Name:________________________________

Instructor:______________________________

Time MWF class meets:_______________

Department of Mathematics
University of Notre Dame
Math 10250 – Elem. of Calc. I
Fall 2022

# Exam 2

**October 11, 2022**

This exam is in 2 parts on 10 pages and contains 12 problems worth a total of 100 points. You have 1 hour and 15 minutes to work on it. **No books, notes, phones or other aids other than calculators are permitted.** Be sure to write your name on this title page, and in case pages become detached put your initials at the top of each.

**Honor Pledge:** As a member of the Notre Dame community, I will not participate in or tolerate academic dishonesty.

Signature:________________________________

**You must record here your answers to the multiple choice problems by placing an × through your answer to each problem.**

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## Multiple Choice

**1.** (5 pts.) The cost of making $x$ supercars is $C(x) = 2\sqrt{x^2 - 16}$ million dollars. What is the marginal cost (in millions of dollars) when 5 cars are produced?

(a) 5

(b) 6

(c) $\frac{5}{3}$

(d) 3

(e) $\frac{10}{3}$

$C(x) = 2(x^2-16)^{1/2}$

$C'(x) = 2\left(\frac{1}{2}\right)(x^2-16)^{-1/2}(2x)$

$= \frac{2x}{\sqrt{x^2-16}}$

$C'(5) = \frac{2(5)}{\sqrt{25-16}} = \frac{10}{\sqrt{9}} = \frac{10}{3}$

**2.** (5 pts.) $y = f(u)$ and $u = h(x)$. What is $\dfrac{dy}{dx}$ when $x = 2$, given that

$$h(2) = 3,\ f(2) = 4,\ h'(2) = 5,\ h'(3) = -5,\ f'(2) = 6,\ f'(3) = 7, \text{ and } f'(4) = 8?$$

(a) 30

(b) 35

(c) 40

(d) 6

(e) -30

$y = f(h(x))$

$\frac{dy}{dx} = f'(h(x))\,h'(x)$

$\left.\frac{dy}{dx}\right|_{x=2} = f'(h(2))\,h'(2)$

$= f'(3)(5)$

$= 7 \cdot 5 = 35$

**3.** (5 pts.) The volume $V$ in cubic centimeters of a sphere is given by

$$V = \frac{4}{3}\pi r^3$$

where $r$ is the radius of the sphere in centimeters.

A child who is blowing up a spherical balloon accidentally lets it go. A few moments after it is released, the balloon has a radius of 10 cm and the air pressure inside the balloon is forcing the air out through the balloon's opening at a rate of 4,000 cubic centimeters per second. At what rate is the radius of the balloon decreasing at that time? (Assume the balloon remains spherical while deflating.)

(a) $10/\pi$ centimeters per second

(b) $1/\pi$ centimeters per second

(c) $5/\pi$ centimeters per second

(d) $30/\pi$ centimeters per second

(e) $20/3\pi$ centimeters per second

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

At that moment $r = 10$ and $\frac{dV}{dt} = -4000$

So $-4000 = 4\pi(10^2)\frac{dr}{dt}$

$$\frac{-4000}{400\pi} = \frac{dr}{dt} = -\frac{10}{\pi}$$

the negative means the quantity is decreasing.

**4.** (5 pts.) Let $f(x) = \sqrt{x}$. Which of the following is the second derivative $f''(x)$ of $f(x)$ ?

(a) $\frac{1}{4}x^{-\frac{3}{2}}$

(b) $-\frac{1}{2}x^{-\frac{1}{2}}$

(c) $\frac{1}{2}x^{-\frac{1}{2}}$

(d) $\frac{3}{8}x^{-\frac{5}{2}}$

(e) $-\frac{1}{4}x^{-\frac{3}{2}}$

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = \frac{-1}{4}x^{-3/2}$

**5.** (5 pts.) Suppose $f(x)$ is a function and you know the following information:

- $f(x)$ is defined for all $x$.
- $f'(x) = 0$ for $x = 1$ and $x = 5$ but no other $x$.
- $f'(0) = -4, \quad f'(3) = 6 \quad$ and $f'(7) = -1$.

Which of the following is **TRUE**? (Hint: draw a number line for the sign of $f'(x)$.)

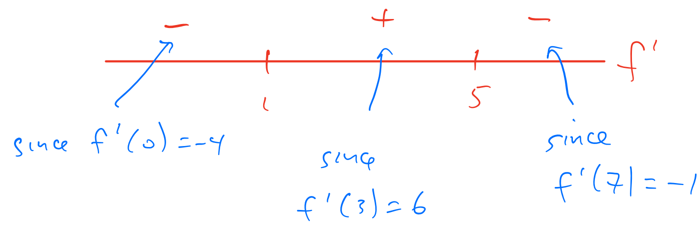
(a) $f(x)$ has a relative max at $x = 1$, and a relative min at $x = 5$.

