**SOLUTIONS TO PRACTICE FINAL EXAM**
**MATH 10250, FALL 2022**

1. Find the equation of the tangent line to the curve $y = x^3$ at the point where $x = 2$.

*Solution:*
We have $y' = 3x^2$ so the slope of the tangent line will be $y'(2) = 12$. The line passes through the point $(2, y(2)) = (2, 8)$ so, by the point-slope formula, the equation is $y - 8 = 12(x - 2)$ or $y = 12x - 16$.

2. Find the slope of the tangent line to the curve

$$y^2 + xe^y = 1$$

at the point $(1, 0)$.

*Solution:*
Using implicit differentiation, we find that $2y\frac{dy}{dx} + e^y + xe^y\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{e^y}{2y + xe^y}$. Plugging in $x = 1$ and $y = 0$ gives $\frac{dy}{dx} = -\frac{e^0}{0 + 1 \cdot e^0} = -1$.

3. A snowball in the shape of a perfect sphere is melting in such a way that its volume is decreasing at a rate of $36\pi$ in$^3$ per minute. How fast is the radius decreasing when the radius is 3 in? Remember that the volume of a sphere of radius $r$ is $\frac{4}{3}\pi r^3$.

*Solution:*
Let $V = \frac{4}{3}\pi r^3$ denote the volume of the ball. Since the ball is melting, we regard both $V$ and $r$ as functions of time. Taking $\frac{d}{dt}$ on both sides of the equation relating $V$ and $r$, we have

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}.$$

In addition, $\frac{dV}{dt} = -36\pi$ (remember that volume is decreasing), so $4\pi r^2 \cdot \frac{dr}{dt} = -36\pi$, which gives $\frac{dr}{dt} = -\frac{36}{4r^2}$. Therefore, when $r = 3$, $\frac{dr}{dt} = -1$. The question already asks at what rate the radius is *decreasing*, so the minus us understood and the answer is 1 in/min.

4. A company that sells canned tuna needs to design a cylindrical can whose volume is $16\pi$ in$^3$. The top and bottom of the can are made from aluminum, which costs \$.01 per in$^2$. However, for structural reasons the side of the can needs to be made from a different metal, which costs \$.08 per in$^2$. What are the radius and height (in inches) of the can which will minimize the total cost of the material needed to construct a can? [Hint: for a cylinder of radius $r$ and height $h$, the volume is $\pi r^2 h$, the area of the side is $2\pi rh$ and the area of the top and bottom is each $\pi r^2$.]

*Solution:*

The cost of making a can with radius $r$ and height $h$ will be: $2\pi rh \cdot 0.08 + 2\pi r^2 \cdot 0.01$. Since the volume of the can is $16\pi = \pi r^2 h$, we find that $h = \frac{16}{r^2}$, so we can rewrite the cost as $C(r) = \frac{2.56\pi}{r} + 0.02\pi r^2$. We then have $C'(r) = -\frac{2.56\pi}{r^2} + 0.04\pi r$. Setting $C'(r) = 0$ gives $0.04r^3 = 2.56$ or $r^3 = 64$, so $r = 4$ and $h = \frac{16}{r^2} = 1$.

5. Solve the initial value problem

$$y' = \frac{2x+3}{x^2+3x+2},$$

$$y(0) = 1 + \ln 2.$$

Assume that $x > 0$.

*Solution:*

We have $y = \displaystyle\int \left(\frac{2x+3}{x^2+3x+2}\right) dx$, which we can solve using the substitution $u = x^2 + 3x + 2$ and $\frac{du}{dx} = 2x + 3$, so $du = (2x+3)dx$ and $y = \displaystyle\int \frac{1}{u} du = \ln|u| + C = \ln(x^2 + 3x + 2) + C$ since $x > 0$. In addition, $y(0) = \ln(2) + C = 1 + \ln(2)$, so $C = 1$ and $y = \ln(x^2 + 3x + 2) + 1$.

6. Evaluate the definite integral

$$\int_1^e \frac{\sqrt{\ln x}}{x}\, dx.$$

*Solution:*

Setting $u = \ln(x)$ gives $\frac{du}{dx} = \frac{1}{x}$ and $\displaystyle\int_1^e \frac{\sqrt{\ln x}}{x}\, dx = \int_{\ln(1)}^{\ln(e)} \sqrt{u} du = \int_0^1 u^{\frac{1}{2}} du = \frac{2}{3} u^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3}.$

7. Compute

$$\lim_{x\to\infty} \left(\frac{x+1}{3+4x}\right).$$

*Solution:*

$$\lim_{x\to\infty} \left(\frac{x+1}{3+4x}\right) = \lim_{x\to\infty} \left(\frac{x+1}{3+4x}\right) \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \left(\frac{1+\frac{1}{x}}{4+\frac{3}{x}}\right) = \frac{1+0}{4+0} = \frac{1}{4}.$$

8. Which of the following graphs could represent the function $ax^4 + bx^3 + cx^2 + dx + e$ where $a, b, c, d, e$ are constants and $a < 0$? [Hint: pay attention to the condition $a < 0$, and think about what happens as $x$ approaches $\infty$ or $-\infty$.]

