Name:____________________________

Instructor:__________________________

Time MWF class meets:______________

Department of Mathematics
University of Notre Dame
Math 10250 – Elem. of Calc. I
Fall 2022

Practice Exam 2

October 11, 2022

This practice exam is in 2 parts on 11 pages and contains 14 problems. It is a little bit longer than the "real" exam will be (it has more partial credit problems). You should be able to finish it in 1.5 hours. **No books, notes, phones or other aids are permitted.**

Honor Pledge: As a member of the Notre Dame community, I will not participate in or tolerate academic dishonesty.

Signature:______________________________

You must record here your answers to the multiple choice problems by placing an $\times$ through your answer to each problem.

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)

MC.	______
9.	______
10.	______
11.	______
12.	______
13.	______
14.	______
Tot.	______

Multiple Choice

1. (5 pts.) Let $f(x)$ be a continuous function and assume that $f(x)$, $f'(x)$, and $f''(x)$ exist for all real numbers. The table below lists where its first and second derivatives are positive and negative.

Interval	$f'(x)$	$f''(x)$
$(-\infty, -6)$	$+$	$-$
$(-6, 0)$	$+$	$+$
$(0, 6)$	$-$	$-$
$(6, 12)$	$-$	$+$
$(12, \infty)$	$+$	$+$

[For example, this reads $f'(x)$ is negative on $(6, 12)$.] Which of the following is **FALSE**?

(a) $f(x)$ has an inflection point at $x = 6$ - true: f'' changes from $-$ to $+$ at $x=6$

(b) $f(x)$ has a relative minimum at $x = 6$ - false: f' is negative on both sides of $x=6$

(c) $f(x)$ has an inflection point at $x = -6$ - true: f'' changes from $-$ to $+$ at $x=-6$

(d) $f(x)$ has a relative maximum at $x = 0$ - true: f' changes from $+$ to $-$ at $x=0$

(e) $f(x)$ has a relative minimum at $x = 12$ - true: f' changes from $-$ to $+$ at $x=12$

2. (5 pts.) Let $P(t)$ denote the population of song sparrows in South Bend in the past decade. The population was increasing, but the rate of increase has becoming smaller over time (i.e. the rate was decreasing). Which of the following is **TRUE**?

(a) $P'(t) > 0$ and $P''(t) < 0$ $P' > 0$ means P is increasing

(b) $P'(t) < 0$ and $P''(t) > 0$ $P'' < 0$ means rate (P') is decreasing

(c) $P'(t) > 0$ and $P''(t) > 0$

(d) $P'(t) < 0$ and $P''(t) < 0$

(e) The graph of $P(t)$ is concave up.

3. (5 pts.) Find all intervals where $f(x) = \frac{1}{30}x^6 - \frac{1}{12}x^4 - 1000$ is concave up.

(a) $(-\infty, 0)$ and $(0, \infty)$

(b) $(-\infty, -1), (0, 1)$ and $(1, \infty)$

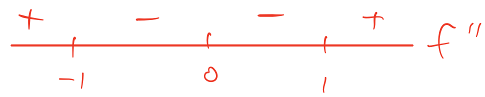
(c) $(-1, 0)$

(d) $(-\infty, -1)$ and $(1, \infty)$

(e) $(-1, 1)$

$f'(x) = \frac{1}{5}x^5 - \frac{1}{3}x^3$

$f''(x) = x^4 - x^2 = x^2(x^2-1) = x^2(x+1)(x-1)$



4. (5 pts.) When analyzing a continuous function $h(x)$, Sally finds that $h'(2) = 0$ and $h''(2) = -3$. What can Sally conclude about $h(x)$? (That is, what *must* be true regardless of what $h(x)$ is?)

(a) $h(x)$ has an inflection point at $x = 2$.

(b) $h(x)$ has a local maximum at $x = 2$.

(c) $h(x)$ has a local minimum at $x = 2$.

(d) $h(x)$ has an absolute minimum at $x = 2$.

(e) The second derivative test is inconclusive.

