Name:____________________

Instructor:____________________

Section number:____________________

Department of Mathematics
University of Notre Dame
Math 10250 – Elem. of Calc. I
Fall 2022

Exam 1

September 15, 2022

This exam is in 2 parts on 10 pages and contains 14 problems worth a total of 100 points. You have 1 hour and 15 minutes to work on it. **No books, notes, phones or other aids are permitted.** Be sure to write your name on this title page, and in case pages become detached put your initials at the top of each.

Honor Pledge: As a member of the Notre Dame community, I will not participate in or tolerate academic dishonesty.

Signature:____________________

You must record here your answers to the multiple choice problems by placing an $\times$ through your answer to each problem.

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)

MC.	______
12.	______
13.	______
14.	______
Tot.	______

Multiple Choice

1. (5 pts.) Suppose that f and g are functions such that $f(2) = 3$, $f(-3) = -2$, $g(3) = 4$ and $g(2) = -3$. What is the value of $(f \circ g)(2)$?

(a) -2

$f(g(2)) = f(-3) = -2$

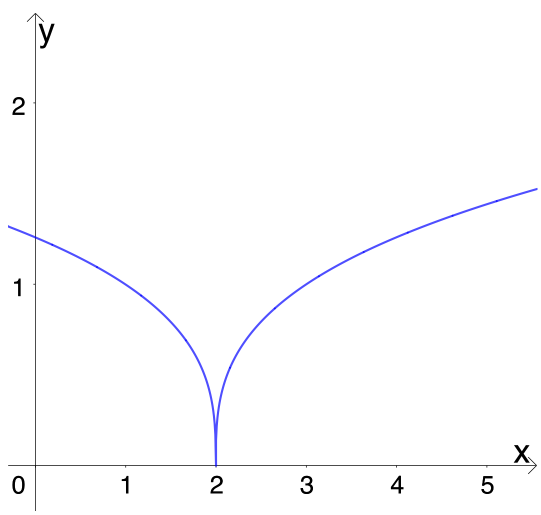
(b) -3

(c) -9

(d) 4

(e) 3

2. (5 pts.) Consider the graph of a function $f(x)$ given below:



$\lim_{x \to 2} f(x) = 0$ and $f(2) = 0$ so $f(x)$ is continuous at $x=2$. But $f(x)$ is not differentrable at $x=2$ since it has a vertical tangent there.

Which of the following is true?

(a) At $x = 2$, $f(x)$ is both continuous and differentiable.

(b) At $x = 2$, $f(x)$ is differentiable but not continuous.

(c) At $x = 2$, $f(x)$ is neither differentiable nor continuous.

(d) At $x = 2$, $f(x)$ is continuous but not differentiable.

(e) $\lim_{x \to 2} f(x)$ is undefined.

3. (5 pts.) Which of the following expressions is equal to

$$\frac{x-1}{x+2} - \frac{x+2}{x-1}?$$

√ (a) $-\dfrac{6x+3}{(x-1)(x+2)}$

(b) $\dfrac{2x+1}{3}$

(c) $-\dfrac{2x^2+2x+5}{(x-1)(x+2)}$

(d) $\dfrac{6x-3}{(x-1)(x+2)}$

(e) $-\dfrac{2x+1}{3}$

$$\frac{x-1}{x+2} - \frac{x+2}{x-1} = \frac{(x-1)^2 - (x+2)^2}{(x+2)(x-1)}$$

$$= \frac{(x^2-2x+1) - (x^2+4x+4)}{(x-1)(x+2)}$$

$$= \frac{x^2-2x+1-x^2-4x-4}{(x-1)(x+2)}$$

$$= \frac{-6x-3}{(x-1)(x+2)} = -\frac{6x+3}{(x-1)(x+2)}$$

4. (5 pts.) Mary has 100 feet of fencing and wants to make a rectangular enclosure. If the width of the rectangle is x, what function measures the area of the rectangle? (*Hint: as a first step you'll have to find the length as a function of* x.)

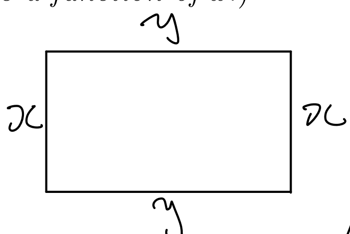
(a) $100x - x^2$

(b) $50x - 2x^2$

√ (c) $50x - x^2$

(d) $x^2 - 50x$

(e) There is not enough information to answer the question.



