemma 5.3. *Let J be an Artinian lex-segment ideal in $T = K[z_1, \ldots, z_c]$ of initial degree α and for which the h-vector ends in degree t. Then there is a unique decomposition*

$$J = \sum_{j=0}^{\alpha} z_1^j \cdot I_j$$

where $I_0 \subset I_1 \subset \cdots \subset I_{\alpha-1} \subsetneq I_\alpha = T$,

$$(z_2, \ldots, z_c)^{t+1-j} \subset I_j \subset (z_2, \ldots, z_c)^{\alpha-j}, \tag{5.1}$$

$$a(I_j) > \operatorname{reg}(I_{j+1}),$$

and $I_j \cap \bar{T}$ is an Artinian lex-segment ideal in $\bar{T} = K[z_2, \ldots, z_c]$ $(0 \leq j \leq \alpha - 1)$.

Proof. The existence of the decomposition is straightforward. Note that the uniqueness comes from the requirement that $I_0 \subset I_1 \subset \cdots \subset I_{\alpha-1} \subsetneq I_\alpha = R$. It remains to show the inequality.

Assume that the regularity of I_{j+1} is $s+1$. This means that $z_c^{s+1} \in I_{j+1}$ but $z_s^s \notin I_{j+1}$. Hence $z_1^{j+1} z_c^{s+1} \in J$ but $z_1^{j+1} z_c^s \notin J$. Therefore $z_1^j z_2^{s+1} < z_1^{j+1} z_c^s$ is also not contained in J, i.e. $z_2^{s+1} \notin I_j$. It follows that $a(J_j) > s + 1 = \operatorname{reg}(I_j)$ which concludes the proof. □

Remark 5.4. Since the I_j we have produced in Lemma 5.3 are Artinian lex-segment ideals and since $I_j \subset I_k$ whenever $j < k$, their Hilbert functions will satisfy the hypothesis of Theorem 5.8 below.

Lemma 5.5. *For any $s \geq 0$, we have*

$$h_{T/J}(s) = \sum_{j=0}^{\alpha-1} h_{\bar{T}/I_j \cap \bar{T}}(s - j)$$

Proof. It is a straightforward computation. □

Remark 5.6. The lemma shows that the decomposition of Lemma 5.3 corresponds to the type-vectors of [19] or [20].

Lemma 5.3 suggests the following definition:

Definition 5.7. Let $\underline{h} = (h_0, h_1, \dots, h_t)$ be an O-sequence. Choose an integer $c \geq h_1$ and let J be the Artinian lex-segment ideal in $T = K[z_1, \dots, z_c]$ such that $\underline{h}$ is the h-vector of T/J. Define $\underline{h}^j$ $(0 \leq j < \alpha)$ to be the h-vector of $\bar{T}/I_j \cap \bar{T}$, where $\sum_{j=0}^{\alpha} z_1^j I_j$ is the unique decomposition according to Lemma 5.3. We call $(\underline{h}^0, \dots, \underline{h}^{\alpha-1})$ the *decomposition of* $\underline{h}$.

Note that in [23], [19] or [20] a purely numerical procedure is given which computes the decomposition of $\underline{h}$, but does not involve the computation of lex-segment ideals. However, we need the relationship to these ideals later on.

Recall that the construction of the scheme $Z_{c,t}$ in Section 4 involves linear forms M_i and L_j in a polynomial ring R of dimension $n+1$ over the field K. We are now ready for a generalization of this construction.

Theorem 5.8. *Let $\underline{h} = (h_0, h_1, \dots, h_v)$ be an O-sequence, where $h_v \neq 0$. Let $c \geq h_1$, $t \geq v$ be integers. Suppose*

$$\mathcal{M}_{c,t} = \{M_0, \dots, M_{t+\lfloor \frac{c-1}{2} \rfloor}, L_0, \dots, L_{t+\lfloor \frac{c-2}{2} \rfloor}\} \subset R$$

is a set of linear forms such that each subset of of $\mathcal{M}_{c,t}$ consisting of $c+1$ elements is linearly independent (over K). Define the ideal $I_{c,t}(\underline{h})$ recursively:

If $c = 1$, put $I_{c,t}(\underline{h}) := (M_{t-v} \cdot M_{t-v+1} \cdot \ldots \cdot M_t)$.
If $c > 1$ put

$$I_{c,t}(\underline{h}) := \begin{cases} \displaystyle\bigcap_{j=0}^{\alpha-1} \left[I_{c-1,t-j}(\underline{h}^j) + (L_{t-j+\frac{c-2}{2}}) \right] & \text{if } c \text{ is even} \\ \displaystyle\bigcap_{j=0}^{\alpha-1} \left[I_{c-1,t-j}(\underline{h}^j) + (M_{t-j+\frac{c-1}{2}}) \right] & \text{if } c \text{ is odd} \end{cases}$$

where $(\underline{h}^0, \dots, \underline{h}^{\alpha-1})$ is the decomposition of $\underline{h}$.

Then $I_{c,t}(\underline{h})$ defines a reduced arithmetically Cohen-Macaulay subscheme $Z_{c,t}(\underline{h}) \subset \mathbb{P}^n$ of codimension c such that $Z_{c,t}(\underline{h}) \subset Z_{c,t}$ and the h-vector of $Z_{c,t}(\underline{h})$ is $\underline{h}$.

If each subset of $c+2$ elements of $\mathcal{M}_{c,t}$ is linearly independent then $Z_{c,t}(\underline{h})$ is a generalized stick figure. Moreover, if $\underline{h}' = (h'_0, \dots, h'_{v'})$ is an O-sequence where $h'_{v'} \neq 0$, such that $h'_i \leq h_i$ for all i with $0 \leq i \leq v' \leq v$, then $Z_{c,t}(\underline{h}') \subset Z_{c,t}(\underline{h})$.

Proof. In view of Theorem 4.1, it suffices to check that $Z_{c,t}(\underline{h})$ is arithmetically Cohen-Macaulay, $Z_{c,t}(\underline{h}) \subset Z_{c,t}$, the h-vector of $Z_{c,t}(\underline{h})$ and the last claim. We induct on $c \geq 1$.

Let $c = 1$. By comparison with Theorem 4.1 we get $Z_{c,t}(\underline{h}) \subset Z_{1,t}$ because $I_{Z_{1,t}} = (M_0 \cdot \ldots \cdot M_t)$.

Now let $c > 1$. Using the notation of Lemma 5.3, $\underline{h}^j$ is the h-vector of $\bar{T}/I_j \cap \bar{T}$, where $\bar{T} = K[z_2, \dots, z_c]$. Let $\underline{h}^j = (h_0^j, h_1^j, \dots, h_{t_j}^j)$. Then $h_i^j \leq \binom{c-2+i}{i}$ for all $i \geq 0$. Moreover, Lemma 5.3 shows that $t_j \leq v - j \leq t - j$. Since $I_0 \subset I_1 \subset \cdots \subset I_{\alpha-1}$, the induction hypothesis provides inclusions of arithmetically Cohen-Macaulay configurations

$$\begin{array}{ccc}
Z_{c-1,t}(\underline{h}^0) & \subset & Z_{c-1,t} \\
\cup & & \cup \\
Z_{c-1,t-1}(\underline{h}^1) & \subset & Z_{c-1,t-1} \\
\cup & & \cup \\
\vdots & & \vdots \\
\cup & & \cup \\
Z_{c-1,t+1-\alpha}(\underline{h}^{\alpha-1}) & \subset & Z_{c-1,t+1-\alpha} \\
 & & \cup \\
 & & \vdots \\
 & & \cup \\
 & & Z_{c-1,0}.
\end{array}$$

Suppose that c is even. Then the configuration $Z_{c-1,t-j}$ is made up using only the linear forms

$$M_0, \dots, M_{t-j+\frac{c-2}{2}}, L_0, \dots, L_{t-j+\frac{c-4}{2}}.$$

Hence the assumption on the set $\mathcal{M}_{c,t}$ and Theorem 4.1 imply

$$I_{Z_{c-1,t-j}} : L_{t-j+\frac{c-2}{2}} = I_{Z_{c-1,t-j}}$$

and hence

$$I_{c-1,t-j}(\underline{h}^j) : L_{t-j+\frac{c-2}{2}} = I_{Z_{c-1,t-j}}.$$

Therefore Corollary 3.5 gives us that the ideal

$$(L_{t+\frac{c-2}{2}} \cdot \ldots \cdot L_{t+1-\alpha+\frac{c-2}{2}}) + \sum_{j=0}^{\alpha-1} I_{c-1,t-j}(\underline{h}^j) \cdot \prod_{i=1}^{j} L_{t+1-i+\frac{c-2}{2}}$$

defines an arithmetically Cohen-Macaulay subscheme having $\underline{h}$ as h-vector and is in fact the ideal $I_{c,t}(\underline{h})$.

If c is odd we conclude similarly because then

$$I_{c-1,t-j}(\underline{h}^j) : M_{t-j+\frac{c-1}{2}} = I_{c-1,t-j}(\underline{h}^j).$$

It remains to show that $I_{c,t}(\underline{h}) \subset I_{c,t}(\underline{h}')$. Let $J' = \sum_{j=0}^{\alpha'} z_1^j I'_j$ be the decomposition of the lex-segment ideal $J' \subset T$ with h-vector $\underline{h}'$. Let $\underline{k}^j$ be the h-vector of $\bar{T}/I'_j \cap \bar{T}$. Lemma 5.3 implies that $I_j \subset I'_j$ for all j with $0 \leq j \leq <'$. Therefore, the induction hypothesis provides

$$I_{c-1,t-j}(\underline{h}^j) \subset I_{c-1,t-j}(\underline{k}^j) \quad \text{if } 0 \leq j < \alpha'.$$

Then using the definition of the ideals we obtain $I_{c,t}(\underline{h}) \subset I_{c,t}(\underline{h}')$ as desired.

Finally, note that if

$$\underline{h} = \left(1, c, \binom{c+1}{2}, \dots, \binom{c-1+t}{t}\right)$$

is a "maximal" h-vector then $Z_{c,t}(\underline{h}) = Z_{c,t}$. Hence we conclude that all of the configurations $Z_{c,t}(\underline{h})$ that we obtain for "smaller" h-vectors are contained in $Z_{c,t}$ as claimed. $\square$

Example 5.9. In Theorem 5.8 we gave the result that if $\underline{h}' = (h'_0, \dots, h'_v)$ and $\underline{h} = (h_0, \dots, h_v)$ are O-sequences, with $v' \le v \le t$ and $h'_i \le h_i$ for all i, then $Z_{c,t}(\underline{h}') \subset Z_{c,t}(\underline{h})$. We want to stress here that the value of t must be the same.

Let $\underline{h}' = (1, 2)$ and let $\underline{h} = (1, 2, 2)$. Then we obtain

$$\begin{array}{rcl} I_{2,1}(\underline{h}') & = & (M_0, L_0) \cap (M_0, L_1) \cap (M_1, L_1) \\ I_{2,2}(\underline{h}') & = & (M_1, L_1) \cap (M_1, L_2) \cap (M_2, L_2) \\ I_{2,2}(\underline{h}) & = & (M_0, L_1) \cap (M_1, L_1) \cap (M_0, L_2) \cap (M_1, L_2) \cap (M_2, L_2) \end{array}$$

In particular, $Z_{2,2}(\underline{h}')$ is contained in $Z_{2,2}(\underline{h})$ (as expected from Theorem 5.8) but $Z_{2,1}(\underline{h}')$ is not contained in $Z_{2,2}(\underline{h})$.

In the following corollary, the use of the term "extremal Betti numbers" will be justified in Theorem 8.12.

Corollary 5.10. *The schemes $Z_{c,t}(\underline{h}) \subset \mathbb{P}^n$ constructed in Theorem 5.8 have extremal Betti numbers, i.e. their graded Betti numbers are the same as those of the Artinian lex-segment ideal $J \subset T = K[z_1, \dots, z_c]$ with h-vector $\underline{h}$.*

Proof. As a preparatory step we consider the lex-segment ideal $J \subset T$. The properties of its decomposition imply

$$J : z_1 = I_0 + \sum_{j=1}^{\alpha} z_1^{j-1} \cdot I_j = \sum_{j=1}^{\alpha} z_1^{j-1} \cdot I_j.$$

Using Lemmas 5.5 and 5.3 we obtain

$$\operatorname{reg}(J : z_1) = \operatorname{reg} I_1 < a(I_0).$$

Since $(J+z_1T)/z_1T = (I_0+z_1T)/z_1T$, the multiplication by z_1 provides the exact sequence

$$0 \to (J : z_1)(-1) \to J \to (I_0 + z_1T)/z_1T \to 0$$

where $a((I_0 + z_1T)/z_1T) = a(I_0) > \operatorname{reg}(J : z_1)$. We claim that this implies for all i

$$\operatorname{Tor}_i^T(J, K) \cong \operatorname{Tor}_i^T(J : z_1, K)(-1) \oplus \operatorname{Tor}_i^T((I_0 + z_1T)/z_1T, K).$$

Indeed, the long exact Tor-sequence provides maps $\varphi_i : \operatorname{Tor}_i^T((I_0 + z_1T)/z_1T, K) \to \operatorname{Tor}_{i-1}^T(J : z_1, K)(-1)$. If $j < i + a(I_0)$ then $[\operatorname{Tor}_i^T((I_0 + z_1T)/z_1T, K)]_j = 0$, and if $j \ge i + a(I_0) > i + \operatorname{reg}(J : z_1)$ then $[\operatorname{Tor}_{i-1}^T(J : z_1, K)(-1)]_j = 0$. Therefore φ is the zero map and the Tor-sequence proves our claim.

Now we consider $Z_{c,t}(\underline{h})$. We induct on c and $a(J) = \alpha$. Again, the case $c = 1$ is easy. Let $c > 1$. If $\alpha = 1$ then J contains a linear form which allows us to conclude using induction on the codimension. Let $\alpha > 1$.