(b) $f(x)$ has a relative min at both $x = 1$ and $x = 5$.

(c) $f(x)$ has a relative min at $x = 1$, and a relative max at $x = 5$.

(d) $f(x)$ has a relative max at both $x = 1$ and $x = 5$.

(e) $f(x)$ is a constant function.




So rel min at $x = 1$

rel max at $x = 5$

**6.** (5 pts.) Let $f(x) = 3x^2 - 6x + 4$. Which of the following is **TRUE** about $f(x)$?

(a) $f(x)$ has a relative min at $(1, 1)$ and is concave down at that point.

(b) $f(x)$ has a relative min at $(1, 0)$ and is concave up at that point.

(c) $f(x)$ has a relative min at $(1, 1)$ and is concave up at that point.

(d) $f(x)$ has a relative min at $(1, 0)$ and is concave down at that point.

(e) $f(x)$ has a relative max at $(1, 1)$ and is concave down at that point.

Note $f(1) = 3 - 6 + 4 = 1$ so the point $(1,1)$ is on the graph and $(1,0)$ isn't

$f'(x) = 6x - 6$

$f''(x) = 6 > 0$ so concave up

then rel. min. by 2nd derivative test

**7.** (5 pts.) Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{4x}{x^2 - x - 6} \qquad = \frac{4x}{(x-3)(x+2)}$$

(a) Horizontal: $y = 4$; Vertical: $x = 2$ and $x = -3$.

(b) Horizontal: $y = 4$; Vertical: None

(c) Horizontal: $y = 2$ and $y = -3$; Vertical: $x = 0$.

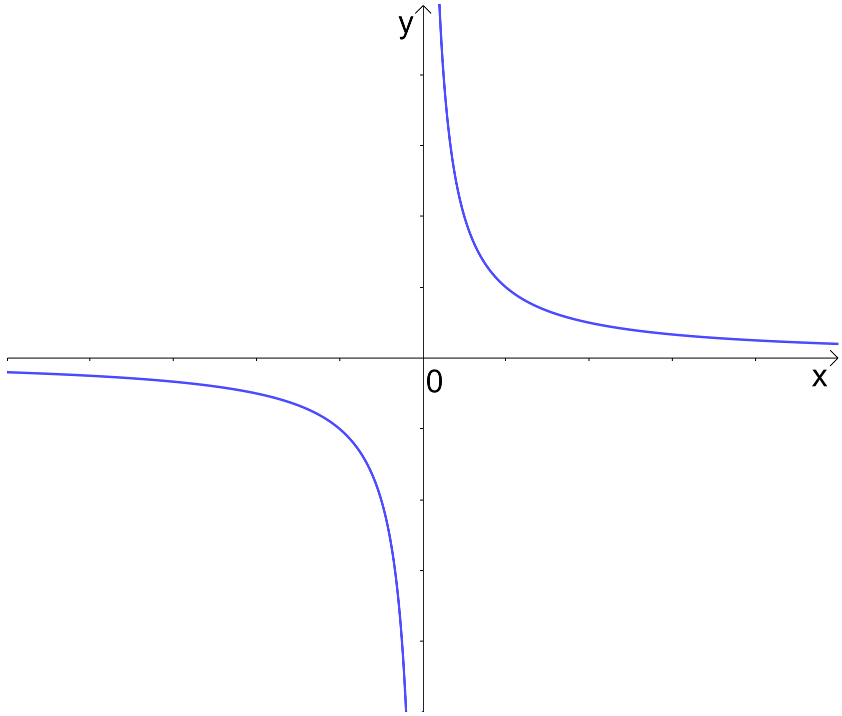
(d) Horizontal: $y = 0$; Vertical: $x = -2$ and $x = 3$.

(e) Horizontal: $y = 0$; Vertical: $x = 0$.

Since the denominator has degree 2 and the numerator only degree 1, we get $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$ so $y=0$ is the horizontal asymptote.

Since the denominator vanishes at $x=3$ and $x=-2$, and the numerator doesn't, we have vertical asymptotes at $x=3$, $x=-2$

**8.** (5 pts.) The graph of a function $f(x)$ is given below.




Which of the following statements is **FALSE**?