*Solution:*

We have that $\lim_{x\to\infty} (ax^4 + bx^3 + cx^2 + dx + e) = \lim_{x\to\infty} (ax^4) = -\infty$ since $a < 0$. Addtionally, since the polynomial has degree 4 (which is even), we know that the limits as $x \to \infty$ and $x \to -\infty$ will be the same. This leaves (a) as the only possibility.

9. The half-life of carbon-14 is approximately 5730 years. This means that if the initial amount is $A_0$ then the amount present after $t$ years is

$$A_0 \left(\frac{1}{2}\right)^{t/5730}.$$

How long does it take an initial amount of carbon-14 to decrease to one fifth of the original amount?

*Solution:*

We want to solve the equation: $A_0 \left(\frac{1}{2}\right)^{t/5730} = \frac{1}{5}A_0$. After cancellation, $\left(\frac{1}{2}\right)^{t/5730} = \frac{1}{5}$ or, equivalently, $e^{\ln(\frac{1}{2})\cdot\frac{t}{5730}} = \frac{1}{5}$. Taking the natural log of both sides gives $\ln(\frac{1}{2}) \cdot \frac{t}{5730} = \ln(\frac{1}{5})$ or $t = 5730 \cdot \frac{\ln(\frac{1}{5})}{\ln(\frac{1}{2})} = 5730 \cdot \frac{-\ln(5)}{-\ln(2)} = 5730 \cdot \frac{\ln(5)}{\ln(2)}$.

10. Which equation below describes the tangent line to the graph of $f(x) = \ln(\ln x)$ at the point $(e, 0)$?

*Solution:*

To find the equation of the line we need to find the slope, which is given by the derivative. By the chain rule,

$$f'(x) = \frac{\left(\frac{1}{x}\right)}{\ln x}$$

The derivative at $x = e$ is $f'(e) = \frac{1}{e}$. Plugging this slope and the point $(e, 0)$ into the point-slope equation for a line, we get $y = \frac{1}{e}(x - e)$.

11. John invests some money into an account, and learns that the amount in the account after $t$ years is given by the formula

$$A(t) = 1,000e^{0.05t}.$$

What is the *average* amount in the account over the first three years?

*Solution:*

The average value of $A(t)$ from $t = 0$ to $t = 3$ is given by the formula

$$\frac{1}{3-0}\int_0^3 1000e^{0.05t}dt = \frac{1}{3}(1000)\int_0^3 e^{0.05t}dt = \frac{1}{3}(1000)(\frac{1}{0.05}e^{0.05t}\big|_0^3)$$
$$= \frac{1}{3}(1000)(\frac{1}{0.05})(e^{1.5} - e^0) = \frac{1}{3}(20000)(e^{1.5} - 1)$$

12. Find the total **area** between the $x$-axis and the graph of $y = x^2 - 1$ from $x = 0$ to $x = 3$. [Hint: this function is not non-negative, so treat areas below the $x$-axis separately.]

*Solution:*

The function is negative (below the $x$-axis) on $[0, 1)$, and is non-negative on $[1, 3]$. Thus, the total area is given by

$$\int_1^3 x^2 - 1dx - \int_0^1 x^2 - 1dx = (\frac{1}{3}x^3 - x)\big|_1^3 - (\frac{1}{3}x^3 - x)\big|_0^1$$
$$= ([9 - 3] - [\frac{1}{3} - 1]) - ([\frac{1}{3} - 1] - [0 - 0]) = \frac{22}{3}$$

13. A ball is thrown into the air and its height in feet after $t$ seconds is given by $h(t) = -16t^2 + 32t + 48$. After how many seconds does the ball reach its peak?

*Solution:*
The ball's peak is its absolute maximum. Searching for a local max, we compute the derivative $h'(t) = -32t + 32$. Now $h'(t) = 0$ when $t = 1$, so $t = 1$ is a critical point. $h''(t) = -32$ which is always negative, so $h(t)$ is always concave down. In particular, $t = 1$ gives the absolute maximum of $h(t)$.

14. What are the critical points of

$$f(x) = \frac{x}{x^2 + 1}$$

*Solution:*
Critical points occur when the derivative is 0 or does not exist. By the quotient rule,

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Now the denominator of $f'(x)$ is always positive, so $f'(x)$ always exists. $f'(x) = 0$ when the numerator is 0, that is when $x = 1$ or $x = -1$. Thus, $x = 1$ and $x = -1$ are the only critical points.

15. Let $f(x)$ be a function whose **derivative** is $f'(x) = x(x - 1)^2(x + 1)$. Where are the local extrema of $f(x)$?

*Solution:*
The local extrema occur when the derivative $f'(x)$ changes sign. Now $f'(x) = 0$ for $x = 0, x = 1$ and $x = -1$. These points define intervals where we will test the sign of $f'(x)$. We find $f'(x)$ is positive on $(-\infty, -1), (0, 1), (1, \infty)$ and negative on $(-1, 0)$. Thus $x = -1$ is a local max and $x = 0$ is a local min.