2nd derivative test

$h''(2) < 0$ and $h'(2) = 0$ means

concave down and critical point,

hence rel. max at $x = 2$

(note "relative max" means the same as "local max")

5. (5 pts.) What are the inflection points of $Q(x) = x^2 + (x+2)^{-1}$?

$Q'(x) = 2x - (x+2)^{-2}$

$Q''(x) = 2 + 2(x+2)^{-3} = 2 + \frac{2}{(x+2)^3}$

(a) $Q(x)$ has only one inflection point, at $x = -2$.

(b) $Q(x)$ has inflection points at $x = -3$ and $x = 0$.

(c) $Q(x)$ has only one inflection point, at $x = -3$.

(d) $Q(x)$ has no inflection points.

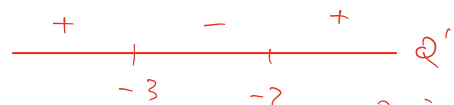
(e) $Q(x)$ has inflection points at $x = -3/2$ and $x = 0$.

Q is undefined at $x = -2$.

Set $Q'' = 0$. $\frac{2}{(x+2)^3} = -2$

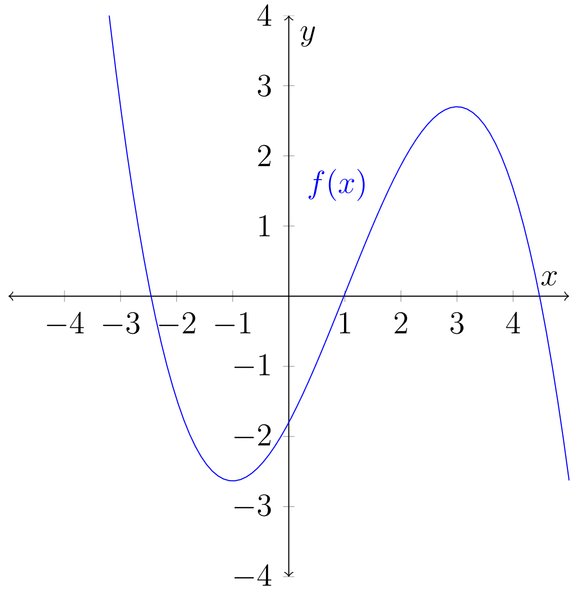
$\frac{1}{(x+2)^3} = -1$

$-(x+2)^3 = 1 \Rightarrow x = -3$



Q is undefined at $x=-2$ so this isn't an inflection point.

6. (5 pts.) Consider a function $f(x)$ having the graph below. Exactly **one** of the following statements is **false**. Which is it?



(a) $f(x)$ is increasing on $(-1, 3)$ and decreasing for $x < -1$ and $x > 3$. true

(b) $f(x)$ has a local minimum at $x = -1$ and a local maximum at $x = 3$. true

(c) $f(x)$ is concave up for $x < 1$ and concave down for $x > 1$. true

(d) $f(x)$ has critical points at $x = -1, x = 1$, and $x = 3$. false — no critical point at $x=1$

(e) $f(x)$ has an inflection point at $x = 1$. true

7. (5 pts.)
Consider the function

$$h(x) = \frac{1}{\sqrt{x-3}}$$

Compute $\lim_{x\to 3^+} h(x)$ and $\lim_{x\to\infty} h(x)$.

(a) $\lim_{x\to 3^+} h(x) = 0$, $\lim_{x\to\infty} h(x) = \infty$

(b) $\lim_{x\to 3^+} h(x) = -\infty$, $\lim_{x\to\infty} h(x) = 3$

(c) $\lim_{x\to 3^+} h(x) = \infty$, $\lim_{x\to\infty} h(x) = \infty$

(d) $\lim_{x\to 3^+} h(x) = -\infty$, $\lim_{x\to\infty} h(x) = 0$

(e) $\lim_{x\to 3^+} h(x) = \infty$, $\lim_{x\to\infty} h(x) = 0$

As $x \to 3$ from the right, the denominator goes to 0 and $h(x)$ is positive so this limit is ∞.

As $x \to \infty$ the denominator gets bigger and bigger so $h(x)$ goes to 0.