Assume that c is even. Let W be defined by

$$I_W = I_{c,t}(\underline{h}) : L_{t+\frac{c-2}{2}} = \bigcap_{j=1}^{\alpha-1} \left[I_{c-1,t-j}(\underline{h}^j) + (L_{t-j+\frac{c-2}{2}}) \right].$$

Then we claim that we have an exact sequence

$$0 \to I_W(-1) \to I_{c,t}(\underline{h}) \to \frac{I_{c-1,t}(\underline{h}^0) + (L_{t+\frac{c-2}{2}})}{(L_{t+\frac{c-2}{2}})} \to 0. \tag{5.2}$$

Indeed, the only question is the cokernel. The sequence clearly holds if we write

$$\frac{I_{c,t}(\underline{h}) + (L_{t+\frac{c-2}{2}})}{(L_{t+\frac{c-2}{2}})} \tag{5.3}$$

for the cokernel. But W is arithmetically Cohen-Macaulay by induction, and $Z_{c,t}(\underline{h})$ is arithmetically Cohen-Macaulay by Theorem 5.8, so sheafifying and taking cohomology gives that (5.3) is a Cohen-Macaulay ideal. But then the generality of $L_{t+\frac{c-2}{2}}$ gives that the components of the corresponding scheme are defined precisely by the ideal given as the cokernel of (5.2).

By induction on α, I_W has the same Betti numbers as $J : z_1$ and, by induction on c, $I_{c-1,t}(\underline{h}^0)$ has the same Betti numbers as I_0. Hence we can conclude as above that the Betti numbers of $I_{c,t}(\underline{h})$ are the sum of the Betti numbers of $I_W(-1)$ and

$$\frac{I_{c-1,t}(\underline{h}^0) + (L_{t+\frac{c-2}{2}})}{(L_{t+\frac{c-2}{2}})}$$

and thus equal to the Betti numbers of J.

The case where c is odd is handled similarly. $\square$

Some arguments in the proofs of this section could be replaced by results from the theory of k-configurations (e.g. [19], [20]). But we prefer to keep the exposition more self-contained.

We would now like to give an explicit (combinatorial) primary decomposition of the ideal $I_{c,t}(\underline{h})$. In order to do this, we will use lexicographic order ideals of monomials. Denote by $S^{(c)}$ the set of monomials in the polynomial ring $S = K[y_1, \ldots, y_c]$. The *reverse lexicographic order* on $S^{(c)}$ is defined by

$y_1^{a_1} \cdot \ldots \cdot y_c^{a_c} <_r y_1^{b_1} \cdot \ldots \cdot y_c^{b_c}$ *if the last non-zero coordinate of the vector*

$$(a_1 - b_1, \ldots, a_c - b_c)$$

is negative.

A non-empty subset $\mathcal{M}$ of $S^{(c)}$ is called an *order ideal of monomials* if $m' \in \mathcal{M}$ and $m|m'$ imply $m \in \mathcal{M}$. It is said to be a *lexicographic set of monomials* if $m' \in \mathcal{M}$, $m <_r m'$ and $\deg m = \deg m'$ imply $m \in \mathcal{M}$. If $\mathcal{M}$ has both properties, it is called a *lexicographic order ideal of monomials (LOIM)*. Observe that a LOIM is *not* an ideal of S.

The *lexicographic order* of the set of monomials $T^{(c)}$ in $T = K[z_1, \ldots, z_c]$ is defined by

$z_1^{a_1} \cdot \ldots \cdot z_c^{a_c} >_\ell z_1^{b_1} \cdot \ldots \cdot z_c^{b_c}$ *if the first non-zero coordinate of the vector*

$$(a_1 - b_1, \ldots, a_c - b_c)$$

is positive.

We define a bijective map $\varphi : T^{(c)} \to S^{(c)}$ by

$$\varphi(z_1^{a_1} \cdot \ldots \cdot z_c^{a_c}) = y_1^{a_c} \cdot \ldots \cdot y_c^{a_1}.$$

Obviously, φ preserves the degree. Moreover, the following is immediate:

Lemma 5.11. *If m, m' are monomials in $T^{(c)}$ then*

$$m >_\ell m' \quad \text{if and only if} \quad \varphi(m) >_r \varphi(m').$$

We also need the following preparatory result.

Lemma 5.12. *Let $\underline{h} = (h_0, h_1, \dots)$ be an O-sequence, where $c \geq h_1$. Let $\mathcal{M}_i$ be the smallest (in the ordering $<_r$) h_i monomials in $S^{(c)}$ of degree i. Then $\mathcal{M} := \bigcup_{i=0}^{\infty} \mathcal{M}_i$ is a LOIM which is called* the LOIM associated to $\underline{h}$ *and denoted by $LOIM(\underline{h})$.*

Proof. This is a special case of Proposition 1 in [5]. See also [55]. □

Now we can relate the lex-segment ideals in T to the LOIM's in S.

Lemma 5.13. *Let $J \subset T$ be a lex-segment ideal. Put $h_i = h_{T/J}(i)$ and $\underline{h} = (h_0, h_1, \dots)$. Define*

$$\mathcal{M} := \{\varphi(m) \mid m \in T^{(c)} \backslash J\}.$$

Then we have $\mathcal{M} = LOIM(\underline{h})$.

Proof. By the definition of h_i, the ideal J does not contain exactly the smallest (with respect to $>_\ell$) h_i monomials in T of degree i. Therefore, by Lemma 5.11, $\mathcal{M}$ contains exactly the smallest (in the ordering $<_r$) h_i monomials of S of degree i. Since $\underline{h}$ is an O-sequence, Lemma 5.12 shows that $\mathcal{M}$ is a LOIM. □

Corollary 5.14. *Let $\underline{h} = (h_0, \dots, h_t)$ be an O-sequence with the decomposition $(\underline{h}^0, \dots, \underline{h}^{\alpha-1})$ and $h_1 \leq c$. Then we have*

$$LOIM(\underline{h}) = \bigcup_{j=0}^{\alpha-1} y_c^j \cdot LOIM(\underline{h}^j).$$

(Note that $LOIM(\underline{h}^j) \subset K[y_1, \dots, y_{c-1}]$.)

Proof. Let $J \subset T$ be the lex-segment ideal having $\underline{h}$ as h-vector. Let $J = \sum_{j=0}^{\alpha-1} z_1^j I_j$ be its decomposition according to Lemma 5.3. Since $z_1^\alpha \in J$ we have for a monomial $z_1^j m$ with $m \in K[z_2, \dots, z_c]$ that

$$\begin{aligned} z_1^j m \notin J \quad &\Leftrightarrow \quad 0 \leq j < \alpha \text{ and } m \notin I_j \\ &\Leftrightarrow \quad \varphi(z_1^j m) = y_c^j \varphi(m) \in LOIM(\underline{h}), \text{ where } 0 \leq j < \alpha \text{ and } \varphi(m) \in LOIM(\underline{h}^j) \end{aligned}$$

by Lemma 5.13. The claim then follows. □

Now we need some more notation.

Definition 5.15. Let $c \geq 1$, $t \geq 0$ be integers and let $m \in S^{(c)}$ be a monomial of degree $k \leq t$. Let $U := K[u_1, \dots, u_{c+2t}]$ be a polynomial ring. Write m as

$$m = y_{e_1} \cdot y_{e_2} \cdot \ldots \cdot y_{e_k} \quad \text{where} \quad 1 \leq e_1 \leq e_2 \leq \cdots \leq e_k \leq c.$$

Define

- $\bar{\beta}_{c,t}(m) := \{u_1, \dots, u_{2(t-k)}\} \cup \bigcup_{j=1}^{k} \{u_{e_j+2(j+t-k)-1}, u_{e_j+2(j+t-k)}\} \subset U$
- $\bar{\mathfrak{p}}_{c,t}(m)$ to be the ideal generated by $\{u_1, \dots, u_{c+2t}\} \backslash \bar{\beta}_{c,t}(m)$

- $\mathfrak{p}_{c,t}(m)$ to be the ideal generated by $\{\mu(u_i) \mid u_i \in \bar{\mathfrak{p}}_{c,t}(m)\}$ where
$$\mu : \{u_1, \dots, u_{c+2t}\} \to \mathcal{M}_{c,t} = \{M_0, \dots, M_{t+\lfloor \frac{c-1}{2} \rfloor}, L_0, \dots, L_{t+\lfloor \frac{c-2}{2} \rfloor}\}$$
is the bijective map defined by
$$\mu(u_i) = \begin{cases} M_{\frac{i-1}{2}} & \text{if } i \text{ is odd;} \\ L_{\frac{i-2}{2}} & \text{if } i \text{ is even.} \end{cases}$$

Theorem 5.16. *Let $\underline{h} = (h_0, \dots, h_t)$ be an O-sequence, where $h_1 \leq c$. Then we have*
$$I_{c,t}(\underline{h}) = \bigcap_{m \in LOIM(\underline{h})} \mathfrak{p}_{c,t}(m).$$

Proof. We induct on $c \geq 1$. First let $c = 1$. Then there is an integer v such that $0 \leq v \leq t$ and
$$h_i = \begin{cases} 1 & \text{if } 0 \leq i \leq v \\ 0 & \text{if } v < i \leq t. \end{cases}$$
Thus we have
$$I_{c,t}(\underline{h}) = (M_{t-v} \cdot \ldots \cdot M_t) \quad \text{and} \quad LOIM(\underline{h}) = \{1, y_1, \dots, y_1^v\}.$$
For $0 \leq k \leq v$ we obtain
$$\bar{\beta}_{c,t}(y_1^k) = \{u_1, \dots, u_{2(t-k)}\} \cup \{u_{2(t-k)+2}, \dots, u_{2t+1}\},$$
and thus $\bar{\mathfrak{p}}_{c,t}(y_1^k) = (u_{2(t-k)+1})$, and $\mathfrak{p}_{c,t}(y_1^k) = M_{t-k}$. It follows that
$$\bigcap_{m \in LOIM(\underline{h})} \mathfrak{p}_{c,t}(m) = \bigcap_{k=0}^{v} (M_{t-k}) = I_{c,t}(\underline{h})$$
as claimed.

Now let $c > 1$. Let $(\underline{h}^0, \dots, \underline{h}^{\alpha-1})$ be the decomposition of $\underline{h}$. Assume that c is even. Then we know by Theorem 5.8 and the induction hypothesis that
$$\begin{aligned} I_{c,t}(\underline{h}) &= \bigcap_{j=0}^{\alpha-1} \left[I_{c-1,t-j}(\underline{h}^j) + (L_{t-j+\frac{c-2}{2}}) \right] \\ &= \bigcap_{j=0}^{\alpha-1} \left[\bigcap_{m \in LOIM(\underline{h}^j)} \mathfrak{p}_{c-1,t-j}(m) + (L_{t-j+\frac{c-2}{2}}) \right]. \end{aligned}$$
Since $LOIM(\underline{h}) = \bigcup_{j=0}^{\alpha-1} y_c^j \cdot LOIM(\underline{h}^j)$, we are done if we can show
<u>Claim</u>: Let $0 \leq j < \alpha$ and let $m \in LOIM(\underline{h}^j)$. Then
$$\mathfrak{p}_{c,t}(y_c^j \cdot m) = \mathfrak{p}_{c-1,t-j}(m) + (L_{t-j+\frac{c-2}{2}}).$$

By Lemma 5.3 we know that $k := \deg m \leq t - j$. Thus we get
$$\bar{\beta}_{c,t}(y_c^j \cdot m) = \bar{\beta}_{c-1,t-j}(m) \cup \{u_{c+2(t-j)+1}, \dots, u_{c+2t}\}.$$

Because $\bar{\mathfrak{p}}_{c-1,t-j}(m)$ is generated by $\{u_1,\dots,u_{c-1+2(t-j)}\}\backslash\bar{\beta}_{c-1,t-j}(m)$ and $\bar{\mathfrak{p}}_{c,t}(y_c^j\cdot m)$ is generated by $\{u_1,\dots,u_{c+2t}\}\backslash\bar{\beta}_{c,t}(y_c^j\cdot m)$, it follows that $\bar{\mathfrak{p}}_{c,t}(y_c\cdot m)=\bar{\mathfrak{p}}_{c-1,t-j}(m)+(u_{c+2(t-j)})$, and so

$$\mathfrak{p}_{c,t}(y_c^j\cdot m)=\mathfrak{p}_{c-1,t-j}(m)+(L_{t-j+\frac{c-2}{2}}),$$

proving the claim.

If c is odd we conclude similarly. □

6. Gorenstein configurations for any SI-sequence

In this section we prove one of the main results of the paper, namely that for every SI-sequence $\underline{h}$ there exists a reduced, arithmetically Gorenstein union of linear varieties whose h-vector is $\underline{h}$. Furthermore, we characterize the linear varieties which are components of this arithmetically Gorenstein configuration, essentially giving the primary decomposition.

We begin by introducing a notation for a shift among the variables u_i:

Definition 6.1. We define the injective map $\tau:\{u_1,u_2,\dots\}\to\{u_1,u_2,\dots\}$ by $\tau(u_i)=u_{i+1}$ for all $i\geq 1$. Abusing notation, we denote by $\tau(\mathfrak{p}_{c-1,t}(m))$ the ideal generated by

$$\{\mu(\tau(u_i))\mid u_i\in\bar{\mathfrak{p}}_{c-1,t}(m)\}$$

and by $\tau(Z_{c-1,t})$ the scheme defined by the intersection of the ideals $\tau(\mathfrak{p})$ where $\mathfrak{p}$ is a minimal prime ideal of $I_{c-1,t}$ (using Theorems 5.8 and 5.16).

In the next technical lemma we collect some facts that we will need later on.

Lemma 6.2. *Let $c\geq 2$, $t\geq 1$ be integers. Then we have*

(a) $\tau(Z_{c-1,t-1})\subset G_{c-1,s,t}$ *for all* $s\geq 2t$.

(b) *If* $m=y_{e_1}\cdot\ldots\cdot y_{e_k}$ *where* $1\leq e_1\leq\cdots\leq e_k\leq c-1$ *and* $k\leq t$ *then*

  (i) $u_1\in\bar{\beta}_{c-1,t}(m)$ *if and only if* $k<t$.

  (ii) $u_{c-1+2t}\in\bar{\beta}_{c-1,t}(m)$ *if and only if* $e_k=c-1$.

  (iii) *If* y_{c-1} *divides* m *then* $\mathfrak{p}_{c-1,t}(m)=\mathfrak{p}_{c-1,t-1}\left(\dfrac{m}{y_{c-1}}\right)$.

Proof. (a) Comparing Theorems 4.1 and 4.3 we observe that $\tau(Z_{c-1,t-1})$ is just defined by $B_{c-1,s,t-1}$ and $B'_{c-1,s,t-1}$ respectively, depending on the divisibility of $c-1$ by 2.

