Solutions to the extra practice problems for Exam 3, Fall 2022, Math 10250

1. Let $f(x) = 2x^3 + 3x^2 - 12x$. Which numbers below are the location(s) of the absolute maximum on the interval $[0, 2]$?

   (a) $x = 2$

   (b) $x = 1$

   (c) $x = 1$ and $x = 2$

   (d) $x = 0$ and $x = 1$

   (e) $x = 0$ and $x = 2$

   Solution:
   $f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$ so the critical numbers are $x = 1, -2$. But only $x = 1$ is in the interval $[0, 2]$. We check $f(0) = 0$, $f(2) = 4$ and $f(1) = -7$. So the absolute max comes for $x = 2$.

2. The owner of a restaurant in Duncan Student Center wants to sell Extra-Fancy Chicken Sandwiches. After some testing the demand function was determined to be

   $$p = \frac{81}{x + 2}$$

   where p is the price in dollars of each sandwich and x is the number of sandwiches sold. The profit function was determined to be

   $$P(x) = \frac{81x}{x + 2} - 2x$$

   where $P(x)$ is measured in dollars. What price should the owner charge to maximize profits? [Hint: the question is asking for the price, not the number sold.]

   (a) 10

   (b) 27

   (c) 3

   (d) 9

   (e) The higher the price he sets the more profit is gained.

   Solution:

   $$P'(x) = \frac{(x + 2)(81) - (81x)(1)}{(x + 2)^2} = \frac{162}{(x + 2)^2} - 2.$$

   This is defined for all $x \geq 0$ (which is the natural domain for $P(x)$). Setting $P'(x) = 0$ we get $81 = (x + 2)^2$ so $x = 7$. Then the demand function gives us $p = \frac{81}{7+2} = 9$. Checking the sign of the derivative to the left and right of $x = 7$ shows that on the domain $x > 0$ this is a maximum, not a minimum.

3. Let $y^2 = xe^{-x}$. Use implicit differentiation to express $\dfrac{dy}{dx}$ as a function of x and y.

   (a) $\dfrac{e^{-x} + xe^{-x}}{2y}$

   (b) $\dfrac{e^{-x} - xe^{-x}}{2y}$

   (c) $x + xe^{-x}$

   (d) $2y(x + xe^{-x})$

   (e) $2y(x - xe^{-x})$

   Solution: Applying the chain rule and the product rule, we have

   $$2y\frac{dy}{dx} = 1 \cdot e^{-x} + x(-e^{-x}) = e^{-x} - xe^{-x}. \text{ Therefore, } \frac{dy}{dx} = \frac{e^{-x} - xe^{-x}}{2y}.$$

4. Find the antiderivative: $\displaystyle\int \frac{1}{x} dx$

   (a) $\dfrac{1}{x^2} + C$

   (b) $\ln(x)$

   (c) $-\dfrac{1}{x^2} + C$

   (d) 1

   (e) $\ln|x| + C$

   Solution: An antiderivative of $f(x) = \dfrac{1}{x}$ is the function $F(x) = \ln|x|$. Therefore, $\displaystyle\int \frac{1}{x} dx = \ln|x| + C$.

5. Consider the function

   $$f(x) = e^x x^3 (x^2 + 1).$$

   (a) Use properties of the natural logarithm to write an expression for $\ln(f(x))$. **Simplify your answer as much as possible.**

      Solution:

$$\ln(f(x)) = \ln(e^x x^3 (x^2 + 1)) = \ln(e^x) + \ln(x^3) + \ln(x^2 + 1) = x + 3\ln(x) + \ln(x^2 + 1).$$

(b) Compute $\dfrac{d}{dx}\ln(f(x))$.

Solution: Using part (a), and applying the chain rule for the third term, we have:

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}(x + 3\ln(x) + \ln(x^2+1)) = 1 + \frac{3}{x} + \frac{2x}{x^2+1}$$

(c) Use part (b) to compute $\dfrac{d}{dx}f(x)$.

Solution: Recall that $\dfrac{d}{dx}(\ln(f(x)) = \dfrac{f'(x)}{f(x)}$, so

$$f'(x) = f(x)\frac{d}{dx}(\ln(f(x)).$$

Therefore,

$$f'(x) = f(x)\frac{d}{dx}(\ln(f(x)) = e^x x^3(x^2+1)\left(1 + \frac{3}{x} + \frac{2x}{x^2+1}\right).$$

6. Aaron invests some sum of money in an account with a 6% annual interest rate. For both of the following problems, what we're looking for is the formula, not the actual numerical answer.

(a) Find the effective rate (also known as APR) if the interest is compounded continuously.

Solution: We're looking for the value of r for which

$$(1 + r_{eff})A_0 = (e^{.06})A_0$$

so $r_{eff} = e^{.06} - 1$, which is about .0618.

(b) Find the effective rate (also known as APR) if the interest is compounded monthly.

Solution: We're looking for the value of r for which

$$(1 + r_{eff})A_0 = \left(1 + \frac{.06}{12}\right)^{12} A_0$$

so

$$r_{eff} = \left(1 + \frac{.06}{12}\right)^{12} - 1$$

which is about .0617.