Math 30710
Exam 2 Solutions
November 12, 2021 **Name** ________________________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 9 & 2 & 10 & 6 & 8 & 1 & 5 & 3 & 7 \end{pmatrix}$$

(a) (5 points) Write σ as a product of disjoint cycles.

Solution:

$(1,4,10,7)(2,9,3)(5,6,8)$

(b) (5 points) Write σ as a product of transpositions.

Solution:

Two possible answers are

$$(1,4)(4,10)(10,7)(2,9)(9,3)(5,6)(6,8)$$

or

$$(1,7)(1,10)(1,4)(2,3)(2,9)(5,8)(5,6).$$

(c) (5 points) Find the order of σ and explain your answer.

Solution:

Part (a) gives σ as a product of **disjoint** *cycles, so for any n, we know that*

$$\sigma^n = (1,4,10,7)^n(2,9,3)^n(5,6,8)^n$$

(this would not be true if the cycles were not disjoint). The first of these cycles has order 4 and the second and third have order 3, so the product, σ, has order that is the least common multiple of 4 and 3, i.e. 12.

2. Let $G = \mathbb{R}$ and $H = \mathbb{Z}$ (both are groups under addition, and H is a subgroup of G). In both parts of this problem, make sure that your answer is an element of G/H and not of G.

   (a) (5 points) Find an element of G/H of order 10. Briefly explain your answer.

   Solution:

   $\frac{1}{10} + \mathbb{Z}$ has order 10 since if you add it to itself 10 times you get $1 + \mathbb{Z}$, which is the same coset as $0 + \mathbb{Z}$ (the identity element of G/H) since $1 - 0 \in \mathbb{Z} = H$, but if you add it to itself any smaller number of times you do not obtain $0 + \mathbb{Z}$.

   (b) (5 points) Find an element of G/H of infinite order. Briefly explain your answer.

   Solution:

   $\pi + \mathbb{Z}$ has infinite order, since adding $\pi + \mathbb{Z}$ to itself n times gives $n\pi + \mathbb{Z}$, and this is never $0 + \mathbb{Z}$ since $n\pi$ is never an integer (other than $n = 0$).

3. Let G be a group and let H be a subgroup of G (not necessarily normal). Let $a, b \in G$.

   (a) (5 points) Assume that $Ha = Hb$. Prove that $b \in Ha$.

   Solution:

   Clearly $b \in Hb$ since $e \in H$. Since $Ha = Hb$, we have $b \in Ha$.

   (b) (8 points) Conversely, assume that $b \in Ha$. Prove that $Ha = Hb$.

   Solution:

   Again we know that $b \in Hb$, so we have b is in both Ha and Hb. But the right cosets form a partition of G, so b can only be in one coset of H. Thus Ha must equal Hb.

   Alternatively, since $b \in Ha$ we have $b = ha$ for some $h \in H$, so $ba^{-1} = h \in H$. But we showed in class that two cosets Hx and Hy are equal if and only if $yx^{-1} \in H$ (actually we proved the version for left cosets and just left the proof for right cosets to you), so we again conclude $Ha = Hb$.

4. This problem involves Lagrange's theorem.

   (a) (5 points) State Lagrange's theorem.

   Solution:

   If G is a finite group and H is a subgroup then $|H|$ divides $|G|$.

   (b) (10 points) Let G be a group (not necessarily abelian) of order pq, where p and q are prime numbers. Prove that every proper subgroup of G is cyclic.

   Solution:

   The integers that divide pq are 1, p, q and pq. Let H be a proper subgroup. Then by definition of "proper," H is not all of G. If $H = \{e\}$ there is nothing to prove, so assume this is not the case. Hence by Lagrange's Theorem, $|H|$ is either p or q, both of which are prime. Suppose without loss of generality that $|H| = p$. Let $g \in H$ be any element other than the identity (which is possible since both p and q are greater than 1) and consider the subgroup $\langle g \rangle$ of H.

Then $|\langle g\rangle|$ has to be > 1 and has to divide p, so it has to be equal to p. Thus H must be equal to $\langle g\rangle$, which means H is cyclic. (Choosing q instead of p makes no difference.)

5. (10 points) Using the Fundamental Theorem of Finitely Generated Abelian Groups, find (up to isomorphism) all abelian groups of order 200 and explain your answer.

Solution:

First note that $200 = 2^3 \cdot 5^2$. This is finite, so there are no copies of $\mathbb{Z}$ in the answer, and the FTFGAG says that the possibilities are

$$\begin{array}{l} \mathbb{Z}_8 \times \mathbb{Z}_{25} \\ \mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\ \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_{25} \\ \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5. \end{array}$$

6. Let G be the group of continuous functions on the closed interval $[0, 1]$, let $G' = \mathbb{R}$, and define

$$\phi : G \to G'$$

by

$$\phi(f) = \int_0^1 f(x)dx.$$

(a) (5 points) Show that ϕ is a group homomorphism, using freely any facts from calculus. (Hint: note that both G and G' are **additive** groups.)

Solution:

Note again that both groups are additive groups.

$$\begin{aligned} \phi(f+g) &= \int_0^1 (f(x)+g(x))dx \\ &= \int_0^1 f(x)dx + \int_0^1 g(x)dx \\ &= \phi(f)+\phi(g). \end{aligned}$$

(b) (6 points) Give a non-trivial element of $\ker(\phi)$ (using a picture if you like). Briefly explain your answer.

Solution:

You just want a continuous function whose integral from 0 to 1 is 0. An example could be $f(x) = x - \frac{1}{2}$.

7. Let $G = \mathbb{Z}_6 \times \mathbb{Z}_4$ and let $H = \langle (2,2) \rangle$.

   (a) (8 points) Write out the elements of H. (Hint: 2 has order 3 in $\mathbb{Z}_6$ and 2 has order 2 in $\mathbb{Z}_4$. So how many elements do you expect H to have?)

   Solution:

   $$H = \{(2,2), (4,0), (0,2), (2,0), (4,2), (0,0)\}.$$

   (b) (8 points) To what familiar group is G/H isomorphic, according to the Fundamental Theorem of Finitely Generated Abelian Groups? Explain your answer.

   Solution:

   Since $|G| = 24$ *and* $|H| = 6$, *we know* $|G/H| = 4$ *so by the FTFGAG we know* G/H *is isomorphic to either* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *(i.e. to the Klein 4-group). We have to decide which of these two possibilities is correct.*

   Note that 4 *and* 6 *are both even, and that* H *contains every possible* (x,y) *where both* x *and* y *are even. So for any* $(x,y) \in G/H$, *we get* $(x,y) + (x,y) \in H$. *So every element has order* 2, *and* G/H *is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

8. (10 points) Consider groups of order 8. Show that up to isomorphism there is exactly one **abelian** group of order 8 that does not contain any cyclic subgroup of order 4. Be sure to identify that group and explain why it has the claimed property. (Hint: think about the Fundamental Theorem of Finitely Generated Abelian Groups.)

   Solution:

   The FTFGAG tells us that the possibilities for an abelian group of order 8 are

   $$\mathbb{Z}_8, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad \text{or} \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

   In order to have a cyclic subgroup of order 4 we would need an element of order 4. Clearly $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *has no such element (all elements other than the identity have order* 2*). On the other hand,* $\mathbb{Z}_4 \times \mathbb{Z}_2$ *has the element* $(1,0)$, *which has order 4, and* $\mathbb{Z}_8$ *has the element* 2, *which has order 4. So* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *is the only such group.*

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