(b) The first two claims follow immediately from the definition of $\bar{\beta}_{c-1,t}(m)$ For (iii) it suffices to note that $e_k=c-1$ implies $\{u_{c-2+2t},u_{c-1+2t}\}\subset\bar{\beta}_{c-1,t}(m)$, and thus

$$\{u_1,\dots,u_{c-1+2t}\}\backslash\bar{\beta}_{c-1,t}(m)=\{u_1,\dots,u_{c-1+2(t-1)}\}\backslash\bar{\beta}_{c-1,t-1}\left(\frac{m}{y_{c-1}}\right)$$

and the assertion follows. □

Now we are ready for the announced construction of the arithmetically Gorenstein schemes and a description of their irreducible components.

Theorem 6.3. *Let $\underline{h}=(h_0,h_1,\dots,h_s)$ be an SI-sequence, where $h_s=1$. Let $c\geq\max\{h_1,2\}$ be an integer and let $t=\min\{i|h_i\geq h_{i+1}\}$. Put $\underline{g}=(g_0,\dots,g_t)$, where $g_i=h_i-h_{i-1}$. Suppose*

$$\mathcal{N}_{c-1,s+1,t}=\{M_0,\dots,M_{t+\lfloor\frac{c-2}{2}\rfloor},L_0,\dots,L_{s-t+\lfloor\frac{c-1}{2}\rfloor}\}\subset R$$

is a subset of linear forms such that each subset of $c+1$ elements is linearly independent. Define the ideal $J_c(\underline{h}) := I_{c-1,t}(\underline{g}) + I_Y$ where $Y = G_{c-1,s+1,t} \backslash Z_{c-1,t}(\underline{g})$. Then $J_c(\underline{h})$ is a Gorenstein ideal in R defining a reduced, arithmetically Gorenstein subscheme $G_c(\underline{h}) \subset \mathbb{P}^n$ of codimension c having $\underline{h}$ as h-vector. Furthermore, $\mathfrak{p}$ is a minimal prime ideal of $J_c(\underline{h})$ if and only if it is generated by a subset of $\mathcal{N}_{c-1,s+1,t}$ consisting of c elements and contains exactly one minimal prime ideal of $I_{c-1,t}(\underline{g})$.

Proof. Notice that $Z_{c-1,t}$ is always contained inside $G_{c-1,s+1,t}$, thanks to Theorem 4.1 and Theorem 4.3. However, even if $\underline{h}$ is maximal and so $Z_{c-1,t} = Z_{c-1,t}(\underline{h})$ (cf. Theorem 5.8), it is not true (if c is odd) that $G_{c,s,t} = G_c(\underline{h})$, or even that $Z_{c,t} \subset G_c(\underline{h})$. See also Remark 6.5.

(Step I) According to Theorem 5.8, we know that $Z_{c-1,t}(\underline{g}) \subset Z_{c-1,t}$. By Theorem 4.3, $G_{c-1,s+1,t}$ is a generalized stick figure containing $Z_{c-1,t}$. Thus $J_c(\underline{h})$ is the sum of the geometrically linked ideals $I_{c-1,t}(\underline{g})$ and I_Y. The fact that $G_c(\underline{h})$ is reduced comes from the fact that $G_{c-1,s+1,t}$ is a generalized stick figure, using Remark 3.2.

We have

$$\underline{h} = (1, c, h_2, \dots, h_{t-1}, h_t, h_t, \dots, h_t, h_t, h_{t+1}, \dots, h_{s-2}, c, 1).$$

Note that the 1's occur in degrees 0 and s and the last h_t occurs in degree $s-t$. Note also that $s \geq 2t$. Consider the sequences $\underline{d} = (d_i)$, $\underline{g} = (g_i)$, $\underline{c} = (c_i)$, $\underline{g}' = (g_i')$, where

$$\begin{array}{rcll}
d_i & = & h_i - h_{i-1} & \text{for } 0 \leq i \leq s+1 \text{ (some of the entries are 0 or} \\
 & & & \text{negative and the sequence is anti-symmetric);} \\
g_i & = & \begin{cases} h_i - h_{i-1} \\ 0 \end{cases} & \begin{array}{l} \text{for } 0 \leq i \leq t \text{ (this is an O-sequence);} \\ \text{for } i > t; \end{array} \\
c_i & = & \begin{cases} \binom{i+c-2}{c-2} \\ \binom{t+c-2}{c-2} \\ \binom{s-i+c-1}{c-2} \end{cases} & \begin{array}{l} \text{for } 0 \leq i \leq t; \\ \text{for } t \leq i \leq s-t+1; \\ \text{for } s-t+1 \leq i \leq s+1 \end{array} \\
 & & & \text{(note that } c_i = c_{s+1-i} \text{ for } 0 \leq i \leq s+1); \\
g_i' & = & c_{s+1-i} - g_{s+1-i} & \text{for } i \geq 0.
\end{array}$$

Note that $\underline{c}$ is the h-vector of $G_{c-1,s+1,t}$. By Lemma 2.5 we know that Y has h-vector $\underline{g}'$, and again by Lemma 2.5 we know that the first difference of the h-vector of $G_c(\underline{h})$ is given by

$$g_i + g_i' - c_i.$$

We now claim that $g_i + g_i' - c_i = d_i$.

- For $0 \leq i \leq t$ we have $g_i = d_i$ and $g_i' = c_i$.
- For $t+1 \leq i \leq s-t$ we have $c_i = \binom{t+c-2}{c-2}$, $g_i = 0 = g_{s+1-i}$ and $g_i' = c_{s+1-i} - 0 = c_i$. However, in this range we have $d_i = 0$ by the symmetry of $\underline{h}$.
- For $s-t+1 \leq i \leq s+1$ we have $g_i = 0$, $g_i' = c_{s+1-i} - g_{s+1-i} = c_i - g_{s+1-i}$. Hence

$$\begin{array}{rcl}
g_i + g_i' - c_i & = & -g_{s+1-i} \\
 & = & -d_{s+1-i} \\
 & = & d_i
\end{array}$$

It follows that $G_c(\underline{h})$ has the desired h-vector $\underline{h}$.

(Step II) It remains to establish the claim on the minimal prime ideals of $J_c(\underline{h})$. Since $G_{c-1,s+1,t}$ is a generalized stick figure and $J_c(\underline{h})$ is unmixed of codimension c, an ideal $\mathfrak{p}$ is a minimal prime of $J_c(\underline{h})$ if and only if $\operatorname{codim}(\mathfrak{p}) = c$ and $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2$ for some minimal prime ideal $\mathfrak{p}_1$ of $I_{c-1,t}(\underline{g})$ and some minimal prime ideal $\mathfrak{p}_2$ of I_Y. Thus the assumption on $\mathcal{N}_{c-1,s+1,t}$ implies that each minimal prime $\mathfrak{p}$ of $J_c(\underline{h})$ is generated by a subset of $\mathcal{N}_{c-1,s+1,t}$ consisting of c elements.

Suppose that $\mathfrak{p}$ contains another minimal prime $\mathfrak{p}_1' \neq \mathfrak{p}_1$ of $I_{c-1,t}(\underline{g})$. Then $\mathfrak{p}_1, \mathfrak{p}_1', \mathfrak{p}_2$ are pairwise distinct minimal prime ideals of $G_{c-1,s+1,t}$ such that $\mathfrak{p}_1 + \mathfrak{p}_1' + \mathfrak{p}_2 = \mathfrak{p}$ has codimension c. This contradicts the fact that $G_{c-1,s+1,t}$ is a generalized stick figure. Thus we have shown the necessity of the conditions for being a minimal prime.

(Step III) In order to prove sufficiency, let $\mathfrak{p}_1 = \mathfrak{p}_{c-1,t}(m)$ be a minimal prime of $I_{c-1,t}(\underline{g})$, where $m \in LOIM(\underline{g})$. We must show that for $H \in \mathcal{N}_{c-1,s+1,t} \backslash \mathfrak{p}_1$, if $\mathfrak{p}_1$ is the only minimal prime of $I_{c-1,t}(\underline{g})$ which is contained in $\mathfrak{p}_1 + (H)$ then $\mathfrak{p}_1 + (H)$ is a minimal prime of $J_c(\underline{h})$.

Our proof will make use of the following observation. Given H and $\mathfrak{p}_1$ as in the last paragraph, consider a specific minimal prime ideal $\mathfrak{q}$ of $I_{G_{c-1,s+1,t}}$ such that $\mathfrak{p}_1 + (H) = \mathfrak{p}_1 + \mathfrak{q}$. Then we know by Step II that $\mathfrak{p}_1 + (H)$ is a minimal prime ideal of $J_c(\underline{h})$ if and only if $\mathfrak{q}$ is not a minimal prime ideal of $I_{c-1,t}(\underline{g})$ (i.e. $\mathfrak{q}$ is a minimal prime of I_Y).

We will distinguish several cases.

Case 1. Suppose that $H \in \mathcal{N}_{c-1,s+1,t} \backslash \mathcal{M}_{c-1,t}$ (see the statement of Theorem 5.8), i.e.

$$H \in \left\{ L_{t+1+\lfloor \frac{c-3}{2} \rfloor}, \dots, L_{s+1-t+\lfloor \frac{c-3}{2} \rfloor} \right\}.$$

We already know that $\mathfrak{p}_1$ is a minimal prime ideal of $I_{G_{c-1,s+1,t}}$. Write $\mathfrak{p}_1$ as in Theorem 4.3 and denote by $\mathfrak{q}'$ the ideal generated by the first $c-2$ linear generators of $\mathfrak{p}_1$. Then $\mathfrak{q} = \mathfrak{q}' + (H)$ is a minimal prime ideal of $I_{G_{c-1,s+1,t}}$ – in fact, it is a primary component of $C_{c-1,s+1,t}$ (if $c-1$ is even) or $C'_{c-1,s+1,t}$ (if $c-1$ is odd). Since $H \notin \mathcal{M}_{c-1,t}$, $\mathfrak{q}$ must be a minimal prime of I_Y. Thus $\mathfrak{p}_1 + (H) = \mathfrak{p}_1 + \mathfrak{q}$ is a primary component of $J_c(\underline{h})$.

Case 2. Suppose $H \in \mathcal{M}_{c-1,t}$, i.e. $H = \mu(u)$ for some $u \in \{u_1, \dots, u_{c-1+2t}\}$. As preparation we need a description of the subsets of $\mathcal{U} = \{u_1, \dots, u_{c-1+2t}\}$ having the form $\bar{\beta}_{c-1,t}(m)$ for some monomial $m \in S^{(c-1)}$ (see Definition 5.15). To this end we call a subset W of $\mathcal{U}$ *consecutive* if $W = \{u_i, u_{i+1}, \dots, u_j\}$ for some $1 \leq i \leq j \leq c-1+2t$. If $i > 1$ then u_{i-1} is called the *predecessor* of W, and if $j < c-1+2t$ then u_{j+1} is called the *successor* of W.

Now let W be an arbitrary non-empty subset of $\mathcal{U}$. It is clear that W has a unique decomposition $W = W_1 \cup \dots \cup W_p$ where $W_1, \dots, W_p$ are consecutive subsets, the successor of W_i is not contained in W if $1 \leq i < p$, the predecessor of W_i is not contained in W if $1 < i \leq p$, and every element of W_i is smaller (in the order $<_r$) than every element of W_{i+1} if $1 \leq i < p$. Using this notation, the definition of $\bar{\beta}_{c-1,t}(m)$ (Definition 5.15) implies:

Observation: *There is a monomial $m \in S^{(c-1)}$ such that $W = \bar{\beta}_{c-1,t}(m)$ if and only if $|W| = 2t$ and the cardinality $|W_i|$ is even for all i with $1 \leq i \leq p$.*

Returning to our previous notation, let $m \in S^{(c-1)}$ again denote a monomial such that $\mathfrak{p}_1 = \mathfrak{p}_{c-1,t}(m)$. As above, write $W = W_1 \cup \cdots \cup W_p$ for the unique decomposition of $W := \bar{\beta}_{c-1,t}(m)$ into consecutive subsets. Since $H = \mu(u) \notin \mathfrak{p}_1$, we obtain $u \in W = \bar{\beta}_{c-1,t}(m)$, i.e. $u \in W_\ell$ for some ℓ $(1 \leq \ell \leq p)$. Write $W_\ell \backslash \{u\} = \tilde{W} \cup \bar{W}$ where $\tilde{W}$ and $\bar{W}$ are both consecutive unless one of them is empty, and each element of $\tilde{W}$ is smaller than each element of $\bar{W}$.

We have to distinguish further cases:

Case 2.1: Suppose that $|\tilde{W}|$ is odd and that $u_1 \notin W$ if $\ell = 1$. Then W_ℓ has a predecessor, say $\bar{u}$. Since $|\bar{W}|$ is even, the observation above implies the existence of a monomial $m' \in S^{(c-1)}$ of degree $\leq t$ such that

$$(W \backslash \{u\}) \cup \{\bar{u}\} = \bar{\beta}_{c-1,t}(m').$$

The prime ideal $\mathfrak{q} := \mathfrak{p}_{c-1,t}(m')$ is a minimal prime of $I_{Z_{c-1,t}}$, and thus of $I_{G_{c-1,s+1,t}}$. Moreover, we have by construction

$$\mathfrak{p}_1 + \mathfrak{q} = \mathfrak{p}_1 + (\mu(u)) = \mathfrak{p}_1 + (H).$$

Hence we are done, as explained at the beginning of Step (III).

Case 2.2: Suppose that $|\tilde{W}|$ is even and that $u_{c-1+2t} \notin W$ if $\ell = p$. Then W_ℓ has a successor, say $\bar{u}$. Since $|\tilde{W}|$ is even, we can find as above a monomial $m' \in S^{(c-1)}$ such that

$$(W \backslash \{u\}) \cup \{\bar{u}\} = \bar{\beta}_{c-1,t}(m')$$

and $\mathfrak{q} = \mathfrak{p}_{c-1,t}(m')$ is a minimal prime of $I_{G_{c-1,s+1,t}}$. We conclude as in Case 2.1.

Case 2.3: Suppose that $|\tilde{W}|$ is odd, $u_1 \in W$ and $\ell = 1$. Then $u_1 \in W \backslash \{u\}$ (since if $u = u_1$ then $|\tilde{W}|$ is not odd) and $p \geq 2$.

