Math 30710
Exam 2
November 12, 2021 **Name** ________________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 9 & 2 & 10 & 6 & 8 & 1 & 5 & 3 & 7 \end{pmatrix}$$

(a) (5 points) Write σ as a product of disjoint cycles.

(b) (5 points) Write σ as a product of transpositions.

(c) (5 points) Find the order of σ and explain your answer.

2. Let $G = \mathbb{R}$ and $H = \mathbb{Z}$ (both are groups under addition, and H is a subgroup of G). In both parts of this problem, make sure that your answer is an element of G/H and not of G.

(a) (5 points) Find an element of G/H of order 10. Briefly explain your answer.

(b) (5 points) Find an element of G/H of infinite order. Briefly explain your answer.

3. Let G be a group and let H be a subgroup of G (not necessarily normal). Let $a, b \in G$.

   (a) (5 points) Assume that $Ha = Hb$. Prove that $b \in Ha$.

   (b) (8 points) Conversely, assume that $b \in Ha$. Prove that $Ha = Hb$.

4. This problem involves Lagrange's theorem.

   (a) (5 points) State Lagrange's theorem.

   (b) (10 points) Let G be a group (not necessarily abelian) of order pq, where p and q are prime numbers. Prove that every proper subgroup of G is cyclic.

5. (10 points) Using the Fundamental Theorem of Finitely Generated Abelian Groups, find (up to isomorphism) all abelian groups of order 200 and explain your answer.

6. Let G be the group of continuous functions on the closed interval $[0, 1]$, let $G' = \mathbb{R}$, and define

$$\phi : G \to G'$$

by

$$\phi(f) = \int_0^1 f(x)dx.$$

(a) (5 points) Show that ϕ is a group homomorphism, using freely any facts from calculus. (Hint: note that both G and G' are **additive** groups.)

(b) (6 points) Give a non-trivial element of $\ker(\phi)$ (using a picture if you like). Briefly explain your answer.

7. Let $G = \mathbb{Z}_6 \times \mathbb{Z}_4$ and let $H = \langle (2,2) \rangle$.

   (a) (8 points) Write out the elements of H. (Hint: 2 has order 3 in $\mathbb{Z}_6$ and 2 has order 2 in $\mathbb{Z}_4$. So how many elements do you expect H to have?)

   (b) (8 points) To what familiar group is G/H isomorphic, according to the Fundamental Theorem of Finitely Generated Abelian Groups? Explain your answer.

8. (10 points) Consider groups of order 8. Show that up to isomorphism there is exactly one **abelian** group of order 8 that does not contain any cyclic subgroup of order 4. Be sure to identify that group and explain why it has the claimed property. (Hint: think about the Fundamental Theorem of Finitely Generated Abelian Groups.)

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