(a) $f'(x) > 0$ for $x > 0$. in fact $f'(x) < 0$ for $x > 0$ (decreasing)

(b) $f(x)$ has a vertical asymptote at $x = 0$. true

(c) $f(x)$ is concave down for $x < 0$. true

(d) $f''(x) > 0$ for $x > 0$. true (concave up)

(e) $f'(x) < 0$ for $x < 0$. true (decreasing)

**Partial Credit**

You must **show your work** on the partial credit problems to receive credit!

**9.** (15 pts.)

The cost of making bikes is $C(x) = 6000 + 200x - x^2$. The price $p$ depends on $x$, the number of units produced, by the formula

$$p(x) = 1000\frac{x+5}{x-5}.$$

- What is the marginal cost when 100 bikes are produced?

$MC = C'(x) = 200 - 2x$

$C'(100) = 0$

- What is the average cost function, $\overline{C}(x)$?

$\overline{C}(x) = \frac{C(x)}{x} = \frac{6000 + 200x - x^2}{x} = \frac{6000}{x} + 200 - x$

$= 6000x^{-1} + 200 - x$

- What is the marginal average cost when 100 bikes are produced?

$\overline{C}'(x) = -6000x^{-2} - 1$

$= \frac{-6000}{x^2} - 1$

so $\overline{C}'(100) = \frac{-6000}{10000} - 1$

$= -0.6 - 1$

$= -1.6$

- What is the revenue function?

$R(x) = xp(x) = 1000x \cdot \left(\frac{x+5}{x-5}\right)$

- What is the profit function?

$P(x) = R(x) - C(x) = 1000x\left(\frac{x+5}{x-5}\right) - (6000 + 200x - x^2)$

**10.** (15 pts.) Consider the curve defined implicitly by the equation $x^3 + 3x + y^3 + 3y = 8$.

(a) By differentiating the equation implicitly, find an expression for $\dfrac{dy}{dx}$ in terms of $x$ and $y$.

$$3x^2 + 3 + 3y^2 \frac{dy}{dx} + 3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(3y^2 + 3\right) = -3x^2 - 3$$

$$\frac{dy}{dx} = -\frac{3x^2+3}{3y^2+3}$$

(b) Explain clearly why the point $(1, 1)$ is on the curve.

Plug in $x=1, y=1$ into the original equation. You get $(1)^3 + 3(1) + (1)^3 + 3(1) = 8$ which is true, so $(1,1)$ is on the curve.

(c) Determine the equation of the tangent line to the curve at the point $(1, 1)$.

slope: $\left.\dfrac{dy}{dx}\right|_{\substack{x=1\\y=1}} = -\dfrac{3(1)^2+3}{3(1)^2+3} = -1$

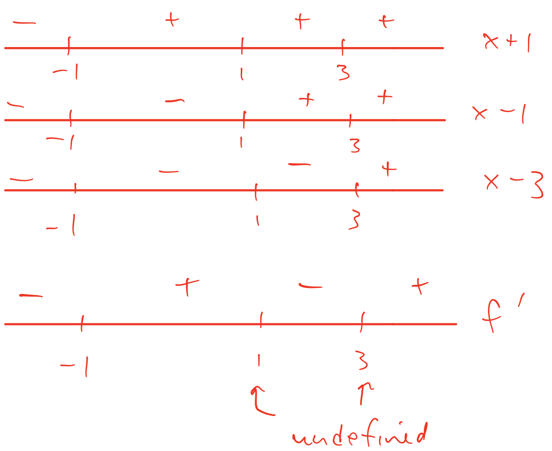
$y-1 = -1(x-1)$ is the desired equation.

**11.** (15 pts.) Let $f(x)$ be some function whose **derivative** is

$$f'(x) = \frac{x+1}{(x-1)(x-3)}.$$

(Note this is already $f'$ – you don't have to differentiate it.) Assume also that the original function $f(x)$ has a vertical asymptote (in particular is undefined) at $x = 1$ and at $x = 3$ but is defined everywhere else.

(a) Tell us on what intervals $f(x)$ is increasing and on what intervals it is decreasing. Be sure to show all necessary work – you must give some justification to get full credit.




increasing on $(-1, 1)$ and $(3, \infty)$

decreasing on $(-\infty, -1)$ and $(1, 3)$

(b) Using your results from (a) and taking into account the asymptotes, identify the values of $x$, if any, where $f(x)$ has relative maxima and relative minima.