Case 2.3.1 Suppose that $u_{c-1+2t} \notin W$. Then we form the subset $X \subset \mathcal{U}$ by replacing u_1 by u_{c-1+2t} and all $u_i \in (W \backslash \{u\}) \cup \{u_{c-1+2t}\}$ with $i > 1$ by u_{i-1}. The observation above shows that there is a monomial $m' \in S^{(c-1)}$ of degree $\leq t$ such that $X = \bar{\beta}_{c-1,t}(m')$. Since $u_{c-1+2t} \in X$, we obtain by Lemma 6.2 that y_{c-1} divides m'. Using Lemma 6.2 again, we conclude that

$$\mathfrak{q} := \mathfrak{p}_{c-1,t}(m') = \mathfrak{p}_{c-1,t-1}\left(\frac{m'}{y_{c-1}}\right)$$

is a minimal prime ideal of $I_{Z_{c-1,t-1}}$. Hence Lemma 6.2 shows that $\tau(\mathfrak{q})$ is a minimal prime of $I_{G_{c-1,s+1,t}}$. It follows that $W \cap \tau(X) = W \backslash \{u, u_1\}$, and thus

$$\mathfrak{p}_1 + \tau(\mathfrak{q}) = \mathfrak{p}_1 + (H)$$

is a minimal prime ideal of $J_c(\underline{h})$.

Case 2.3.2: Suppose that $u_{c-1+2t} \in W$. Let $\bar{u}$ be the predecessor of W_p. As in Case 2.3.1, construct X out of $(W \backslash \{u\}) \cup \{\bar{u}\}$ by replacing all $u_i \in (W \backslash \{u\}) \cup \{\bar{u}\}$ by u_{i-1} if $i > 1$ and by u_{c-1+2t} if $i = 1$. Again we get the existence of a monomial $m' \in S^{(c-1)}$ such that $X = \bar{\beta}_{c-1,t}(m')$ and

$$\mathfrak{q} = \mathfrak{p}_{c-1,t}(m') = \mathfrak{p}_{c-1,t-1}\left(\frac{m'}{y_{c-1}}\right)$$

is a minimal prime of $I_{Z_{c-1,t-1}}$, and thus $\mathfrak{p}_1 + \tau(\mathfrak{q}) = \mathfrak{p}_1 + (H)$ is a minimal prime of $J_c(\underline{h})$.

Case 2.4: Suppose that $|\tilde{W}|$ is even, $u_{c-1+2t} \in W$ and $\ell = p$. Then $u_{c-1+2t} \in \bar{W}$ and $|\bar{W}|$ is odd. (Note that $u = u_{c-1+2t}$ is impossible since otherwise $\tilde{W} = W_\ell \setminus \{u_{c-1+2t}\}$ has odd cardinality.)

Case 2.4.1: Suppose that $u_1 \notin W$. Similarly as above we conclude that there is a monomial $m' \in S^{(c-1)}$ such that $\bar{\beta}_{c-1,t}(m') = (W \backslash \{u\}) \cup \{u_1\}$, and

$$\mathfrak{q} = \mathfrak{p}_{c-1,t}(m') = \mathfrak{p}_{c-1,t-1}\left(\frac{m'}{y_{c-1}}\right)$$

is a minimal prime of $I_{Z_{c-1,t-1}}$, and thus $\mathfrak{p}_1 + \tau(\mathfrak{q}) = \mathfrak{p}_1 + (H)$ is a minimal prime of $J_c(\underline{h})$.

Case 2.4.2: Suppose that $u_1 \in W$. Let $\bar{u}$ be the successor of W_1. Then there is a monomial $m' \in S^{(c-1)}$ such that y_{c-1} divides m', $\bar{\beta}_{c-1,t}(m') = (W \backslash \{u\}) \cup \{\bar{u}\}$ and

$$\mathfrak{p}_1 + \tau\left(\mathfrak{p}_{c-1,t-1}\left(\frac{m'}{y_{c-1}}\right)\right) = \mathfrak{p}_1 + (H)$$

is a minimal prime of $J_c(\underline{h})$.

Therefore we have shown the claim in all cases, and the proof of Theorem 6.3 is complete. □

We wish to illustrate the methods of the last proof and of Section 5 by a simple example.

Example 6.4. We want to construct a reduced arithmetically Gorenstein subscheme of $\mathbb{P}^n$ with h-vector

$$\underline{h} = (1, 3, 5, 3, 1).$$

Thus we have $s = 4$ and $t = 2$. Moreover, we take $c = 3$ and $n \geq 3$. Thus, we want to produce the Gorenstein scheme $G_3(\underline{h})$.

Our construction uses the set of sufficiently general linear forms

$$\mathcal{N}_{2,5,2} = \{M_0, M_1, M_2, L_0, L_1, L_2, L_3\}.$$

Following the proof of Theorem 6.3 we get

$$\underline{g} = (1, 2, 2).$$

First, we have to construct the scheme $Z_{2,2}(\underline{g})$ using the methods of Section 5. The starting point is the Artinian lex-segment ideal $J \subset T := K[z_1, z_2]$ corresponding to the h-vector $\underline{g}$. We obtain

$$\begin{aligned} J &= (z_1^2, z_1 z_2^2, z_2^3) \\ &= (z_2^3) + z_1(z_2^2) + z_1^2 T. \end{aligned}$$

Hence we get for the decomposition of $\underline{g}$ (cf. Definition 5.7)

$$\underline{g} = (\underline{g}^0, \underline{g}^1)$$

where

$$\underline{g}^0 = (1, 1, 1) \quad \text{and} \quad \underline{g}^1 = (1, 1).$$

Therefore Theorem 5.8 provides for the ideal of $Z_{2,2}(\underline{g})$

$$\begin{aligned} I_{2,2}(\underline{g}) &= (I_{1,2}(\underline{g^0}) + (L_2)) \cap (I_{1,1}(\underline{g^1}) + (L_1)) \\ &= (M_0M_1M_2, L_2) \cap (M_1M_2, L_1) \\ &= (M_0M_1M_2, M_1M_2L_2, L_1L_2). \end{aligned}$$

Second, we want to link $Z_{2,2}(\underline{g})$ by $G_{2,5,2}$ to the scheme Y. Theorem 4.3 gives the primary decomposition of the ideal of $G_{2,5,2}$ which we group somewhat for the sake of transparency

$$\begin{aligned} I_{G_{2,5,2}} =& (M_0, L_0L_1L_2) \cap (M_1, L_1L_2) \cap (M_2, L_2) \\ & \cap (L_0, M_1M_2) \cap (L_1, M_2) \\ & \cap (M_0M_1M_2, L_3). \end{aligned}$$

It is easy to see that $I_{G_{2,5,2}}$ is just the complete intersection $(M_0M_1M_2, L_0L_1L_2L_3)$.

For the linked scheme $Y = G_{2,5,2} \setminus Z_{2,2}(\underline{g})$ we get (grouping again)

$$\begin{aligned} I_Y &= (M_0, L_0L_1) \cap (L_0, M_1M_2) \cap (M_0M_1M_2, L_3) \\ &= (M_0M_1M_2, M_0L_0L_3, L_0L_1L_3). \end{aligned}$$

Now, on the one hand the construction of the proof of Theorem 6.3 provides for the ideal of $G_3(\underline{h})$

$$\begin{aligned} J_3(\underline{h}) &= I_{2,2}(\underline{g}) + I_Y \\ &= (M_0M_1M_2, M_0L_0L_3, M_1M_2L_2, L_0L_1L_3, L_1L_2). \end{aligned}$$

On the other hand Theorem 6.3 describes directly the components of $G_3(\underline{h})$. We get (grouping again)

$$J_3(\underline{h}) = (L_0L_3, M_0M_1M_2, L_2) \cap (M_0L_0L_3, M_1M_2, L_1) \cap (L_1, M_0, L_2)$$

which in this special case we also could have obtained easily from the description of the generators of the ideal above.

Remark 6.5. We conjecture that the arithmetically Gorenstein configurations produced in Theorem 6.3 are in fact themselves generalized stick figures. The obstruction to proving this comes from the possibility of situations like the hypothetical one described in the proof of Theorem 4.3 (see page 16). There we were able to conclude because of the very explicit nature of the primary decomposition, which is missing here. Note that in codimension 3 the result is known [25].

This conjecture would follow easily if we knew, for example, that $G_c(\underline{h}) \subset G_{c,s,t}$. There is a subtle point, to begin with. The scheme $G_c(\underline{h})$ is constructed using the set

$$\mathcal{N}_{c-1,s+1,t} = \{M_0, \ldots, M_{t+\lfloor \frac{c-2}{2} \rfloor}, L_0, \ldots, L_{s+1-t+\lfloor \frac{c-3}{2} \rfloor}\}$$

whereas $G_{c,s,t}$ is constructed using

$$\mathcal{N}_{c,s,t} = \{M_0, \ldots, M_{t+\lfloor \frac{c-1}{2} \rfloor}, L_0, \ldots, L_{s-t+\lfloor \frac{c-2}{2} \rfloor}\}.$$

These agree if and only if c is even. In order to remedy this situation we can proceed as we did in Theorem 4.3 and define

$$G'_{c,s,t} = \begin{cases} G_{c,s,t} \text{ if } c \text{ is even;} \\ \text{The scheme defined by the ideal which is obtained from } I_{G_{c,s,t}} \\ \text{by re-naming } M_{t+\frac{c-1}{2}} \text{ as } L_{s-t+\frac{c-1}{2}}. \end{cases}$$

With this re-naming, $\mathcal{N}_{c,s,t}$ becomes $\mathcal{N}_{c-1,s+1,t}$, and of course $G'_{c,s,t}$ is still arithmetically Gorenstein.

The "right" question is thus whether $G_c(\underline{h}) \subset G'_{c,s,t}$. In fact, if $\underline{h}$ is maximal, i.e. if

$$h_i = \begin{cases} \binom{c-1+i}{c-1} & \text{if } 0 \le i \le t; \\ \binom{c-1+t}{c-1} & \text{if } t \le i \le s-t; \\ \binom{s-i+c-1}{c-1} & \text{if } s-t \le i \le s \end{cases}$$

then it follows from the constructions of Theorem 4.3 and Theorem 6.3 that $G_c(\underline{h}) = G'_{c,s,t}$.

However, we can not expect, in general, that $G_c(\underline{h}) \subset G'_{c,s,t}$. The reason is the following. Suppose we have $Z_{c-1,t}(\underline{g}) \subsetneq Z_{c-1,t} \subset G$, where G is the suitable arithmetically Gorenstein scheme from the construction. The components of $G_{c,s,t}$ are obtained by intersecting a component of $Z_{c-1,t}$ with a component of $G \backslash Z_{c-1,t}$. The components of $G_c(\underline{h})$ are obtained by intersecting a component of $Z_{c-1,t}(\underline{g})$ with a component of $G \backslash Z_{c-1,t}(\underline{g})$. Hence there are components of $G_c(\underline{h})$ which consist of the intersection of two components of $Z_{c-1,t}$, and no such intersection is a component of $G'_{c,s,t}$.

For a specific counterexample, let $\underline{h} = (1,3,5,3,1)$, so $s = 4$, $t = 2$ and we put $c = 3$. Then (L_1, M_0, L_2) is a minimal prime ideal of $J_3(\underline{h})$ (cf. Example 6.4) but not of $I_{G'_{3,4,2}}$. Hence $G_3(\underline{h}) \not\subset G'_{3,4,2}$.

7. The Subspace Property

In this section we want to show that the arithmetically Gorenstein schemes constructed in the previous section have the Weak Lefschetz Property. Passing to the Artinian reduction we would lose our useful combinatorial description. Thus, we introduce the so-called *subspace property*. It is even defined for arithmetically Cohen-Macaulay schemes and does not require taking consecutive hyperplane sections. However, for arithmetically Gorenstein subschemes the subspace property implies the Weak Lefschetz Property. Thus, we conclude by showing that our arithmetically Gorenstein schemes do possess the subspace property.

Recall (cf. Remark 2.10) that $\operatorname{reg}(X)$ denotes the Castelnuovo-Mumford regularity of the scheme X.

Notation 7.1. For a graded module M we denote by $a(M)$ the initial degree:

$$a(M) := \inf\{t \in \mathbb{Z} | [M]_t \neq 0\}.$$

Definition 7.2. A homogeneous ideal $I \subset R = K[x_0, \ldots, x_n]$ of codimension c is said to have the *subspace property* if there is a linear form $\ell \in R$ such that $(I + \ell R)/\ell R$ is a perfect ideal in $R/\ell R$ of codimension $c-1$ and the initial degree of $(I :_R \ell)/I$ is at least $\frac{\operatorname{reg}(I)-1}{2}$. $X \subset \mathbb{P}^n$ is said to have the subspace property if I_X does.

Remark 7.3. The condition that $(I + \ell R)/\ell R$ have codimension $c - 1$ in $R/\ell R$ means that ℓ has to vanish on some components of the scheme defined by I. The conditions that $(I+\ell R)/\ell R$ is perfect and that the initial degree of $(I :_R \ell)/I$ is "not too small" are what is difficult to verify, in general.

The main example for us is the following. Let I_S be the saturated ideal of an arithmetically Cohen-Macaulay subscheme S of codimension $c-1$ in $\mathbb{P}^n$. Let I_C be the saturated ideal of an arithmetically Cohen-Macaulay subscheme C of S of codimension c in $\mathbb{P}^n$. Let ℓ be a linear form not vanishing on any component of S or of C. By Lemma 3.4, $I = I_S + \ell \cdot I_C$ is the saturated ideal of an arithmetically Cohen-Macaulay subscheme of codimension c in $\mathbb{P}^n$, and this subscheme consists of the union of C and a hyperplane section of S. The ideal $I + \ell R/\ell R$ is just the ideal of this hyperplane section *inside the hyperplane*, and hence is perfect and has codimension $c-1$ inside the hyperplane as required. The ideal $I :_R \ell$ is just I_C, so if we start with C which is rather large inside S, the condition on the regularity is easy to verify.

