Math 10250 Exam 3 Solutions November 17, 2022

1. What is the maximum value for $f(x) = x^3 - 3x^2$ in the interval [0,4]. (Notice this same function and interval appears in problem 2 as well. The other time it asks for the minimum value.)

Solution:

The derivative of f is $f'(x) = 3x^2 - 6x$. The critical points are where f'(x) = 0 or f'(x) does not exist. Here f'(x) always exists so that is a non-issue. We see f'(x) = 0 when $0 = 3x^2 - 6x = 3x(x-2)$. Or when x = 0 or x = 2. We also have to include the end points, x = 0 and x = 4. Hence we have three points to check: x = 0, 2, 4. f(0) = 0, f(2) = -4, and f(4) = 16. The maximum value is 16 when x = 4.

2. What is the minimum value for $f(x) = x^3 - 3x^2$ in the interval [0,4]. (Notice this same function and interval also appears in problem 1. The other time it asks for the maximum value.)

Solution:

This is the same function as in the first problem, so most of the work was done there. The minimum value is -4 when x = -2.

3. Which of the following real numbers x satisfies the equation

$$\frac{e^x}{1-e^x} = 2?$$

Solution:

Multiplying $\frac{e^x}{1-e^x} = 2$ by $1-e^x$ gives $e^x = 2(1-e^x) = 2-2e^x$. So $3e^x = 2$, $e^x = \frac{2}{3}$ and $x = \ln(\frac{2}{3}) = \ln 2 - \ln 3$.

4. Use logarithmic differentiation to find the derivative of the function $f(x) = x^{2x}$. Solution: Taking logarithms,

$$\ln f(x) = \ln(x^{2x}) = 2x \ln x.$$

Differentiating with respect to x, using the chain rule and product rule, gives

$$\frac{f'(x)}{f(x)} = 2\ln x + 2x\frac{1}{x} = 2\ln x + 2.$$

Multiplying both sides by f(x) gives

$$f'(x) = (2\ln x + 2)x^{2x}.$$

5. Bob's family has a tradition that upon finishing college, the graduate gives his or her parents a check for \$10,000. Bob plans to graduate from Notre Dame in May of 2026, so **exactly four years** in advance (May, 2022) he decided to invest some sum of money so that in four years he would have exactly \$10,000 in the account. He found an account bearing 5% interest, compounded continuously. How much should he have invested in May, 2022, in order to fulfill his plans of reaching \$10,000 in four years?

Solution:

The formula for the amount present at time t is $A(t) = A_0 e^{rt}$ where r is the interest rate, t is the number of years and A_0 is the initial investment. We know r = .05 and t = 4, and A(4) = 10,000 and we want to compute A_0 . This gives

$$10,000 = A_0 e^{0.05(4)} = A_0 e^{0.2},$$

so one computes $A_0 = \$8,187.31$.

6. Find the critical numbers for the function $f(x) = x^3 e^{-x}$ (i.e. find the values of x where f'(x)is either undefined or equal to 0).

Solution:

First, notice that the domain of f(x) is all x. Now we differentiate to get

$$f'(x) = x^{3}(-e^{-x}) + 3x^{2}e^{-x} = x^{2}e^{-x}(3-x).$$

Again this is defined for all x, so the only issue is to find where f'(x) = 0. Since e^x is never zero for any x, we conclude x = 0 or x = 3.

7. Find the equation of the tangent line to the function $f(x) = x^2 \ln(x)$ at x = 1.

Solution:

The slope of the tangent line is given by the derivative, for which we need to use the product rule:

$$f'(x) = (2x)\ln(x) + x^2\left(\frac{1}{x}\right) = 2x\ln(x) + x$$

The slope is then $f'(1) = 2\ln(1) + 1 = 1$ (because $\ln(1) = 0$). When x = 1, the corresponding y-value is $f(1) = \ln(1) = 0$. Thus, the equation of the tangent line is y - 0 = 1(x - 1) or y = x - 1.

8. The number of students who contracted the "freshman flu" this year is modeled by

$$Q(t) = \frac{500}{1 + 99e^{-0.8t}}$$

where t is the number of days since the start of the semester. How many students eventually contracted the disease? (That is, what is the limit when t approaches infinity?)

Solution:

$$\lim_{t \to \infty} Q(t) = \lim_{t \to \infty} \frac{500}{1 + 99e^{-0.8t}} = \frac{500}{1 + 99\lim_{t \to \infty} e^{-0.8t}}$$

As t approaches ∞ , $-0.8t$ approaches $-\infty$. Since $\lim_{x \to -\infty} e^x = 0$,

$$\lim_{t \to \infty} Q(t) = \frac{500}{1 + 99(0)} = 500$$

9. We are asked to design a Petri dish with volume 27π cm³ with **minimal** external surface area. Note that a Petri dish has the shape of a cylinder with no top. What are the radius r and height h of the resulting Petri dish? You may use the following formulas:

> volume $= V = \pi r^2 h$, external surface area $= A = \pi r^2 + 2\pi rh.$

Solution:

We have that $27\pi = \pi r^2 h$. It is best to solve for h. $h = \frac{27}{r^2}$. Notice r > 0. We can write the Area as a function of r alone:

$$A = \pi r^{2} + 2\pi r \frac{27}{r^{2}} = \pi r^{2} + 2\pi \frac{27}{r} = \pi r^{2} + 54\pi r^{-1}.$$

Hence $A' = 2\pi r - \frac{54\pi}{r^2}$. A potential max/min occurs when A'(r) = 0, so we get

$$0 = 2\pi r - \frac{54\pi}{r^2}$$
, i.e. $\frac{54\pi}{r^2} = 2\pi r$, i.e. $\frac{27}{r^2} = r$.

