Math 30710
Exam 1
October 3, 2018 **Name** ______________________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. Let $S = \{a, b, c\}$ be a set consisting of three elements. S is not necessarily a group (yet). Consider a table of the form

$*$	a	b	c
a			
b			
c			

For each of the following, clearly explain your answer. In this problem I'm only looking for a number (with explanation), not a list of tables.

How many binary operations $*$ are possible (i.e. in how many ways can you fill this table) if ...

(a) (5 points) there are no conditions other than $*$ being a binary operation?

(b) (5 points) a is an identity element (no other conditions)?

(c) (5 points) a is not necessarily an identity element but $*$ is commutative?

(d) (5 points) S is a group with identity element a?

[redacted]

3. (10 points) The following is the group table for S_3 (with a reminder of the book's notation). To the right is a set of six vertices. Complete it to a Cayley digraph, labelling all six vertices, and using

$\longrightarrow$ for ρ_1 and - - - - - - - - - - - - - - - for μ_1

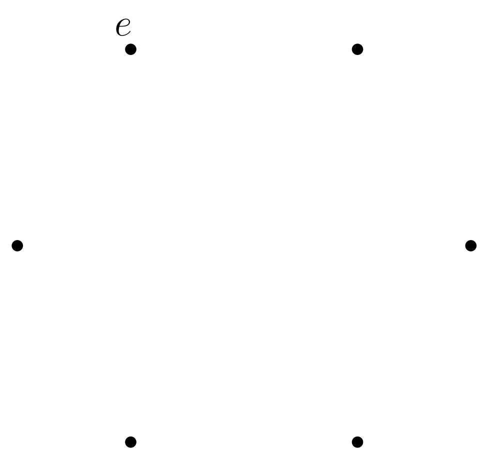
as your generators. **Note that the solid line has an arrow and the dashed line doesn't!! Pay attention to the direction of the arrow.** Don't draw any unjustified conclusions from the fact that I put the vertices in the shape of a regular hexagon!

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
e	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	e	μ_3	μ_1	μ_2
ρ_2	ρ_2	e	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	e	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	e	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	e



4. (10 points) True or false? If true, give a proof. If false, give a counterexample.

If G is an infinite group then G has infinitely many **distinct** *subgroups.*

[Hint: think about cyclic subgroups.]

5. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 9 & 5 & 4 & 10 & 3 & 1 & 6 & 7 & 8 \end{pmatrix}$$

(a) [redacted]

(b) [redacted]

(c) [redacted]

(d) (5 points) Is σ an element of D_{10} (the 10^{th} dihedral group, i.e. the group of symmetries of a regular 10-gon)? Explain your answer. (It does not have to be a rigorous proof.)

(e) (5 points) Find σ^{-1}.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ & & & & & & & & & \end{pmatrix}$$

6. (a) (5 points) Let $G = \langle a \rangle$ be a cyclic group and let $\phi : G \to G'$ be a group homomorphism. Suppose that $\phi(a)$ is known. Explain how you then know $\phi(x)$ for any $x \in G$.

(b) (5 points) Give an example of a one-to-one homomorphism ϕ from $(\mathbb{Z}_4, +)$ to $(\mathbb{C}^*, \times)$. Thanks to part (a), you can answer this question by just giving me $\phi(1)$. But be sure to explain why this ϕ is well-defined.

7. Let G be a finite group of even order $2n$.

   (a) (5 points) Prove that G must contain at least one element of order 2. [Hint: explain why another way to say this is that other than the identity element e, G has to contain at least one other element that is equal to its own inverse. Then pair off each element with its inverse.]

   (b) (9 points) Suppose that G is an **abelian** group. If a and b are distinct elements of order 2, prove the following. [Remember that the identity element has order 1, not 2. The fact that G has order $2n$ is not relevant to this part.]

   - ab also has order 2.
   - $ab \neq a$ (or b).
   - $ab \neq e$.

   (c) (5 points) Now suppose that G is an **abelian** group of order $2n$, **where** n **is odd**. Using the previous parts of this problem (whether you were able to solve them or not), show that G contains *exactly* one element of order 2. [Hint: Lagrange's theorem.]

(Extra sheet.)