Lemma 7.4. *Let R/I be an Artinian Gorenstein algebra. Then R/I has the weak Lefschetz property with respect to a linear form g (cf. Definition 2.3) if and only if I has the subspace property with respect to g.*

Proof. Assume that R/I has the weak Lefschetz property with respect to g. The fact that we get a perfect ideal comes from the Artinian property. Let

$$(h_0, h_1, \ldots, h_{t-1}, h_t, \ldots, h_t, \ldots, h_1, h_0)$$

be the h-vector of R/I where $h_{t-1} < h_t$ and let $s := \operatorname{reg}(R/I)$ (so the second $h_0 = 1$ occurs in degree s). Consider the exact sequence

$$0 \to \frac{I:g}{I}(-1) \to \frac{R}{I}(-1) \xrightarrow{g} \frac{R}{I} \to \frac{R}{I+gR} \to 0. \tag{7.1}$$

Since the multiplication by g on R/I is injective if $h_{i-1} \leq h_i$, we get

$$s - t + 1 = a\left(\frac{I:g}{I}(-1)\right)$$

where $a(M) := \min\{j \in \mathbb{Z} \mid [M]_j \neq 0\}$ denotes the initial degree. Since $s \geq 2t$ we conclude

$$a\left(\frac{I:g}{I}\right) = s - t \geq \frac{s}{2} = \frac{\operatorname{reg}(R/I)}{2} = \frac{\operatorname{reg} I - 1}{2}.$$

Conversely, if I has the subspace property then the exact sequence (7.1) gives injectivity for the multiplication by g on R/I in degrees $\leq s/2$, and the self-duality of R/I (up to shift) gives the surjectivity for the second half. $\square$

Lemma 7.5. *Suppose that I has the subspace property with respect to g. Let ℓ be a general linear form. If* $\operatorname{depth} R/I > 0$ *then $I + \ell R$ has the subspace property with respect to g.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \frac{I:g}{I}(-2) & \to & \frac{R}{I}(-2) & \xrightarrow{g} & \frac{R}{I}(-1) & \to & \frac{R}{I+gR}(-1) & \to & 0 \\
 & & \downarrow \ell & & \downarrow \ell & & \downarrow \ell & & \downarrow \ell & & \\
0 & \to & \frac{I:g}{I}(-1) & \to & \frac{R}{I}(-1) & \xrightarrow{g} & \frac{R}{I} & \to & \frac{R}{I+gR} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \frac{(I+\ell R):g}{I+\ell R}(-1) & \to & \frac{R}{I+\ell R}(-1) & \xrightarrow{g} & \frac{R}{I+\ell R} & \to & \frac{R}{I+(g,\ell)R} & \to & 0
\end{array}$$

It follows that

$$a\left(\frac{(I+\ell R):g}{I+\ell R}\right) = a\left(\frac{I:g}{I}\right) \geq \frac{\operatorname{reg} I - 1}{2} = \frac{\operatorname{reg}(I+\ell R)-1}{2}.$$

□

In particular, we see that if X is an arithmetically Cohen-Macaulay subscheme with the subspace property then its general hyperplane section also has this property, provided that $\dim X \geq 1$. In view of the last part of Definition 2.3, where we extended the weak Lefschetz property to the non-Artinian case, we have the following corollary.

Corollary 7.6. *Suppose that an ideal I has the subspace property. Then I has the weak Lefschetz property.*

Proof. It follows from Lemma 7.4 and Lemma 7.5. □

Lemma 7.7. *The ideals constructed in the proof of Theorem 6.3 have the subspace property.*

Proof. In the proof of Theorem 6.3 we use $Z \subset Z_{c-1,t} \subset G_{c-1,s+1,t}$, where $s \geq 2t$. Thus we get for $g := L_{s+1-t+\lfloor \frac{c-3}{2} \rfloor} = L_{s-t+\lfloor \frac{c-1}{2} \rfloor}$ that

$$I_{G_{c-1,s+1,t}} : g = I_{G_{c-1,s,t}} =: K' \qquad \text{and} \qquad I_Z : g = I_Z$$

(this follows from Theorem 4.3 and Theorem 4.1).

Put $K := I_{G_{c-1,s+1,t}}$, $I := I_Z$, $J := K : I$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \frac{R}{K:g}(-1) & \to & \left(\frac{R}{I} \oplus \frac{R}{J:g}\right)(-1) & \to & \left(\frac{R}{(I+J):g}\right)(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \frac{R}{K} & \to & \frac{R}{I} \oplus \frac{R}{J} & \to & \frac{R}{I+J} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \frac{R}{K+gR} & \to & \frac{R}{I+gR} \oplus \frac{R}{J+gR} & \to & \frac{R}{I+J+gR} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since I and $J : g$ are linked by $K' = K : g$, the ideal $(I + J) : g = I + (J : g)$ is Gorenstein of codimension c and has the h-vector $(1, c, h_2, \dots, h_t, \dots, h_t, h_{t-1}, \dots, c, 1)$ with the final

1 occurring in degree $s-1$. Hence the rightmost column shows that $I+J+gR$ is a perfect ideal of codimension c in R and

$$a\left(\frac{(I+J):g}{I+J}\right) = s-t \geq \frac{s}{2} = \frac{\operatorname{reg}(I+J)-1}{2}$$

Therefore $I+J$ has the subspace property with respect to g. $\square$

Remark 7.8. The fact that our arithmetically Gorenstein schemes possess the subspace property shows that in some sense the approach of Harima [29] has an analog in higher dimension. We find a large subset of $G_{c,s,t}$ lying on a hyperplane, and after it is removed we are left with $G_{c,s-1,t}$, which is the higher dimensional analog of his ideal quotient trick. See Remark 2.7.

On the other hand, the choice of the hyperplane which gives the subspace property for a subscheme X can be counter-intuitive, and in particular it is not necessarily the hyperplane containing the "largest" subset of X, as is shown in the following example.

Example 7.9. Let X consist of six points in $\mathbb{P}^2$, three of which are generically chosen on a line λ_1, and three more generally chosen points on a line λ_2. Note that X is a complete intersection, and $\operatorname{reg}(I_X) = 4$. Let ℓ be a linear form. We claim that if ℓ is the form defining λ_1 or λ_2 then it does not give the subspace property, but that if ℓ vanishes on one point of X that lies on λ_1 and one point of X that lies on λ_2 then ℓ does give the subspace property. Indeed, in all three cases $(I+\ell R)/\ell R$ is perfect in $R/\ell R$, but in the first two cases the initial degree of $(I :_R \ell)/I$ is 1 (violating the second condition of the definition), while in the last case this initial degree is 2, satisfying the condition.

8. Extremal Graded Betti Numbers

The goal of this section is to show that the arithmetically Gorenstein schemes constructed in Section 6 have the maximal graded Betti numbers (among the class of arithmetically Gorenstein schemes with the same Hilbert function and the Weak Lefschetz Property). To this end we first compute the Betti numbers of the sum of two geometrically (Gorenstein) linked ideals. Second, in order to get upper bounds for the Betti numbers we compare the Betti numbers of a scheme with those of a hyperplane section. This is done in the more general framework of modules, i.e. we compare Betti numbers of M and $M/\ell M$ where M is a graded module and ℓ is any linear form.

Let $X, Y \subset \mathbb{P}^n$ be geometrically linked subschemes of codimension $c \leq n$. We have already seen that the Hilbert function of $X \cap Y$ is determined by the Hilbert function of X alone. The same is true of the graded Betti numbers under an additional hypothesis.

Lemma 8.1. *If $X, Y \subset \mathbb{P}^n$ are geometrically linked and $\operatorname{reg}(X \cup Y) \geq 2 \cdot \operatorname{reg}(X)$ then (for all i)*

$$\operatorname{Tor}_i^R(R/(I_X+I_Y), K) \cong \operatorname{Tor}_i^R(R/I_X, K) \oplus \operatorname{Tor}_{i-1}^R(K_X, K)(-\operatorname{reg}(X\cup Y) - c + n + 2)$$

where $K_X := \operatorname{Ext}_R^c(R/I_X, R)(-n-1)$ denotes the canonical module of X.

Proof. Following [57], we define for a Noetherian graded module M,

$$\begin{aligned} r(M) &:= \inf\{n \in \mathbb{Z} \mid p_M(m) = h_M(m) \text{ for all } m \geq n\} \\ e(M) &:= \sup\{n \in \mathbb{Z} \mid [M]_n \neq 0\}, \end{aligned}$$

where $p_M(t)$ is the Hilbert polynomial of M. In [57], $r(M)$ is called the *index of regularity* of M. Since $X \cup Y$ is arithmetically Gorenstein by assumption, we get for its index of regularity

$$\begin{array}{rcl} r(R/I_{X\cup Y}) & = & e(H^{n+1-c}_{\mathfrak{m}}(R/I_{X\cup Y})) + 1 \\ & = & \operatorname{reg}(R/I_{X\cup Y}) - (n+1-c) + 1 \\ & = & \operatorname{reg}(X \cup Y) + c - n - 1 \end{array}$$

Thus we obtain (cf. for instance [48], Lemma 2.5)

$$K_X(-\operatorname{reg}(X\cup Y) - c + n + 2) \cong I_Y/(I_X \cap I_Y) \cong (I_X + I_Y)/I_X$$

and the exact sequence

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & G_s & & F_t & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \vdots & & \vdots & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & G_1 & & F_1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & G_0 & & R & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \rightarrow & K_X(-\operatorname{reg}(X\cup Y) - c + n + 2) & \rightarrow & R/I_X & \rightarrow & R/(I_X + I_Y) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

(note that X is not assumed to be arithmetically Cohen-Macaulay), where the vertical sequences denote the corresponding minimal free resolutions.

Now we observe $K_X \cong H^{n+1-c}_{\mathfrak{m}}(R/I_X)^\vee$ and

$$e(H^{n+1-c}_{\mathfrak{m}}(R/I_X)) + n + 1 - c \leq \operatorname{reg}(R/I_X) = \operatorname{reg}(X) - 1.$$

It follows that

$$a(K_X) = -e(H^{n+1-c}_{\mathfrak{m}}(R/I_X)) \geq -\operatorname{reg}(X) - c + n + 2,$$

Thus

$$a(K_X(-\operatorname{reg}(X\cup Y) - c + n + 2)) \geq \operatorname{reg}(X \cup Y) - \operatorname{reg}(X) \geq \operatorname{reg}(X)$$

where the last inequality is by assumption. We conclude that $a(G_i) \geq \operatorname{reg}(X) + i$ for all $i \geq 0$.

On the other hand, it is well-known that the modules F_i are generated in degrees $\leq \operatorname{reg}(R/I_X) + i = \operatorname{reg}(X) - 1 + i$. It follows that the mapping cone procedure provides a free resolution of $R/I_X + I_Y$ which is minimal because cancellation cannot occur. This proves the claim. □

If X is arithmetically Cohen-Macaulay, the graded Betti numbers of K_X can be computed from the graded Betti numbers of X. Thus as a corollary we obtain Theorem 7.1 of [19]:

Corollary 8.2. *If $X, Y \subset \mathbb{P}^n$ are geometrically linked arithmetically Cohen-Macaulay schemes such that* $\operatorname{reg}(X \cup Y) \geq 2 \cdot \operatorname{reg}(X)$ *then*

$$\operatorname{Tor}_i^R(R/(I_X + I_Y), K) \cong \operatorname{Tor}_i^R(R/I_X, K) \oplus \operatorname{Tor}_{c-i+1}^R(R/I_X, K)^\vee(-\operatorname{reg}(X \cup Y) - c + 1).$$

Proof. Since R/I_X is Cohen-Macaulay, we get by applying $\operatorname{Hom}_R(-, R)$ to a minimal free resolution of R/I_X a minimal free resolution of $\operatorname{Ext}_R^c(R/I_X, R)$, i.e.

$$\operatorname{Tor}_i^R(\operatorname{Ext}_R^c(R/I_X, R), K) \cong \operatorname{Tor}_{c-i}^R(R/I_X, K)^\vee.$$

Thus the Lemma implies our assertion. □

Lemma 8.3. *Let M be a graded R-module, $\ell \in R$ a linear form. Then there is an exact sequence of graded R-modules (where $\bar{R} := R/\ell R$):*

$$\begin{aligned} \cdots \to \operatorname{Tor}_{i-1}^{\bar{R}}((0 :_M \ell), K)(-1) \to \operatorname{Tor}_i^R(M, K) \to \operatorname{Tor}_i^{\bar{R}}(M/\ell M, K) \to \cdots \\ \cdots \to \operatorname{Tor}_1^R(M, K) \to \operatorname{Tor}_1^{\bar{R}}(M/\ell M, K) \to 0. \end{aligned}$$

Proof. Consider the exact sequence

$$0 \to R(-1) \xrightarrow{\ell} R \to \bar{R} \to 0.$$

Tensoring with M and taking homology provides $\operatorname{Tor}_i^R(M, \bar{R}) = 0$ for all $i \geq 2$ and

$$\operatorname{Tor}_1^R(M, \bar{R}) \cong \ker\left(M(-1) \xrightarrow{\ell} M\right) = (0 :_M \ell)(-1).$$

Now we compute $\operatorname{Tor}_i^R(M, \bar{R})$ using a minimal free resolution of M:

$$\mathbf{F}_\bullet \qquad 0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to M \to 0.$$

Tensoring by $\bar{R}$ gives the complex (with $\bar{F}_i := F_i \otimes_R \bar{R}$)

$$\mathbf{F}_\bullet \otimes_R \bar{R} \qquad 0 \to \bar{F}_s \to \cdots \to \bar{F}_2 \xrightarrow{\alpha} \bar{F}_1 \xrightarrow{\beta} \bar{F}_0 \to M/\ell M \to 0.$$

Its homology is

$$H_i(\mathbf{F}_\bullet \otimes \bar{R}) = \begin{cases} 0 & \text{if } i \leq 0 \\ \operatorname{Tor}_i^R(M, \bar{R}) & \text{if } i \geq 1 \end{cases} \cong \begin{cases} 0 & \text{if } i \neq 1 \\ (0 :_M \ell)(-1) & \text{if } i = 1. \end{cases}$$

Since $\mathbf{F}_\bullet$ is a minimal free resolution, the maps in $\mathbf{F}_\bullet \otimes \bar{R}$ are minimal maps too. Thus we obtain a minimal free resolution of $\bar{F}_1/\operatorname{im}\alpha$ as an $\bar{R}$-module:

$$0 \to \bar{F}_s \to \cdots \to \bar{F}_2 \xrightarrow{\alpha} \bar{F}_1 \to \bar{F}_1/\operatorname{im}\alpha \to 0.$$

It follows that

$$\operatorname{Tor}_i^{\bar{R}}(\bar{F}_1/\operatorname{im}\alpha, K) \cong \bar{F}_{i+1} \otimes_{\bar{R}} K \cong \operatorname{Tor}_{i+1}^R(M, K) \qquad \text{for all } i \geq 0. \tag{8.1}$$

The exact sequence

$$0 \to \bar{F}_1/\ker\beta \to \bar{F}_0 \to M/\ell M \to 0$$

implies

$$\operatorname{Tor}_i^{\bar{R}}(\bar{F}_1/\ker\beta, K) \cong \operatorname{Tor}_{i+1}^{\bar{R}}(M/\ell M, K) \qquad \text{for all } i \geq 0. \tag{8.2}$$

Using $\ker\beta/\operatorname{im}\alpha = H_1(\mathbf{F}_\bullet \otimes \bar{R}) \cong (0 :_M \ell)(-1)$ we obtain the exact sequence

$$0 \to (0 :_M \ell)(-1) \to \bar{F}_1/\operatorname{im}\alpha \to \bar{F}_1/\ker\beta \to 0.$$

The associated long exact Tor sequence reads as

$$\cdots \to \operatorname{Tor}_1^{\bar R}(\bar F_1/\ker\beta, K) \to \operatorname{Tor}_0^{\bar R}((0:_M \ell), K)(-1) \to \operatorname{Tor}_0^{\bar R}(\bar F_1/\operatorname{im}\alpha, K) \\ \to \operatorname{Tor}_0^{\bar R}(\bar F_1/\ker\beta, K) \to 0.$$

Taking into account the isomorphisms (8.1) and (8.2), our claim follows. □

Notation 8.4. We denote $\left[\operatorname{tor}_i^R(M,K)\right]_j := \operatorname{rank}_K \left[\operatorname{Tor}_i^R(M,K)\right]_j$.