8. (5 pts.) Consider the following equation in x and y:

$$x^4 + y^4 = y^2 + 3.$$

Using implicit differentiation, find $\dfrac{dy}{dx}$ as a function of x and y.

(a) $\dfrac{2-4x^3}{4y^3-2y}$

(b) $\dfrac{2y-4x^3}{4y^3}$

(c) $\dfrac{-4x^3}{4y^3-2y}$

(d) $\dfrac{4x^3+4y^3}{2y}$

(e) $\dfrac{2y+2-4x^3}{4y^3}$

$$4x^3 + 4y^3\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[4y^3 - 2y\right] = -4x^3$$

$$\frac{dy}{dx} = \frac{-4x^3}{4y^3-2y}$$

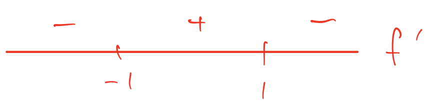
Partial Credit

You must **show your work** on the partial credit problems to receive credit!

9. (15 pts.) Consider the function $y = f(x) = -x^3 + 3x$.

(a) Determine the intervals where the graph of $f(x)$ is increasing and where it is decreasing.

$f'(x) = -3x^2 + 3 = -3(x^2 - 1) = -3(x+1)(x-1)$



increasing on $(-1, 1)$

decreasing on $(-\infty, -1)$ and $(1, \infty)$

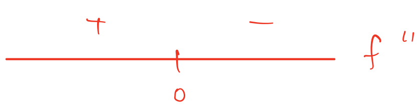
(b) Find the points (x, y) which are relative extrema of $f(x)$. (*Don't forget the y-value.*) Label which are maxima and which are minima.

rel min at $x = -1$. Since $f(-1) = -(-1)^3 + 3(-1) = -2$, the point is $(-1, -2)$

rel max at $x = 1$. Since $f(1) = -(1)^3 + 3(1) = 2$, the point is $(1, 2)$

(c) Determine the intervals where the graph of $f(x)$ is concave up and where it is concave down.

$f''(x) = -6x$



concave up on $(-\infty, 0)$

Concave down on $(0, \infty)$

(d) Find the points (x, y) that are inflection points of $f(x)$.