16. An automobile company finds that their marginal profit is given by the function $P' = x^2 + 3x$, where $x$ is the number of cars produced per week (in units of one thousand) and $P(x)$ is given in dollars. What is the total change in profit (in dollars) as they raise production from $x = 0$ to $x = 6$ thousand cars per week?

*Solution:*
The total change in profit from $x = 0$ to $x = 6$ is $P(6) - P(0)$ which, by the fundamental theorem of calculus, is the same as $\int_0^6 P'(x)dx$. We evaluate the integral:

$$\int_0^6 P'(x)dx = \int_0^6 x^2 + 3xdx = (\frac{1}{3}x^3 + \frac{3}{2}x^2)\Big|_0^6$$
$$= (72 + 54) - (0 + 0) = 126$$

17. How many inflection points does $\frac{1}{2}x^4 - \frac{1}{5}x^6$ have?

*Solution:*
If we call the function $f(x)$ then $f'(x) = 2x^3 - \frac{6}{5}x^5$ and

$$f''(x) = 6x^2 - 6x^4 = 6x^2(1 - x^2) = 6x^2(1 + x)(1 - x).$$

$f''(x)$ is zero at $x = -1, 0, 1$, but it does not change sign at $x = 0$. It does change sign at $x = \pm 1$. So only the latter two give inflection points.

18. Using logarithmic differentiation, find $\dfrac{dy}{dx}$ for

$$y = (2x+1)^4(3x+1)^5(4x+1)^6.$$

[Hint: don't forget the chain rule!]

*Solution:*
We get $\ln(y) = 4\ln(2x+1) + 5\ln(3x+1) + 6\ln(4x+1)$. Differentiating both sides we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{4 \cdot 2}{2x+1} + \frac{5 \cdot 3}{3x+1} + \frac{6 \cdot 4}{4x+1}.$$

Then

$$\frac{dy}{dx} = y\left[\frac{8}{2x+1} + \frac{15}{3x+1} + \frac{24}{4x+1}\right] = (2x+1)^4(3x+1)^5(4x+1)^6\left[\frac{8}{2x+1} + \frac{15}{3x+1} + \frac{24}{4x+1}\right]$$

19. If $y = \ln\left(\dfrac{11}{x+5}\right)$, solve for $x$ in terms of $y$.

*Solution:*
The given equation is equivalent to $e^y = \frac{11}{x+5}$ so $x + 5 = \frac{11}{e^y}$ and $x = \frac{11}{e^y} - 5$.

20. What is the domain of the function $f(x) = \dfrac{\sqrt{x}}{x^5 - 32}$?

*Solution:*
In order for the numerator to be valid, we need $x \geq 0$ (in particular, 0 is included). But we also have to avoid the denominator being 0, so we need $x \neq 2$ as well. So we get $[0, 2) \cup (2, \infty)$.

21. Find the derivative of $xe^{x^2+1}$.

*Solution:*
Use the product rule and the chain rule.

$$x\left(e^{x^2+1} \cdot (2x)\right) + e^{x^2+1}(1) = e^{x^2+1}(2x^2+1).$$

22. To evaluate the definite integral

$$\int_0^4 \frac{dx}{1+\sqrt{x}}$$

you decide to use the $u$-substitution $u = \sqrt{x}$. When you make this substitution, what does the definite integral become? [Hint: notice that $u^2 = x$.]

*Solution:*

Since $u^2 = x$ we get $(2u)du = dx$. When $x = 0$, $u = 0$. When $x = 4$, $u = 2$. So remembering to change the limits of integration we have

$$\int_0^2 \frac{2u}{1+u} du.$$

23. Bob invests a certain sum of money into an account giving interest at an annual rate of 5%, compounded **continuously**. How long will it take his investment to **triple** (all answers are in years)?

*Solution:*

The amount after $t$ years is $A(t) = A_0 e^{0.05t}$, where $A_0$ is the initial amount. We want to know for what $t$ is it true that $A(t) = 3A_0$. So we get

$$3A_0 = A_0 e^{0.05t}, \qquad \text{or} \qquad 3 = e^{0.05t}.$$

We take the natural log on both sides and solve for $t$ to get $t = \frac{\ln(3)}{0.05}$.

24. Bob's sister, Bobbie, invests a certain sum of money into an account giving interest at an annual rate of 5%, compounded **annually** (i.e. compounded just once a year). How long will it take her investment to **triple** (all answers are in years)?

*Solution:*

Now the formula is $A(t) = A_0(1 + \frac{0.05}{1})^t$ so we get $3A_0 = A_0(1.05)^t$ or $3 = (1.05)^t$. Again using the natural log we get

$$\ln(3) = t \ln(1.05) \qquad \text{or} \qquad t = \frac{\ln(3)}{\ln(1.05)}.$$

25. What are **all** the value(s) of $x$ where the tangent line to $y = x^3 + x$ has slope 13?

*Solution:*

$\frac{dy}{dx} = 3x^2 + 1$ so we solve $3x^2 + 1 = 13$ and get $x^2 = 4$ or $x = \pm 2$.