Corollary 8.5. *With the notation of Lemma 8.3 we have for all $i \geq 1$ and all $j \in \mathbb{Z}$:*

$$\left[\operatorname{tor}_i^R(M,K)\right]_j \leq \left[\operatorname{tor}_i^{\bar R}(M/\ell M,K)\right]_j + \left[\operatorname{tor}_{i-1}^{\bar R}((0:_M \ell),K)\right]_{j-1}.$$

Furthermore,

$$\left[\operatorname{Tor}_i^R(M,K)\right]_j \cong \left[\operatorname{Tor}_i^{\bar R}(M/\ell M,K)\right]_j \quad \begin{array}{l} \text{if } i = 1 \text{ and } j \leq a+i-1 \\ \text{or } i \geq 2 \text{ and } j \leq a+i-2 \end{array}$$

where $a := a(0:_M \ell)$.

Proof. Using $[\operatorname{Tor}_i^{\bar R}((0:_M \ell),K)]_j = 0$ if $j < a+i$ the claim follows by analyzing the sequence given in Lemma 8.3. □

Remark 8.6. In the special case where ℓ is a non-zero divisor for M we have $a(0:_M \ell) = \infty$. Then we get back the well-known fact that graded Betti numbers do not change under such hyperplane sections.

Now we specialize the previous results to Gorenstein K-algebras.

Proposition 8.7. *Let $A = R/I$ be a graded Artinian Gorenstein K-algebra and let $c = \operatorname{rank}_K[R]_1$. Let $\ell \in R$ be any linear form. Put $s := \operatorname{reg}(A)$ and $a := a(0:_A \ell)$. Then we have for all $i \in \mathbb{Z}$*

$$\left[\operatorname{tor}_i^R(A,K)\right]_j =$$

$$\begin{cases} \left[\operatorname{tor}_i^{\bar R}(A/\ell A,K)\right]_j & \text{if } j \leq a+i-2 \\ \leq \left[\operatorname{tor}_i^{\bar R}(A/\ell A,K)\right]_j + \left[\operatorname{tor}_{c-i}^{\bar R}(A/\ell A,K)\right]_{s+c-j} & \text{if } a+i-1 \leq j \leq s-a+i+1 \\ \left[\operatorname{tor}_{c-i}^{\bar R}(A/\ell A,K)\right]_{s+c-j} & \text{if } j \geq s-a+i+2 \end{cases}$$

Proof. We denote by $^\vee := \operatorname{Hom}_R(-,K)$ the dualizing functor with respect to $K \cong R/\mathfrak{m}$. It is exact. Thus, the exact sequence

$$0 \to (0:_A \ell)(-1) \to A(-1) \to A \to A/\ell A \to 0$$

provides the exact sequence

$$0 \to (A/\ell A)^\vee(-1) \to A^\vee(-1) \to A^\vee \to (0:_A \ell)^\vee \to 0.$$

Since A is Artinian and Gorenstein, we have $A^\vee \cong A(s)$. Consider the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & (0:_A \ell)(-1) & \to & A(-1) & \xrightarrow{\ell} & A & \to & A/\ell A & \to & 0 \\ & & & & \downarrow\wr & & \downarrow\wr & & & & \\ 0 & \to & (A/\ell A)^\vee(-s-1) & \to & A^\vee(-s-1) & \xrightarrow{\ell} & A^\vee(-s) & \to & (0:_A \ell)^\vee(-s) & \to & 0. \end{array}$$

We conclude that

$$0 :_A \ell \cong (A/\ell A)^\vee(-s). \tag{8.3}$$

Put $\bar{R} = R/\ell R$. Since $A/\ell A$ is Artinian we get by local duality

$$(A/\ell A)^\vee \cong H^0_{\mathfrak{m}}(A/\ell A)^\vee \cong \operatorname{Ext}^{c-1}_{\bar{R}}(A/\ell A, \bar{R})(-c).$$

It follows that

$$\operatorname{Tor}_i^{\bar{R}}((A/\ell A)^\vee, K) \cong \left(\operatorname{Tor}^{\bar{R}}_{c-1-i}(A/\ell A, K)\right)^\vee(-c)$$

and therefore

$$\operatorname{Tor}_i^{\bar{R}}(0 :_A \ell, K) \cong \left(\operatorname{Tor}^{\bar{R}}_{c-1-i}(A/\ell A, K)\right)^\vee(-s-c) \qquad \text{(as } \bar{R}\text{-modules).}$$

In particular, we get for all $i, j \in \mathbb{Z}$ isomorphisms of K-vector spaces

$$\left[\operatorname{Tor}_i^{\bar{R}}(0 :_A \ell, K)\right]_j \cong \left[\operatorname{Tor}^{\bar{R}}_{c-1-i}(A/\ell A, K)\right]_{s+c-j}.$$

Hence Corollary 8.5 proves the claim for all pairs (i, j) with $j \le s - a + i + 1$. Since the minimal free resolution of A is self-dual we have

$$\operatorname{Tor}_i^R(A, K) \cong \left(\operatorname{Tor}^R_{c-i}(A, K)\right)^\vee(-s-c).$$

Using what we have already proved we get for $j \ge s - a + i + 2$

$$\left[\operatorname{tor}_i^R(A, K)\right]_j = \left[\operatorname{tor}^R_{c-i}(A, K)\right]_{s+c-j} = \left[\operatorname{tor}^{\bar{R}}_{c-i}(A/\ell A, K)\right]_{s+c-j}$$

concluding the proof. □

Remark 8.8. Proposition 8.7 provides a somewhat quantitative version of the following fact: The more general ℓ is, the more the graded Betti numbers of A as an R-module are determined by the graded Betti numbers of $A/\ell A$ as an $R/\ell R$-module.

If $\operatorname{reg}(A) \le 2 \cdot a(0 :_A \ell) - 3$, the formula in Proposition 8.7 simplifies as follows:

Corollary 8.9. *Using the notation and assumptions of Proposition 8.7, suppose in addition that $s \le 2a - 3$. Then we have for all $i \in \mathbb{Z}$*

$$[\operatorname{tor}_i^R(A, K)]_j = \begin{cases} [\operatorname{tor}_i^{\bar{R}}(A/\ell A, K)]_j & \text{if } j \le a + i - 2 \\[2ex] \left[\operatorname{tor}^{\bar{R}}_{c-i}(A/\ell A, K)\right]_{s+c-j} & \text{if } j \ge s - a + i + 2. \end{cases}$$

Proof. If $s \le 2a - 3$ then there is no integer j satisfying $a + i - 1 \le j \le s - a + i + 1$. Thus Proposition 8.7 proves the claim. □

Notation 8.10. Let $\underline{h} = (1, h_1, \ldots, h_s)$ be the h-vector of an Artinian K-algebra. Let $c \ge h_1$ be an integer. Then there is a uniquely determined lex-segment ideal $I \subset K[z_1, \ldots, z_c] =: T$ such that T/I has $\underline{h}$ as its Hilbert function. We define

$$\beta_{i,j}(\underline{h}, c) := \left[\operatorname{tor}_i^T(T/I, K)\right]_j.$$

If $c = h_1$ we simply write $\beta_{i,j}(\underline{h})$ instead of $\beta_{i,j}(\underline{h}, h_1)$.

Remark 8.11. The numbers $\beta_{i,j}(\underline{h}, c)$ can be computed numerically without considering lex-segment ideals. Explicit formulas can be found in [16].

Theorem 8.12 ([3], [35])**.** *If $I \subset R = K[x_0, \dots, x_n]$ is a Cohen-Macaulay ideal of codimension c and $\underline{h}$ is the h-vector of R/I then we have for all $i, j \in \mathbb{Z}$*

$$\left[\operatorname{tor}_i^R(R/I, K)\right]_j \leq \beta_{i,j}(\underline{h}, c).$$

Let $X \subset \mathbb{P}^n$ be a reduced arithmetically Gorenstein scheme of codimension c with h-vector $(1, h_1, \dots, h_s)$. Then we must have

$$n \geq c \geq h_1.$$

Moreover, X is non-degenerate if and only if $c = h_1$.

We are now ready for the main result of this paper.

Theorem 8.13. *Let $\underline{h} = (1, h_1, h_2, \dots, h_t, \dots, h_s)$ be an SI-sequence where $h_{t-1} < h_t = \dots = h_{s-t} > h_{s-t+1}$. Put $\underline{g} = (1, h_1 - 1, h_2 - h_1, \dots, h_t - h_{t-1})$. Then we have*

(a) *If $A = R/I$ is a Gorenstein K-algebra with $c = \operatorname{codim} I$ and having $\underline{h}$ as h-vector and an Artinian reduction which has the weak Lefschetz property, then*

$$\left[\operatorname{tor}_i^R(A, K)\right]_j \leq \begin{cases} \beta_{i,j}(\underline{g}, c-1) & \text{if } j \leq s - t + i - 1 \\ \beta_{i,j}(\underline{g}, c-1) + \beta_{c-i,s+c-j}(\underline{g}, c-1) & \text{if } s - t + i \leq j \leq t + i \\ \beta_{c-i,s+c-j}(\underline{g}, c-1) & \text{if } j \geq t + i + 1 \end{cases}$$

(b) *Suppose that K has sufficiently many elements. Let n, c be positive integers such that $n \geq c \geq h_1$. Then there is a reduced, non-degenerate arithmetically Gorenstein subscheme $X \subset \mathbb{P}^n = Proj(R)$ of codimension c with the subspace property (and hence the Artinian reduction of $A = R/I_X$ has the weak Lefschetz property) and h-vector $\underline{h}$, and with equality holding for all $i, j \in \mathbb{Z}$ in* (a)*. Indeed, we can take $X = G_c(\underline{h})$ whenever the set $\mathcal{N}_{c-1,s+1,t}$ is chosen sufficiently general, i.e. satisfies the condition given in Theorem 6.3.*

Proof. We first prove (a). We may assume that A is Artinian. Then it has the weak Lefschetz property, i.e. there is an $\ell \in [R]_1$ such that the multiplication map

$$[A]_{i-1} \xrightarrow{\ell} [A]_i$$

has maximal rank for all $i \in \mathbb{Z}$. It follows that the Hilbert function of $A/\ell A$ is $\underline{g}$. Thus Theorem 8.12 gives

$$\left[\operatorname{tor}_i^{\bar{R}}(A/\ell A, K)\right]_j \leq \beta_{i,j}(\underline{g}, c-1)$$

where $\bar{R} = R/lR$. Therefore Proposition 8.7 provides for all $i, j \in \mathbb{Z}$

$$\left[\operatorname{tor}_i^R(A, K)\right]_j \leq \beta_{i,j}(\underline{g}, c-1) + \beta_{c-i,s+c-j}(\underline{g}, c-1).$$

The claim then follows because $\beta_{i,j}(\underline{g}, c-1) = 0$ if $j > t + i$.

For (b), according to Theorem 6.3 and Lemma 7.5 it suffices to compute the graded Betti numbers of the subscheme $X = G_c(\underline{h}) \subset \mathbb{P}^n$ constructed in Theorem 6.3. The starting point was an arithmetically Cohen-Macaulay subscheme $Z \subset \mathbb{P}^n = \operatorname{Proj}(R)$ of codimension $c-1$ with h-vector $\underline{g}$ which satisfies

$$\left[\operatorname{tor}_i^R(R/I_Z, K)\right]_j = \beta_{i,j}(\underline{g}, c-1) \text{ for all } i, j \in \mathbb{Z}$$

thanks to Corollary 5.10. Let Y be the residual to Z in $G_{c-1,s+1,t}$. Then we have $I_X = I_Z + I_Y$. The graded Betti numbers were computed in Corollary 8.2:

$$\begin{aligned} \left[\operatorname{tor}_i^R(R/I_X, K)\right]_j &= \left[\operatorname{tor}_i^R(R/I_Z, K)\right]_j + \left[\operatorname{tor}_{c-i}^R(R/I_Z, K)\right]_{s+c-j} \\ &= \beta_{i,j}(\underline{g}, c-1) + \beta_{c-i,s+c-j}(\underline{g}, c-1). \end{aligned}$$

Using again the fact that $\beta_{i,j}(\underline{g}, c-1) = 0$ if $j > t+i$, we get the result. $\square$

Next we prove that if $I \subset R$ is a Gorenstein ideal of codimension c such that R/I has the Weak Lefschetz property and maximal h-vector $\underline{h}$ of socle degree s, with $\underline{h}$ bounded above by $\binom{c-1+t}{c-1}$, then the graded Betti numbers of R/I are determined by c, t, s (and equal to the ones of $R/J_c(\underline{h})$), except in the case $s = 2t+1$.