Inflection point at $(0, 0)$ since f changes concavity there

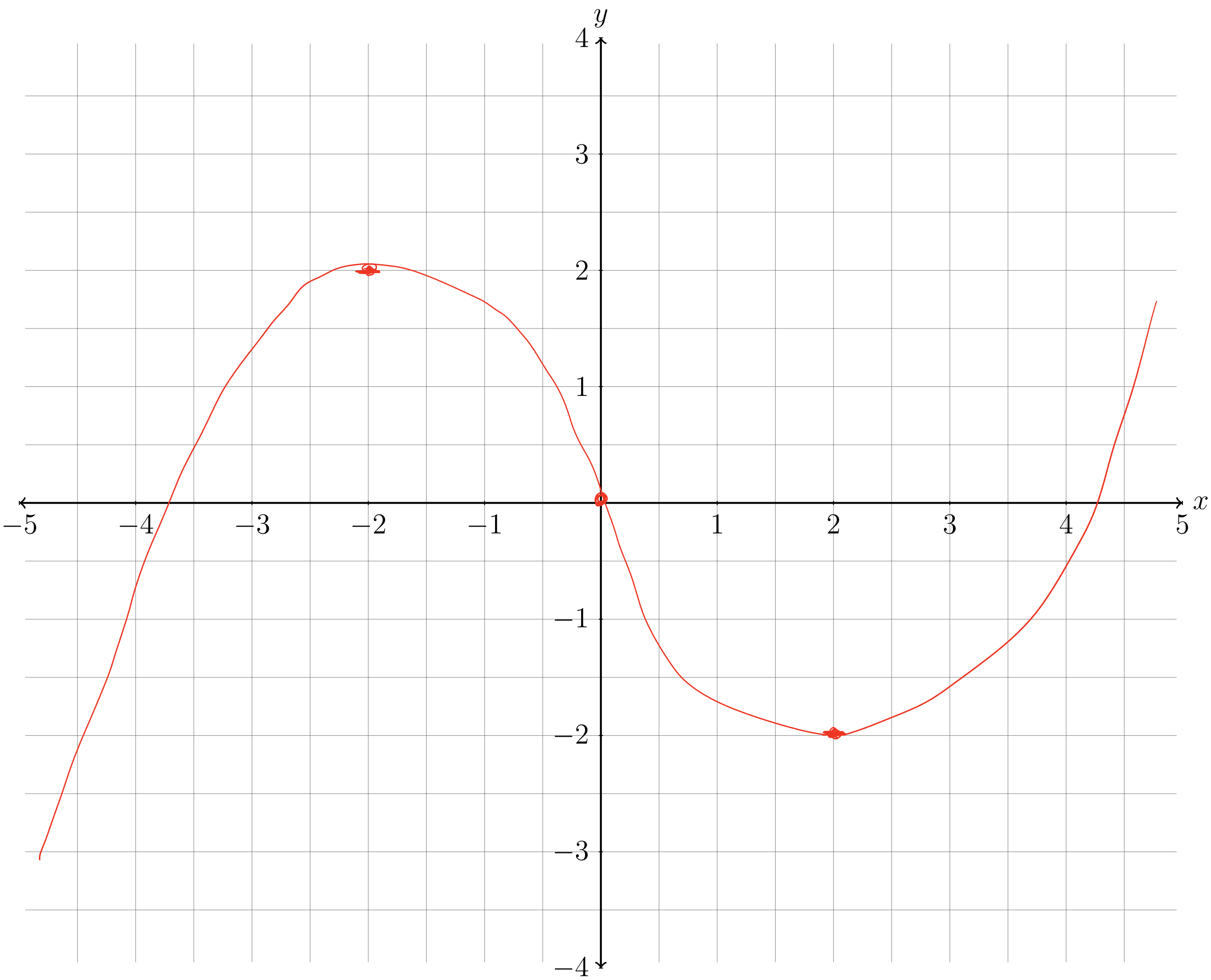
10. (15 pts.) Draw the graph of a continuous function $y = f(x)$ satisfying the following list of properties. Specify the coordinates (x, y) of any relative minima, relative maxima, and inflection points in the blanks below (write NONE if there are none). Draw any asymptotes with a dotted line.

- The domain is all real numbers.
- $f(-2) = 2, f(0) = 0$, and $f(2) = -2$.
- $f'(x) < 0$ on $(-2, 2)$.
- $f'(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$.
- $f'(x) = 0$ at $x = -2$ and $x = 2$
- $f''(x) < 0$ on $(-\infty, 0)$
- $f''(x) > 0$ on $(0, \infty)$
- $\lim_{x\to-\infty} f(x) = -\infty$
- $\lim_{x\to\infty} f(x) = \infty$

Relative maxima at ____________________

Relative minima at ____________________

Inflection point(s) at ____________________

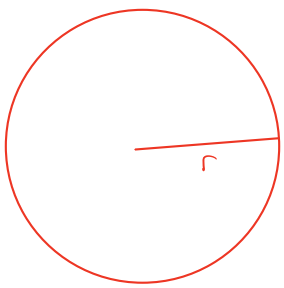


increasing decreasing increasing

concave down concave up

11. (15 pts.) A bowling ball is dropped from a helicopter into a perfectly calm lake, causing a circular ripple whose radius expands at a rate of 5 feet per second. When the radius reaches 10 feet, at what rate is the area of the circle increasing? (Remember that the area of a circle is $A = \pi r^2$, where r is the radius and A is the area.)

Be sure to explain your work, including a discussion of what is true at all moments and what is true at a certain moment. You might lose points for insufficient explanation even if your final answer is correct.



$A = \pi r^2$

$\frac{dA}{dt} = \pi (2r) \frac{dr}{dt}$ we know $\frac{dr}{dt} = 5$ always.

At the moment when $r=10$, we get

$\frac{dA}{dt} = 2(10)\pi(5) = 100\pi \ ft^2/sec$

12. (15 pts.) Consider the function

$$R(x) = \frac{3x^2}{x^2 - 9}$$

(a) Find all the horizontal asymptotes of $R(x)$.