Multipling both sides by r^2 we get $27 = r^3$ or r = 3. Hence $h = \frac{27}{3^2} = 3$.

10. (a) Evaluate the indefinite integral

$$\int \left(\frac{1}{\sqrt{x}} - \frac{3}{x} + 2x - 2 + e^x\right) \, dx.$$

Solution:

$$f(x) = \int \left(\frac{1}{\sqrt{x}} - \frac{3}{x} + 2x - 2 + e^x\right) dx = \int x^{-1/2} dx - 3\int \frac{1}{x} dx + 2\int x dx - \int 2 dx + \int e^x dx$$
$$= 2x^{1/2} - 3\ln x + 2 \cdot \frac{1}{2}x^2 - 2x + e^x + C$$
$$= 2\sqrt{x} - 3\ln x + x^2 - 2x + e^x + C$$

(b) Solve the initial value problem

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{3}{x} + 2x - 2 + e^x, \qquad f(1) = e^x$$

Solution:

The initial condition gives

$$e = f(1) = 2 \cdot \sqrt{1} - 3\ln 1 + 1^2 - 2 \cdot 1 + e^1 + C = 1 + e + C$$

since $\ln 1 = 0$. So C = e - (1 + e) = -1 and

$$f(x) = 2\sqrt{x} - 3\ln x + x^2 - 2x + e^x - 1.$$

- 11. Sally invests \$5,000 in a bank account with an annual interest rate of 3%. In all of the following questions, we are looking for the formula. The numerical answer is not important.
 - (a) (6 points) If the interest is compounded quarterly (meaning 4 times a year), how much will there be after 10 years?

Solution:

The general formula for a deposit of A_0 with annual interest rate r, compounding n times a year for t years, is

$$A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

so we get

$$A(10) = 5,000 \left(1 + \frac{.03}{4}\right)^{4 \cdot 10} = 5,000 \left(1 + \frac{.03}{4}\right)^{40} = 6,741.74$$

(b) (6 points) If the interest is compounded continuously, how much will there be after 10 years?

Solution:

The general formula for a deposit of A_0 with annual interest rate r, compounding continuously for t years, is $A(t) = A_0 e^{rt}$. So we get

$$A(10) = 5,000e^{10(0.03)} = 6,749.29$$

(c) (3 points) In the situation of part (b), write a formula for the effective rate (APR). *Solution:*

In one year the interest factor is $e^{.03}$ so we set $1 + r = e^{.03}$, i.e. $r = e^{.03} - 1 = .03045$.

12. Americium-241, a radioactive isotope, is used in smoke detectors. The amount of Americium-241 present in a new smoke detector after t years is modeled by the function

$$Q(t) = 0.3 \left(\frac{1}{2}\right)^{t/432}$$

where the quantity Q(t) is measured in micrograms.

(a) What is the half-life of Americium-241? (That is, how long does it take for the amount present to reduce to half of its initial amount?)

Since 0.3 is the initial amount, it is reduced by half when $\frac{t}{432} = 1$, i.e. when t = 432. Alternatively, one can solve the equation

$$0.15 = 0.3(1/2)^{t/432}$$

to get the same result.

(b) How long will it take for there to be 0.27 micrograms of Americium-241 remaining in the smoke detector?

Solution:

We solve for the t such that

$$Q(t) = 0.27 = 0.3(1/2)^{t/432}$$

Dividing by 0.3 and taking the natural log gives

$$\ln(0.9) = \ln((1/2)^{t/432}) = \frac{t}{432}\ln(0.5)$$

Solving for t gives

$$t = \frac{432\ln(0.9)}{\ln(0.5)} \approx 65.67$$
 years

(c) Find the rate of decay Q'(t). (Hint: you can write $\frac{1}{2}$ as $e^{\ln(\frac{1}{2})}$) Solution:

Noting that $\frac{1}{2} = e^{\ln(\frac{1}{2})}$, we can rewrite Q(t) as

$$Q(t) = 0.3(e^{\ln(0.5)})^{t/432} = 0.3e^{\frac{\ln(0.5)t}{432}}$$

We can take the derivative using the chain rule:

$$Q'(t) = (0.3) \left(\frac{\ln(0.5)}{432}\right) e^{\frac{\ln(0.5)t}{432}} = (0.3) \left(\frac{-\ln(2)}{432}\right) (1/2)^{t/432}$$