Corollary 8.14. *Let c, s, t be positive integers, where either $s = 2t$ or $s \geq 2t+2$. Define $\underline{h} = (h_0, \ldots, h_s)$ by*

$$h_i = \begin{cases} \binom{c-1+i}{c-1} & \text{if } 0 \leq i \leq t; \\ \binom{c-1+t}{c-1} & \text{if } t \leq i \leq s-t; \\ \binom{s-i+c-1}{c-1} & \text{if } s-t \leq i \leq s \end{cases}$$

Let $I \subset R$ be a Gorenstein ideal of codimension c such that R/I has the weak Lefschetz property and h-vector $\underline{h}$. Then the minimal free resolution of R/I has the shape

$$0 \to R(-s-c) \to F_{c-1} \to \cdots \to F_1 \to R \to R/I \to 0$$

where

$$F_i = R(-t-i)^{\alpha_i} \oplus R(t-s-i)^{\gamma_i} \quad \text{and} \quad \alpha_i = \binom{c+t-1}{i+t}\binom{t-1+i}{t} = \gamma_{c-i}.$$

Proof. We may assume that R/I is Artinian and has the weak Lefschetz property with respect to $\ell = x_{c-1}$. Thus $A/\ell A \cong \bar{R}/(x_0, \ldots, x_{c-2})^{t+1}$. Its graded Betti numbers are easily computed. For the reader's convenience we include the argument.

Let $\varphi : F = \bar{R}^{c+t-1} \to G = \bar{R}^{t+1}$ be the graded homomorphism given by the matrix

$$M = \begin{pmatrix} x_0 & x_1 & \ldots & x_{c-2} & & & & 0 \\ & x_0 & x_1 & \ldots & x_{c-2} & & & \\ & & \ddots & & & \ddots & & \\ 0 & & & x_0 & x_1 & \ldots & & x_{c-2} \end{pmatrix} \in \bar{R}^{t+1,c+t}.$$

The ideal of maximal minors of M is $(x_0, \ldots, x_{c-2})^{t+1}$ and it is resolved by the Eagon-Northcott complex

$$\begin{aligned} 0 \to \wedge^{c+t-1} F \otimes S_{c-2}(G) \to \wedge^{c+t-2} F \otimes S_{c-3}(G) \to \ldots \\ \to \wedge^{t+1} F \otimes S_0(G) \to (x_0, \ldots, x_{c-2})^{t+1} \to 0 \end{aligned}$$

In particular, we get for all $i, j \in \mathbb{Z}$:

$$\left[\operatorname{tor}_i^{\bar{R}}(A/\ell A, K)\right]_j = \beta_{i,j}(\underline{g}), \quad \text{where } \underline{g} = (g_0, \dots, g_t) \quad \text{and } g_i = \binom{c-2+i}{i}.$$

If $s \geq 2t+3$ the claim follows by Corollary 8.9 because $a := a(0 :_A \ell) = s - t$ which then forces $s \leq 2a - 3$. If $s = 2t+2$ then we apply Proposition 8.7. It proves our assertion since we have for $i > 0$ that $[\operatorname{tor}_i^{\bar{R}}(A/\ell A, K)]_j = 0$ if $j \neq i + t$. If $s = 2t$ then Theorem 8.13 shows that $[\operatorname{tor}_i^R(A, K)]_j = 0$ unless $(i, j) = (0, 0)$, $(i, j) = (c, s + c)$ or $j = i + t$ with $1 \leq i \leq c - 1$. But then $[\operatorname{tor}_i^R(A, K)]_{i+t}$ can be computed recursively from the Hilbert function of A. (A similar computation can be found on page 4386 of [49].) □

Remark 8.15. If we have in the preceeding corollary that $s = 2t$ then R/I automatically has the Weak Lefschetz property, since there is just one peak in the Hilbert function, and up to that point the ring R/I agrees with R. (We are grateful to R. Miró-Roig for pointing this out to us.) Furthermore, when $s = 2t$ the result about the resolution also follows from [17], Proposition 16(iii) or [15], Theorem A1.

Example 8.16. Corollary 8.14 cannot be extended to the case $s = 2t + 1$. Indeed, consider the case $c = 4$, $s = 5$ and $t = 2$. Then

$$\underline{h} = (1, 4, 10, 10, 4, 1).$$

The Betti numbers predicted by Corollary 8.14 are represented by the Macaulay [2] diagram

```
; total:      1    16    30    16     1
; ------------------------------------
;      0:      1     -     -     -     -
;      1:      -     -     -     -     -
;      2:      -    10    15     6     -
;      3:      -     6    15    10     -
;      4:      -     -     -     -     -
;      5:      -     -     -     -     1
```

However, Boij [8] has shown that the generic Betti numbers of a Gorenstein Artin algebra with this Hilbert function are

```
; total:      1    10    18    10     1
; ------------------------------------
;      0:      1     -     -     -     -
;      1:      -     -     -     -     -
;      2:      -    10     9     -     -
;      3:      -     -     9    10     -
;      4:      -     -     -     -     -
;      5:      -     -     -     -     1
```

At the suggestion of the referee, we gave some thought to the question of whether there exists a reduced arithmetically Gorenstein scheme with these Betti numbers, but we were unable to construct one.

Also, it was pointed out to us by T. Harima that in Example 4.4 of [38], H. Ikeda constructed an interesting example of an Artinian Gorenstein algebra with the above h-vector, not having the weak Lefschetz property. It is not known whether this Artinian Gorenstein ideal can be lifted to a zero-dimensional reduced, arithmetically Gorenstein subscheme in $\mathbb{P}^4$ or not.

9. Simplicial Polytopes

If the dimension of the ambient space is large enough we can choose the linear forms in the sets $\mathcal{M}_{c,t}$ and $\mathcal{N}_{c,s,t}$ as variables. Then the resulting ideals in Theorems 5.8 and 6.3 are monomial ideals. Thus they correspond to simplicial complexes. Relating our results to the constructions of Billera and Lee in [5], we will show that the simplicial complexes we are getting are very particular. They are triangulations of balls or polytopes and have extremal properties with respect to their Betti numbers.

For the convenience of the reader we recall some basic definitions. Further details can be found in [11], [33] or [54].

A simplicial complex on a vertex set $V = \{v_0, \dots, v_n\}$ is a collection Δ of subsets of V such that all $\{v_i\}$ belong to Δ and $F \in \Delta$ whenever $F \subset G$ for some $G \in \Delta$. The elements of Δ are called the *faces* of Δ. The faces $\{v_i\}$ are called *vertices*, and the maximal faces under inclusion are called *facets*. The dimension of a face $F \in \Delta$ is $\dim F := |F| - 1$. The dimension of Δ is defined to be $\dim \Delta = \max\{\dim F | F \in \Delta\}$. The complex is called *pure* if all its facets have the same dimension.

Let Δ be a $(d-1)$-dimensional simplicial complex. The number of i-dimensional faces of Δ is denoted by f_i. Putting $f_{-1} := 1$, the vector of positive integers $(f_{-1}, f_0, \dots, f_{d-1})$ is called the *f-vector* of Δ. The complex Δ is called *shellable* if Δ is pure and its maximal faces can be ordered $F_1, \dots, F_p$ so that the complex $(\bigcup_{j=1}^{i-1} \bar{F}_j) \cap \bar{F}_i$ is pure of dimension $d-2$ for every i with $2 \leq i \leq p$. Here $\bar{F}_j$ is the complex

$$\bar{F}_j := \{\sigma \in \Delta | \sigma \subset F_j\}. \tag{9.1}$$

Let Δ be a simplicial complex on the vertex set $V = \{v_0, \dots, v_n\}$. The ideal $I_\Delta \subset R = K[x_0, \dots, x_n]$ which is generated by all square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_q}$ such that $\{v_{i_1}, \dots, v_{i_q}\} \notin \Delta$, is called the *Stanley-Reisner ideal of* Δ. The Stanley-Reisner ring $K[\Delta]$ of Δ is the factor ring R/I_Δ. The Hilbert series $H_{K[\Delta]}(z) := \sum_{j=0}^{\infty} h_{K[\Delta]}(j) \cdot z^j$ can be written as

$$H_{K[\Delta]}(z) = \frac{Q(z)}{(1-z)^d}$$

where $d = \dim K[\Delta] = \dim \Delta + 1$ and $Q(z) = \sum_{i=0}^{t} h_i z^i$ is a polynomial in $\mathbb{Q}[z]$. The vector of integers $(h_0, \dots, h_t)$ is called the *h-vector* of Δ. If $K[\Delta]$ is Cohen-Macaulay then the h-vector of Δ and the h-vector of $K[\Delta]$ agree. The f-vector and the h-vector of Δ are related by the equality of polynomials

$$\sum_{j=0}^{t} h_j z^j = \sum_{j=0}^{d} f_{j-1} \cdot z^j (1-z)^{d-j}.$$

It follows in particular that $t \leq d = \dim \Delta + 1$ and

$$n + 1 = f_0 = h_1 + d.$$

Moreover, one can compute the h-vector from the f-vector and vice versa. The Stanley-Reisner ideal I_Δ is determined by the facets of Δ. In fact, one has (cf. for instance [11], Theorem 5.1.4).

Lemma 9.1. *The Stanley-Reisner ideal of a simplicial complex Δ is*

$$I_\Delta = \bigcap_F \mathfrak{p}_F$$

where the intersection is taken over all facets of Δ and $\mathfrak{p}_F$ denotes the prime ideal generated by all variables x_i such that $v_i \notin F$.

The simplicial complex Δ is called Cohen-Macaulay if $K[\Delta]$ is a Cohen-Macaulay ring. Such a complex is pure. A shellable simplicial complex is Cohen-Macaulay (cf., for example, [11], Theorem 5.1.13). Nevertheless, the possible h-vectors of shellable complexes are the same as the h-vectors of Cohen-Macaulay algebras according to the following result of Stanley [53] (cf. also [6]).

Theorem 9.2. *Let $\underline{h} = (h_0, \ldots, h_t)$ be a sequence of positive integers. Then the following conditions are equivalent:*

(a) *$\underline{h}$ is the h-vector of a shellable simplicial complex.*
(b) *$\underline{h}$ is the h-vector of a Cohen-Macaulay algebra.*
(c) *$\underline{h}$ is an O-sequence.*

We want to show that a similar phenomenon occurs for the maximal Betti numbers of shellable simplicial complexes. Let Δ be a simplicial complex on the vertex set $V = \{v_0, \ldots, v_n\}$. The graded Betti numbers of Δ are defined to be

$$\beta_{ij}(\Delta) := \left[\operatorname{tor}_i^R(K[\Delta], K\right]_j.$$

Remark 9.3. The Betti numbers of $K[\Delta]$ may depend on the characteristic of K. In fact, Reisner exhibited in [51] the minimal triangulation of the projective plane. Then $K[\Delta]$ is Cohen-Macaulay if and only if $char(K) \neq 2$. This possible dependence on the characteristic is not made explicit in the notation $\beta_{ij}(\Delta)$ but we will mention assumptions on the characteristic if they are necessary.

Let $\underline{h}$ be the h-vector of Δ. If Δ is Cohen-Macaulay then it is known (cf. Theorem 8.12) that we have for all $i, j \in \mathbb{Z}$

$$\beta_{ij}(\Delta) \leq \beta_{ij}(\underline{h}).$$

The next result shows that this estimate is the best possible, even for shellable complexes.

Proposition 9.4. *Let $\underline{h} = (h_0, \ldots, h_t)$ be an O-sequence. Then there is a shellable simplicial complex Δ of dimension $2t-1$ with h-vector $\underline{h}$ having maximal graded Betti numbers, i.e. $\beta_{ij}(\Delta) = \beta_{ij}(\underline{h})$ for $i, j \in \mathbb{Z}$.*

Proof. Define the integer n by $n+1 = c+2t$ where $c = h_1$. We will use the notation of Definition 5.15. We list the monomials in LOIM($\underline{h}$) in order, $m_1 <_r m_2 <_r \cdots <_r m_p$. Using the bijection $V = \{v_0, \dots, v_n\} \to \{u_1, \dots, u_{c+2t}\}$, given by $v_i \mapsto u_{i+1}$, we define $F_i \subset V$ as the subset corresponding to $\bar{\beta}_{c,t}(m_i)$. Finally, we put

$$\Delta := \bigcup_{i=1}^{p} \bar{F}_i$$

where $\bar{F}_i$ is defined in (9.1). This complex is shellable with shelling order $F_1, F_2, \dots, F_p$ according to [5].

As the next step we define the set $\mathcal{M}_{c,t} \subset R = K[x_0 \dots, x_n]$ by

$$\mathcal{M}_{c,t} := \left\{ M_0, \dots, M_{t+\lfloor \frac{c-1}{2} \rfloor}, L_0, \dots, L_{t+\lfloor \frac{c-2}{2} \rfloor} \right\},$$

with $M_i := x_{2i}$ and $L_i := x_{2i+1}$. We claim that $I_{c,t}(\underline{h}) \subset R$ is the Stanley-Reisner ideal of Δ. Indeed, the definitions immediately imply $\mathfrak{p}_{F_i} = \mathfrak{p}_{c,t}(m_i)$. But Lemma 9.1 provides $I_\Delta = \bigcap_{i=1}^{p} \mathfrak{p}_{F_i}$ and Theorem 5.16 gives

$$I_{c,t}(\underline{h}) = \bigcap_{i=1}^{p} \mathfrak{p}_{c,t}(m_i).$$

Thus we conclude that $I_\Delta = I_{c,t}(\underline{h})$. Hence Theorem 5.8 shows that $\underline{h}$ is the h-vector of Δ, and Corollary 5.10 gives $\beta_{ij}(\Delta) = \beta_{ij}(\underline{h}, c)$. Finally, we get $\dim \Delta = n + 1 - h_1 - 1 = 2t - 1$. □

Now we turn to polytopes. A *d-polytope* P is the d-dimensional convex hull of a finite set of points in $\mathbb{R}^d$. It is called *simplicial* if all its proper faces are simplices. Then the collection of the proper faces of P together with the empty set is called the *boundary complex* $\Delta(P)$. Identifying j-dimensional simplices with their $j+1$ vertices, we consider $\Delta(P)$ as the $(d-1)$-dimensional simplicial complex on the vertex set V, where $V = \{v_0, \dots, v_n\}$ is the set of vertices of P. The h-vector and the graded Betti numbers of the polytope P are defined to be the h-vector and the graded Betti numbers of $\Delta(P)$, i.e. $\beta_{ij}(P) := \beta_{ij}(\Delta(P))$.

We now recall the famous g-theorem, which was conjectured by McMullen [43]. Sufficiency of the condition was proved by Billera and Lee in [5] and the necessity by Stanley in [56].

Theorem 9.5. *A sequence $\underline{h} = (h_0, \dots, h_s)$ of positive integers is the h-vector of a simplicial polytope if and only if $\underline{h}$ is an SI-sequence.*

The main result of this section gives an optimal upper bound for the graded Betti numbers of simplicial polytopes.

