Math 30710
Practice Exam 1-1
October 3, 2018 **Name** ____________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. (5 points) Let D be a Cayley digraph, consisting of some number of vertices (corresponding to the elements of a given finite group G) and some number of different kinds of arrows (arcs) (corresponding to a given set of generators for G). Let g_1 and g_2 be vertices of D. Prove that *at most* one arrow (i.e. either one arrow or zero arrows) can go from g_1 to g_2. (Hint: find an equation.)

2. (5 points) Let G be a group of order pq, where p and q are distinct prime numbers. Prove that every proper subgroup of G is cyclic.

3. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 10 & 5 & 1 & 4 & 2 & 3 & 6 & 7 & 8 \end{pmatrix}$$

(a) (5 points) Find σ^{-1} and write it in the space below:

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ & & & & & & & & & \end{pmatrix}$$

(b) (5 points) Write σ as a product of disjoint cycles. (Make sure you look at σ and not σ^{-1}.)

(c) (5 points) Write σ as a product of transpositions.

(d) (5 points) Find the order of σ, i.e. the smallest positive power n such that $\sigma^n = e$ (the identity permutation).

4. (10 points) The following is the group table for an **abelian** group of order 8 with identity element e. Fill in the blanks. No justification needed, but don't forget that G is **abelian**.

	e	a_1	a_2	a_3	a_4	a_5	a_6	a_7
e	____	____	____	____	a_4	a_5	a_6	a_7
a_1	____	e	____	a_7	a_2	a_6	a_5	a_3
a_2	____	a_4	____	e	a_5	a_7	____	a_1
a_3	____	a_7	____	a_6	____	a_4	a_2	a_5
a_4	a_4	a_2	____	a_1	____	a_3	a_7	e
a_5	a_5	a_6	____	a_4	____	____	a_1	a_2
a_6	a_6	a_5	____	a_2	a_7	a_1	e	a_4
a_7	a_7	a_3	____	a_5	e	a_2	a_4	____

5. Provide the following examples. You do not have to justify your answers.

   (a) (5 points) An infinite cyclic group other than $\mathbb{Z}$. (It's OK that your answer will be isomorphic to $\mathbb{Z}$.)

   (b) (5 points) A finite, non-cyclic abelian group.

   (c) (5 points) An infinite, non-abelian group.

   (d) (5 points) A non-abelian group G of order 8 and a subgroup H of G such that the index is $(G : H) = 2$.

6. (10 points) Let G be a group. Prove that if G has only finitely many subgroups then G must be a finite group. (Hint: think about why $\mathbb{Z}$ has infinitely many subgroups.)

7. (5 points) Prove that if G is a finite group of order n with identity element e then $a^n = e$ for all $a \in G$.

8. (5 points) Consider the relation

$$z_1 \mathcal{R} z_2 \text{ in } \mathbb{C} \text{ if } |z_1| = |z_2|.$$

You can assume without proof that this is an equivalence relation. Find the equivalence classes (i.e. the partition) arising from this equivalence relation and describe them geometrically. (Remember that this is in $\mathbb{C}$, not in $\mathbb{R}$.) Explain your answer.

9. (5 points) Prove that the Klein 4-group and $\langle Z_4, + \rangle$ are not isomorphic.

10. (5 points) How many *different* subgroups does $\mathbb{Z}_{19}$ have? Explain your answer.

11. (10 points) Let G be a finite group with identity element e, and assume that $g^2 = e$ for all $g \in G$. Prove that G is abelian.

(Extra sheet.)