Math 30710
Practice Exam 1-1 Solutions
October 3, 2018 **Name** ____________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. (5 points) Let D be a Cayley digraph, consisting of some number of vertices (corresponding to the elements of a given finite group G) and some number of different kinds of arrows (arcs) (corresponding to a given set of generators for G). Let g_1 and g_2 be vertices of D. Prove that *at most* one arrow (i.e. either one arrow or zero arrows) can go from g_1 to g_2. (Hint: find an equation.)

   Solution:

   An arrow from g_1 to g_2 corresponds to a generator, x, with the property that $g_1 x = g_2$. Since this equation has exactly one solution for x, there is at most one arrow. Since this solution may or may not be one of the generators, it's possible that there is no arrow.

2. (5 points) Let G be a group of order pq, where p and q are distinct prime numbers. Prove that every proper subgroup of G is cyclic.

   Solution:

   By Lagrange's theorem, the order of a subgroup of a finite group divides the order of the group. If H is a subgroup of G, in this case we must have $|H| = 1, p, q$, or pq. Since H is proper, $|H|$ is not 1 or pq. Thus $|H| = p$ or q, hence is prime. But we saw in class that any group of prime order is cyclic, so we are done.

3. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 10 & 5 & 1 & 4 & 2 & 3 & 6 & 7 & 8 \end{pmatrix}$$

   (a) (5 points) Find σ^{-1} and write it in the space below:

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 6 & 7 & 5 & 3 & 8 & 9 & 10 & 1 & 2 \end{pmatrix}$$

   (b) (5 points) Write σ as a product of disjoint cycles. (Make sure you look at σ and not σ^{-1}.)

   Solution:

   $\sigma = (1, 9, 7, 3, 5, 4)(2, 10, 8, 6)$.

(c) (5 points) Write σ as a product of transpositions.

Solution:

$\sigma = (1,4)(1,5)(1,3)(1,7)(1,9)(2,6)(2,8)(2,10)$.

(d) (5 points) Find the order of σ, i.e. the smallest positive power n such that $\sigma^n = e$ (the identity permutation).

Solution:

$\mathrm{ord}(\sigma) = LCM(6,4) = 12$.

4. (10 points) The following is the group table for an **abelian** group of order 8 with identity element e. Fill in the blanks. No justification needed, but don't forget that G is **abelian**.

	e	a_1	a_2	a_3	a_4	a_5	a_6	a_7
e	e	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_1	a_1	e	a_4	a_7	a_2	a_6	a_5	a_3
a_2	a_2	a_4	a_6	e	a_5	a_7	a_3	a_1
a_3	a_3	a_7	e	a_6	a_1	a_4	a_2	a_5
a_4	a_4	a_2	a_5	a_1	a_6	a_3	a_7	e
a_5	a_5	a_6	a_7	a_4	a_3	e	a_1	a_2
a_6	a_6	a_5	a_3	a_2	a_7	a_1	e	a_4
a_7	a_7	a_3	a_1	a_5	e	a_2	a_4	a_6

5. Provide the following examples. You do not have to justify your answers.

(a) (5 points) An infinite cyclic group other than $\mathbb{Z}$. (It's OK that your answer will be isomorphic to $\mathbb{Z}$.)

Solution:

$2\mathbb{Z}$, or $\langle \pi \rangle$ (in $\mathbb{R}$) or ...

(b) (5 points) A finite, non-cyclic abelian group.

Solution:

The Klein group.

(c) (5 points) An infinite, non-abelian group.

Solution:

$GL_2(\mathbb{R})$, i.e. the group of invertible 2×2 matrices with real entries, under multiplication.

(d) (5 points) A non-abelian group G of order 8 and a subgroup H of G such that the index is $(G : H) = 2$.

Solution:

$G = D_4$ and H is the subgroup generated by a rotation of $90°$.

6. (10 points) Let G be a group. Prove that if G has only finitely many subgroups then G must be a finite group. (Hint: think about why $\mathbb{Z}$ has infinitely many subgroups.)

   Solution:

   Suppose that G were infinite. Let $a \in G$ and consider the subgroup $\langle a \rangle$. If this is infinite then it is isomorphic to $\mathbb{Z}$, which has infinitely many subgroups $(2\mathbb{Z}, 3\mathbb{Z}, 4\mathbb{Z}, \dots)$; hence G has infinitely many subgroups, contradicting the hypothesis. So G has no infinite cyclic subgroup. But now every element of G generates a cyclic subgroup (these subgroups are not necessarily distinct), so G is the union of all its cyclic subgroups. Since there are only finitely many of them, and since each of them is finite, we get that G must be finite.

7. (5 points) Prove that if G is a finite group of order n with identity element e then $a^n = e$ for all $a \in G$.

   Solution:
   Let $a \in G$ and consider the subgroup $\langle a \rangle$. Let $m = |\langle a \rangle|$. We know that $a^m = e$ because for a finite cyclic group, the order of a generator is equal to the order of the group. We also know that m divides n by Lagrange's theorem. Thus $n = mk$ for some positive integer k, so

   $$a^n = a^{mk} = (a^m)^k = e^k = e.$$

8. (5 points) Consider the relation

   $$z_1 \mathcal{R} z_2 \text{ in } \mathbb{C} \text{ if } |z_1| = |z_2|.$$

   You can assume without proof that this is an equivalence relation. Find the equivalence classes (i.e. the partition) arising from this equivalence relation and describe them geometrically. (Remember that this is in $\mathbb{C}$, not in $\mathbb{R}$.) Explain your answer.

   Solution:

   The equivalence classes are the concentric circles centered at the origin, together with the origin itself as the entire equivalence class of the complex number 0.

9. (5 points) Prove that the Klein 4-group and $\langle Z_4, + \rangle$ are not isomorphic.

   Solution:

   The Klein 4-group has three elements of order 2, while $\mathbb{Z}_4$ has only one element of order 2.

10. (5 points) How many *different* subgroups does $\mathbb{Z}_{19}$ have? Explain your answer.

    Solution:

    Every subgroup of a cyclic group is cyclic, so we only have to look for cyclic subgroups. Clearly $\langle 0 \rangle$ is one such subgroup. Any other element $a \in \mathbb{Z}_{19}$ is relatively prime to 19, so it generates a subgroup of order 19. So the only two subgroups are $\langle 0 \rangle$ and $\mathbb{Z}_{19}$ itself.

11. (10 points) Let G be a finite group with identity element e, and assume that $g^2 = e$ for all $g \in G$. Prove that G is abelian.

    Solution:

    Let $x, y \in G$. We want to show that $xy = yx$. We know that

    $$e = (xy)^2 = (xy)(xy).$$

    Multiplying both sides on the left by x and on the right by y, we get by associativity that

    $$xy = xey = x(xy)(xy)y = (x^2)(yx)(y^2) = e(yx)e = yx.$$