$\lim_{x\to\infty} R(x) = \frac{3}{1} = 3$ since numerator and denominator have the same degree.

Similarly $\lim_{x\to -\infty} R(x) = 3$.

So horiz. asymptotes is $y=3$ (only)

(b) Find all the vertical asymptotes of $R(x)$.

Vertical asymptote comes at $x = \pm 3$ since only there is the denominator zero but the numerator isn't zero.

(c) For each vertical asymptote $x = a$ found in part (b), compute $\lim_{x\to a^+} R(x)$ and $\lim_{x\to a^-} R(x)$.

$\lim_{x\to -3^-} R(x) = +\infty$ $\qquad \frac{+}{+}$

$\lim_{x\to -3^+} R(x) = -\infty$ $\qquad \frac{+}{-}$

$\lim_{x\to 3^-} R(x) = -\infty$ $\qquad \frac{+}{-}$

$\lim_{x\to 3^+} R(x) = +\infty$ $\qquad \frac{+}{+}$

13. (15 pts.) In this problem, y is a function of x, and x is a function of t. Specifically,

$$y = x^5$$

$$x = \sqrt{t^3+1} = (t^3+1)^{1/2}$$

(a) Find $\frac{dy}{dx}$ and $\frac{dx}{dt}$ and put your answer in the appropriate space in the box:

$$\frac{dy}{dx} = 5x^4$$

$$\frac{dx}{dt} = \frac{1}{2}(t^3+1)^{-1/2}(3t^2)$$

(b) Find $\left.\frac{dy}{dt}\right|_{t=2}$ (your answer should be a number).

when $t=2$, $x = \sqrt{8+1} = 3$

so $\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{dy}{dx}\right|_{x=3}$

$= 5 \cdot 81 = 405$

$$\left.\frac{dy}{dt}\right|_{t=2} = 810$$

$$\left.\frac{dy}{dt}\right|_{t=2} = \left.\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)\right|_{t=2} = \left.\frac{dy}{dx}\right|_{x=3}\cdot\left.\frac{dx}{dt}\right|_{t=2}$$

$$= (405)\left(\tfrac{1}{2}\right)(8+1)^{-1/2}(3\cdot 2^2)$$

$$= 810$$

14. (15 pts.) Schwartz & Sons makes solid gold model cars. Let $R(x)$ be their revenue function, where x is the number of model cars produced.

(a) If the company charges \$500 per car, no matter what x is, what is $R(x)$ and what is the marginal revenue function?

$R(x) = 500x$ so $MR = R'(x) = 500$

(b) The executive in charge of setting prices has been sacked, and his replacement now has established a price function

$$p(x) = 200 + \frac{500}{1+x}.$$

What is the new revenue function and what is the new marginal revenue function?

$$R(x) = x\,p(x) = 200x + \frac{500x}{1+x}$$

$$MR = R'(x) = 200 + \frac{(1+x)(500) - 500x}{(1+x)^2} = 200 + \frac{500}{(1+x)^2}$$

(c) The company realizes that it is in their interest to make very many of their cars (i.e. x will get very large). What is the limiting value of the **price** they charge per car, using the formula for $p(x)$ in (b)?

$\lim_{x\to\infty} p(x) = 200$ since $1+x$ gets arbitrarily large

(d) If the cost function is $C(x) = \sqrt{2x+1}$, find the average cost and the marginal average cost functions.

$$\text{avg cost} = \bar{C}(x) = \frac{C(x)}{x} = \frac{\sqrt{2x+1}}{x} = \frac{(2x+1)^{1/2}}{x}$$

$$\text{Marginal avg cost} = \bar{C}'(x) = \frac{x\left(\frac{1}{2}\right)(2x+1)^{-1/2}(2) - (2x+1)^{1/2}}{x^2}$$

$$= \frac{\frac{x}{\sqrt{2x+1}} - \sqrt{2x+1}}{x^2} = \frac{1}{x^2}\left(\frac{x}{\sqrt{2x+1}} - \frac{2x+1}{\sqrt{2x+1}}\right) = \frac{-x-1}{x^2\sqrt{2x+1}}$$

(don't worry if you didn't simplify as much as I did)