$x + x + y + y = 100$

$2y = 100 - 2x$

$y = \dfrac{100-2x}{2} = 50 - x$

Area $= xy = x(50-x) = 50x - x^2$

5. (5 pts.) Suppose $f(x) = x^2 + 1$ and $g(x) = x^2 - 1$. Find $(f \circ g)(x)$

(a) $(f \circ g)(x) = x^4 - 2x^2 + 3$

√ (b) $(f \circ g)(x) = x^4 - 2x^2 + 2$

(c) $(f \circ g)(x) = x^4 - 1$

(d) $(f \circ g)(x) = x^4 + 2x^2 + 1$

(e) $(f \circ g)(x) = x^4 + 2x^2$

$$(f \circ g)(x) = f(g(x)) = f(x^2-1) = (x^2-1)^2 + 1 = (x^4 - 2x^2 + 1) + 1 = x^4 - 2x^2 + 2$$

6. (5 pts.) Suppose that the function

$$h(t) = \frac{125t^2}{5t^2 + 25t + 100}$$

describes the height (in meters) of a certain species of tree after t years. The tree can live for centuries. What is the approximate height of a very old member of this species? (*Hint: think about the limit as t approaches infinity.*)

(a) 5 meters

(b) 0 meters

(c) 125 meters

√ (d) 25 meters

(e) The tree grows arbitrarily high if you wait long enough.

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \frac{125t^2}{5t^2+25t+100} = \lim_{t\to\infty} \frac{125t^2}{5t^2\left(1+\frac{5}{t}+\frac{20}{t^2}\right)} = \lim_{t\to\infty} 25 \cdot \lim_{t\to\infty} \frac{1}{1+\frac{5}{t}+\frac{20}{t^2}} = 25 \cdot 1 = 25$$

7. (5 pts.) Let $f(x)$ be the function

$$f(x) = \begin{cases} \dfrac{x^2-3x+2}{x-1} & \text{if } x \neq 1 \\ c & \text{if } x = 1. \end{cases}$$

Which value of c will make $f(x)$ continuous on all real numbers?

(a) -2

(b) 3

(c) 0

(d) There is no such c.

√ (e) -1

$$f(x) = \begin{cases} \dfrac{x^2-3x+2}{x-1} = \dfrac{(x-2)(x-1)}{x-1} = x-2 & \text{if } x \neq 1 \\ c & \text{if } x = 1 \end{cases}$$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} (x-2) = 1-2 = -1$

If f is continuous,

$\lim_{x\to 1} f(x) = f(1)$ ie $-1 = c$

8. (5 pts.) Let $f(x)$ be a function such that $f(0) = 0$, $f(2) = 10$ and $f'(2) = 3$. Which of the following is **FALSE**?

√ (a) The average rate of change of $f(x)$ on $[0, 2]$ is 3. Average rate of change $\Rightarrow \dfrac{f(2)-f(0)}{2-0} = \dfrac{10-0}{2} = 5 \neq 3$

(b) The slope of the tangent line to $f(x)$ at $x = 2$ is 3. Slope $= f'(2) = 3$

(c) $f(x)$ is continuous at $x = 2$. f is differentiable at $x=2$ so is continuous at $x=2$

(d) The instantaneous rate of change of $f(x)$ at $x = 2$ is 3. Instant. rate of change $= f'(2) = 3$

(e) $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = 3$ The left hand side is $f'(2) = 3$.

9. (5 pts.)

The slope of the tangent line to $f(x) = \frac{x^3}{16} + 2\sqrt{x} + 27$ at $x = 4$ is

(a) 3

(b) 4

√(c) 3.5

(d) 0

(e) −3

Slope is $f'(4)$.

$f(x) = \frac{1}{16}x^3 + 2x^{1/2} + 27$

$f'(x) = \frac{1}{16}\cdot 3x^2 + 2\cdot\frac{1}{2}x^{-1/2} = \frac{3x^2}{16} + \frac{1}{\sqrt{x}}$

$f'(4) = \frac{3\cdot 4^2}{16} + \frac{1}{\sqrt{4}} = 3 + \frac{1}{2} = 3\frac{1}{2}$

10. (5 pts.) Find $h'(2)$ when $h(x) = f(x)g(x)$, $f(2) = -1$, $f'(2) = \frac{1}{3}$, $g(2) = 3$, and $g'(2) = 5$.

(a) 0

(b) $-\frac{5}{3}$

√(c) −4

(d) 2

(e) There is not enough information given

$h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(2) = f'(2)g(2) + f(2)g'(2)$

$= \frac{1}{3}\cdot 3 + (-1)\cdot 5 = 1 - 5 = -4$

11. (5 pts.)
Evaluate the following limit:

$$\lim_{h\to 0} \frac{2(1+h)^3-2}{h}$$

√(a) 6

(b) -2

(c) 2

(d) 7

(e) The limit does not exist.

Method 1 Compare with $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$.