Theorem 9.6. *Suppose that K is a field of characteristic zero. Let $\underline{h} = (h_0, \dots, h_s)$ be an SI-sequence where*

$$h_{t-1} < h_t = \cdots = h_{s-t} > h_{s-t+1}.$$

Put $\underline{g} = (1, h_1 - h_0, \dots, h_t - h_{t-1})$. Then we have

(a) *If P is a simplicial d-polytope with h-vector $\underline{h}$ then*

$$\beta_{ij}(P) \le \begin{cases} \beta_{ij}(\underline{g}) & \text{if } j \le s-t+i-1; \\ \beta_{ij}(\underline{g}) + \beta_{h_1-i,s+h_1-j}(\underline{g}) & \text{if } s-t+i \le j \le t+i; \\ \beta_{h_1-i,s+h_1-j}(\underline{g}) & \text{if } j \ge t+i+1. \end{cases}$$

(Observe that necessarily $d = s$.)

(b) *The bounds given in* (a) *are sharp: there is an s-polytope P with h-vector $\underline{h}$ for which all of the bounds given in* (a) *are attained.*

Proof. (a) It is well known that the boundary complex $\Delta(P)$ of P is Gorenstein (cf. for instance [11], Corollary 5.5.6). Moreover, Stanley proves in [56] one direction of the g-theorem by showing that $K[\Delta(P)]$ in fact has the Weak Lefschetz property. Since the Stanley-Reisner ideal $I_{\Delta(P)}$ has codimension h_1, we conclude by Theorem 8.13. Note that the only place in this proof where we need the assumption of characteristic zero is the application of Stanley's result guaranteeing the Weak Lefschetz property.

We now turn to (b). We will first describe the boundary complex following Billera and Lee [5] and then verify that it has the required properties.

Put $c = h_1$. We are looking for a polytope with vertex set $V = \{v_0, \dots, v_{s+c-1}\}$. Let $V' := \{v_0, \dots, v_{c-2+2t}\}$ and $V'' = V \backslash V'$. Now we proceed in a manner similar to our proof of Proposition 9.4. We list the monomials in $LOIM(\underline{g})$ in order, $m_1 <_r m_2 <_r \cdots <_r m_p$. Using the bijection $V' \to \{u_1, \dots, u_{c-1+2t}\}$, which sends $v_i \mapsto u_{i+1}$, we define $F' \subset V$ as the subset corresponding to $\tilde{\beta}_{c-1,t}(m_i)$. Put $F_i := F_i' \cup V''$ and define

$$\Delta := \bigcup_{i=1}^{p} \bar{F}_i$$

where again, $\bar{F}_i$ is defined in (9.1). It follows immediately that Δ is a pure s-dimensional simplicial complex. Let $\partial\Delta$ denote the pure $(s-1)$-dimensional simplicial complex on the vertex set V whose facets are the $(s-1)$-dimensional faces of Δ which are contained in exactly one facet of Δ.

Billera and Lee have shown in [5], section 7 (using the construction of section 5) that $\partial\Delta$ is indeed the boundary complex of a simplicial s-polytope. In order to complete the proof we relate the Stanley-Reisner ideals I_Δ and $I_{\partial\Delta}$ to the ideals constructed in our sections 5 and 6, respectively.

Put $n = s + c - 1$ and define the sets

$$\mathcal{N}_{c-1,s+1,t} = \left\{ M_0, \dots, M_{t+\lfloor \frac{c-1}{2} \rfloor}, L_0, \dots, L_{s-t+\lfloor \frac{c-1}{2} \rfloor} \right\} \subset R = K[x_0, \dots, x_n]$$

and

$$\mathcal{M}_{c-1,t} = \left\{ M_0, \dots, M_{t+\lfloor \frac{c-2}{2} \rfloor}, L_0, \dots, L_{t+\lfloor \frac{c-3}{2} \rfloor} \right\}$$

by $M_i := x_{2i}$ and

$$L_i := \begin{cases} x_{2i+1} & \text{if } 0 \le i \le t + \lfloor \frac{c-3}{2} \rfloor \\ x_{s+c-1-i} & \text{if } t + \lfloor \frac{c-1}{2} \rfloor \le i \le s-t+\lfloor \frac{c-1}{2} \rfloor. \end{cases}$$

Thus, we have $\mathcal{N}_{c-1,s+1,t} = \{x_0, \dots, x_n\}$ and $\mathcal{M}_{c-1,t} = \{x_0, \dots, x_{c-2+2t}\}$.

Using the notation of Lemma 9.1, we see that the ideals $\mathfrak{p}_{F_i}$ and $\mathfrak{p}_{c,t}(m_i)$ have the same minimal generators. It follows that

$$I_\Delta = \bigcap_{i=1}^{p} \mathfrak{p}_{F_i} = \bigcap_{i=1}^{p} \mathfrak{p}_{c,t}(m_i) \cdot R.$$

Therefore, Theorem 5.16 shows that

$$I_\Delta = I_{c-1,t}(\underline{g}) \cdot R,$$

which means in particular that I_Δ defines a scheme in $\mathbb{P}^n$ which is a cone over $Z_{c-1,t}(\underline{g}) \subset \mathbb{P}^{c-2+2t}$. Using Lemma 9.1 again, we observe that $\mathfrak{p}$ is a minimal prime of $I_{\partial\Delta}$ if and only if it is generated by c of the variables $x_0, \ldots, x_n$ and contains exactly one minimal prime ideal of $I_\Delta = I_{c-1,t}(\underline{g}) \cdot R$. Hence Theorem 6.3 provides $I_{\partial\Delta} = J_c(\underline{h})$. Since the graded Betti numbers of $K[\partial\Delta] = R/J_c(\underline{h})$ were computed in Theorem 8.13, the proof is complete. □

Remark 9.7. (i) It is interesting to observe that the passage from Δ to $\partial\Delta$ has an interpretation using Gorenstein liaison. Indeed, the proof above shows that $I_{\partial\Delta} = I_\Delta + I_Y$ where Y is the residual of $I_\Delta = I_{c-1,t}(\underline{g})$ in $I_{G_{c-1,s+1,t}}$.

(ii) Carl Lee pointed out to us a conjecture of Kalai, Kleinschmidt and Lee (cf. [39], Conjecture 2) which states another extremal property of the Billera-Lee polytopes. It seems interesting to explore if this conjecture is related to Theorem 9.6.

Using Corollary 8.14 instead of Theorem 8.13, the proof above provides:

Corollary 9.8. *Let P be a simplicial polytope with h-vector $\underline{h}$ as given in Corollary 8.14. Then the shape of the minimal free resolution of $K[\Delta(P)]$ is the one described in Corollary 8.14 if K has characteristic zero.*

Remark 9.9. A simplicial d-polytope P is called *stacked* if it admits a triangulation Γ which is a $(d-1)$-tree, i.e. Γ is a shellable $(d-1)$-dimensional simplicial complex with h-vector $(1, c-1)$ (cf. [32], Corollary 1.3). The graded Betti numbers of stacked polytopes have been computed by Hibi and Terai in [34] (cf. also [32], Theorem 3.3). Since the h-vector of a stacked d-polytope with $s+c$ vertices is $(1, c, \ldots, c, 1)$, this result may be considered as a special case of Corollary 9.8 (with $t = 1$) if $s \neq 3$.

We want to point out one more instance where Corollary 9.8 applies. Recall that a *cyclic polytope* $C(n+1, d)$ is a d-dimensional simplicial polytope which is the convex hull of $n+1$ distinct points on the moment curve

$$\{(t, t^2, \ldots, t^d) \mid t \in \mathbb{R}\}.$$

The notation $C(n+1, d)$ is justified since the combinatorial type depends only on $n+1$ and d.

The next statement can be viewed as an uniformity result.

Corollary 9.10. *All simplicial polytopes of even dimension d with the same f-vector as the cyclic polytope $C(n+1, d)$ have the same graded Betti numbers.*

Proof. Let P be such a polytope. According to McMullen's Upper Bound theorem ([44]) the cyclic polytope $C(n+1,d)$ has the maximal f-vector among all d-polytopes with $n+1$ vertices. Thus, the h-vector of P is $(h_0,\dots,h_d)$ where

$$h_i = \binom{n-d+i}{i} \quad \text{if } 0 \leq i \leq \frac{d}{2}.$$

Since d is even Corollary 9.8 proves the claim. Note that the assumption on the characteristic in Corollary 9.8 is not needed, by Remark 8.15. □

The shape of the minimal free resolution of $K[\Delta(P)]$ is the one described in Corollary 8.14.

The last result partially answers a question posed to the second author by Jürgen Eckhoff. The case where d is odd, which was the other part of Eckhoff's question, is still open.

10. Final comments

Remark 10.1. Our approach in Theorem 6.3 (and hence Theorem 8.13(b)) is similar to that of [25] in that we construct our linking scheme X to be a generalized stick figure, thus automatically guaranteeing that any choice of an arithmetically Cohen-Macaulay subscheme $W \subset X$ will be geometrically linked to its residual, so we can add the linked ideals to produce our desired arithmetically Gorenstein scheme. There are two important differences between our approach and that of [25]. First, in [25] it was enough to choose X to be a complete intersection (in fact, X had codimension two and so was forced to be a complete intersection). In our situation, complete intersections do not give all possible Hilbert functions! (See Example 10.2.) Also, we had to apply the "sums of linked ideals" method more than once. Second, in [25], as in most applications of liaison, the authors started with the scheme W and found the "right" X to geometrically link it and produce the desired residual. In our case we start with a very reducible X and show that there must exist the "right" W inside it!

We should also remark that our method is quite different from that of Harima [29]. In that paper the author heavily uses the Artinian property. He forms a geometric link of two finite sets of points using a complete intersection, takes the sum of the linked ideals, and considers an ideal quotient on the resulting Artinian ideal which modifies the Hilbert function in the desired way. To mimic that approach we would have to first construct a larger Gorenstein scheme Y' by linking with a complete intersection, find a large number of components of Y' lying on a hyperplane and forming an arithmetically Cohen-Macaulay union of linear spaces with certain Hilbert function, remove these components, repeat this process a certain number of times, and show that the result of this procedure preserves the Gorenstein property. It is possible that such an approach would also give a construction. With the advantage of hindsight, though, something like this does happen. See Remark 7.8.

Example 10.2. As mentioned above, if we try to apply the approach of this paper but use only complete intersections for our links, we cannot obtain all SI-sequences. For example, we consider Remark 3.5 of [21]. There, the authors say, they cannot use their method (basically sums of certain linked ideals using complete intersection links) to obtain Gorenstein rings with Hilbert function

$$1 \;\; 4 \;\; 10 \;\; n \;\; 10 \;\; 4 \;\; 1$$

where $14 \le n \le 20$, although they remark that it is known how to construct them for $n = 20$ using other methods. Here we show that the cases $n = 20$ and $n = 19$ cannot be constructed as the sum of complete intersection linked ideals, but the other cases *can* be so constructed. Of course all of them can be constructed by our method using arithmetically Gorenstein links (and obtaining not only graded Artinian algebras but also reduced arithmetically Gorenstein subschemes of projective space). In addition, as remarked in the Introduction, they have also been obtained by Harima [29] in the Artinian case. This special type of Hilbert function has also been constructed by Iarrobino ([36] Proposition 4.7) in the Artinian case.

Now we will use the notation $h_G(t)$ for the entry in degree t of the h-vector of G, and we will use the same notation as in Remark 4.2 for the linked schemes. We know that we want a codimension 4 Gorenstein scheme G_4 with regularity 7, so the complete intersection G_3 would have to have regularity 8. Hence the sum of the degrees of the generators must be 10. Furthermore, one can check that because we seek $\Delta h_{G_4}(2) = 6$, we must have $h_{G_3}(2) = 6$. That is, G_3 has no quadric generator. Hence G_3 is the complete intersection of two cubics and a quartic. Furthermore, the linked scheme Z_3 must also have $h_{Z_3}(2) = 6$ and $h_{Y_3}(2) = 6$. Hence the regularity of Z_3 is at most 5. We get the following diagram:

	degree:								
	0	1	2	3	4	5	6	7	8
G_3	1	3	6	8	8	6	3	1	0
Z_3	1	3	6	a	b	0			
Y_3	1	3	6	$8-b$	$8-a$	0			
ΔG_4	1	3	6	$n-10$	$10-n$	-6	-3	-1	0
G_4	1	4	10	n	10	4	1	0	0

where a and b still have to be determined. As a result, we have

$$n - 10 = (8 - b) + a - 8 = a - b.$$

Since clearly $a \le 8$, we get that $n = 20$ and $n = 19$ cannot be achieved in this way. Furthermore, any n in the range $14 \le n \le 18$ *can* be constructed in this way. It is enough to take Z_3 to be a set of n general points in $\mathbb{P}^3$ (hence $a = n - 10$ and $b = 0$) and G_3 the complete intersection of two cubics and one quartic containing Z_3.

Remark 10.3. In the process of thinking about the construction of the arithmetically Gorenstein schemes in Theorem 6.3, we were led to consider the following question (as a special case), which is of independent interest. Suppose that X is an arithmetically Cohen-Macaulay union of lines in projective space. Is it possible to remove lines from X one by one such that at each step the union of the remaining lines is arithmetically Cohen-Macaulay, and the Hilbert function of the general hyperplane section is the truncated Hilbert function in the sense of [22]? At first glance the answer would seem to be yes (and we believe it to be yes) since the hyperplane section is a finite set of points, and it is known that one can take away a point at a time giving truncated Hilbert functions. But the problem is that removing the corresponding line from X does not guarantee that the remaining union of lines will be arithmetically Cohen-Macaulay.

For example, let X be the union of three lines A, B and C such that B meets each of A and C but A and C are disjoint. This is arithmetically Cohen-Macaulay (of arithmetic genus 0). Clearly removing either A or C preserves the property of being arithmetically Cohen-Macaulay while removing B does not.

We end with a question posed by the referee.

Question 10.4. *Consider the stratum of the arithmetically Gorenstein schemes of given codimension, dimension and h-vector $\underline{h}$ (an SI-sequence), that have the Weak Lefschetz property and maximal Betti numbers, as described in this paper. Is this stratum in the closure of all other graded Betti number strata for arithmetically Gorenstein subschemes of $\mathbb{P}^n$ whose h-vector is $\underline{h}$ and whose graded Betti numbers are bounded by the maximal ones for Weak Lefschetz graded Gorenstein algebras?*

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