rel min at $x = -1$

no rel. max

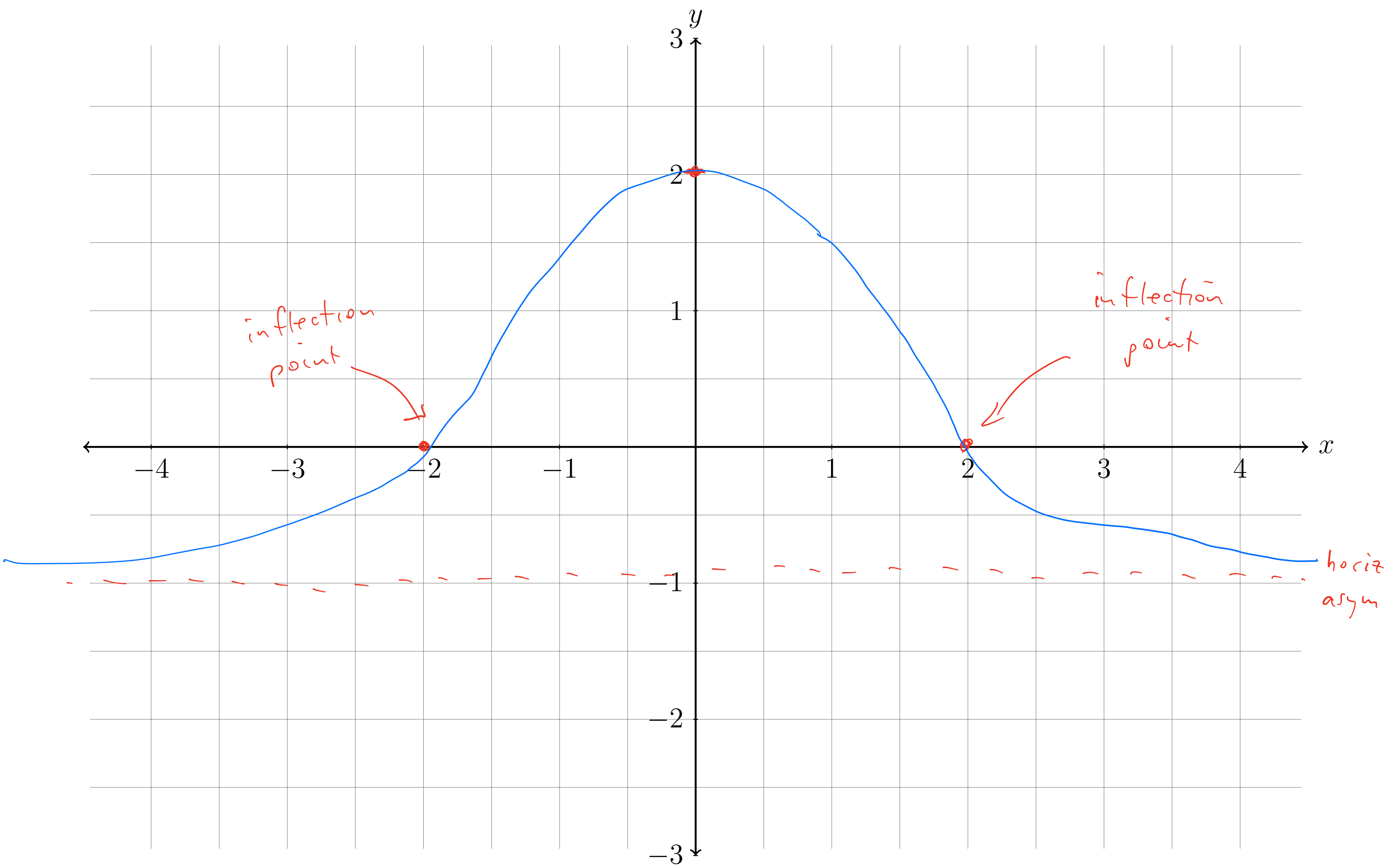
**12.** (15 pts.) Draw the graph of a function $y = f(x)$ satisfying the following list of properties. Specify the coordinates $(x, y)$ of any relative minima, relative maxima, and inflection points in the blanks below (write NONE if there are none). Draw any asymptotes with a dotted line.

- The domain is all real numbers.
- $f(-2) = 0$, $f(0) = 2$, and $f(2) = 0$.
- $\lim_{x\to-\infty} f(x) = -1$
- $\lim_{x\to\infty} f(x) = -1$
- $f'(x) < 0$ on $(0, \infty)$.
- $f'(x) > 0$ on $(-\infty, 0)$.
- $f'(0) = 0$.
- $f''(x) < 0$ on $(-2, 2)$.
- $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$.

Inflection point(s) at ____________________

Relative maxima at ____________________

Relative minima at ____________________




increasing

rel max at (0,2)

decreasing

conc up

conc down

conc up

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