Try to choose function f and number a so $\frac{f(a+h)-f(a)}{h} = \frac{2(1+h)^3-2}{h}$. Try $\begin{cases} f(a+h) = 2(1+h)^3 \\ f(a) = 2. \end{cases}$

Guess $f(x) = 2x^3$. $f(a) = 2a^3 = 2$ gives $a = 1$.
$f(a+h) = f(1+h) = 2(1+h)^3$ so this guess works.
So the limit is $f'(1)$ where $f(x) = 2x^3$.
Then $f'(x) = 6x^2$, and $f'(1) = 6 \cdot 1^2 = 6$.

$$\lim_{h\to 0} \frac{2(1+h)^3-2}{h} = 6$$

Method 2 $(1+h)^3 = (1+h)(1+h)^2 = (1+h)(1+2h+h^2)$
$= 1(1+2h+h^2) + h(1+2h+h^2)$
$= 1 + 2h + h^2 + h + 2h^2 + h^3$
$= 1 + 3h + 3h^2 + h^3$.

$$\lim_{h\to 0} \frac{2(1+h)^3-2}{h} = \lim_{h\to 0} \frac{2(1+3h+3h^2+h^3)-2}{h}$$

$$= \lim_{h\to 0} \frac{2+6h+6h^2+2h^3-2}{h} = \lim_{h\to 0} \frac{6h+6h^2+2h^3}{h}$$

$$= \lim_{h\to 0} (6+6h+2h^2) = 6.$$

Partial Credit

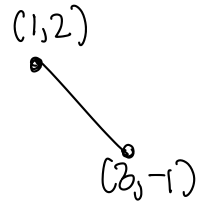
You must **show your work** on the partial credit problems to receive credit!

12. (15 pts.) (Again, be sure to show all work.)

(a) What is the slope of the line passing through the points $(1, 2)$ and $(3, -1)$?

has slope $\dfrac{y_2 - y_1}{x_2 - x_1}$.

For



, slope $= \dfrac{2-(-1)}{1-3} = \dfrac{3}{-2} = -\dfrac{3}{2}$

(b) What is the slope of a line which is perpendicular to the line in (a)?

If a line is not horizontal or vertical and has slope p, a line perpendicular to it has slope $-\frac{1}{p}$.

If $p = -\frac{3}{2}$ as in (a), $-\dfrac{1}{m} = -\dfrac{1}{-\frac{3}{2}} = \dfrac{2}{3}$

(c) Find the equation of the line which passes through the point $(1, 1)$ and is perpendicular to the line in (a). Put your final answer for the equation of the line in slope-intercept form ($y = mx + b$).

The line through (x_1, y_1) with slope m has equation $y - y_1 = m(x - x_1)$.

We have $m = \frac{2}{3}$ from (b) and $(x_1, y_1) = (1, 1)$.

The equation is

$$y - 1 = \frac{2}{3}(x - 1)$$
$$y - 1 = \frac{2}{3}x - \frac{2}{3}$$
$$y = \frac{2}{3}x + \frac{1}{3}$$

The last form is slope-intercept form $\left(m = \frac{2}{3}, b = \frac{1}{3}\right)$.

13. (15 pts.)
Consider the function $f(x) = 5x^2$.

(a) Use the limit definition to compute the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

by completing the following steps:

(i) Compute $f(x+h)$.

$$f(x+h) = 5(x+h)^2 = 5(x^2 + 2hx + h^2) = 5x^2 + 10hx + 5h^2$$

(ii) Compute the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

$$\frac{f(x+h) - f(x)}{h} = \frac{(5x^2 + 10hx + 5h^2) - 5x^2}{h} = \frac{10hx + 5h^2}{h} = 10x + 5h$$

(iii) Compute the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

The limit is

$$\lim_{h\to 0} (10x + 5h) = \lim_{h\to 0} 10x + \lim_{h\to 0} 5h = 10x + 0 = 10x$$

(b) What is the derivative of $5x^2 + 120$?
(*Hint*: Use derivative rules. No need to compute a limit.)

$$\frac{d}{dx}(5x^2 + 120) = \frac{d}{dx}(5x^2) + \frac{d}{dx}(120) = 10x + 0 = 10x$$

from (a) or differentiation rules

from rule for derivative of a constant.

14. (15 pts.) Let

$$f(x) = \frac{x^2 - 4x + 7}{2x - 1}.$$

(a) Find the derivative of $f(x)$. There is no need to simplify.

$$f'(x) = \frac{(2x-1)(x^2-4x+7)' - (x^2-4x+7)(2x-1)'}{(2x-1)^2}$$

$$= \frac{(2x-1)(2x-4) - (x^2-4x+7)\cdot 2}{(2x-1)^2}$$

(b) Find the equation of the tangent line to the graph of $y = f(x)$ at $x = 1$.

The tangent has slope $f'(1) = \frac{(2\cdot 1-1)(2\cdot 1-4) - (1^2-4\cdot 1+7)\cdot 2}{(2\cdot 1-1)^2}$

$= \frac{1\cdot -2 - 4\cdot 2}{1^2} = \frac{-10}{1} = -10.$

Then $x=1$, $y = f(1) = \frac{1^2-4\cdot 1+7}{2\cdot 1-1} = \frac{4}{1} = 4.$

The tangent passes through $(1, 4)$ with slope -10.
Its equation is

$y - 4 = -10(x-1)$, $y - 4 = -10x + 10$

or $y = -10x + 14$.