Math 30710
Exam 1 Solutions
September 18, 2020 **Name** ______________________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. (10 points) Let G be a finite group, not necessarily cyclic. Let a be part of a generating set for G and let

$$\xrightarrow{\quad a \quad}$$

be the arc representing multiplication by a in a Cayley digraph for G. Prove that any vertex, g, of the Cayley digraph has exactly one arc of this kind starting at g, and one arc ending at g.

Solution:

At the vertex g, the arc going out from g goes to the element ga. The arc coming into g comes from the element ga^{-1} since $(ga^{-1}) \cdot a = g$.

2. (a) (10 points) Recall that $\mathbb{R}^{>0}$ is the multiplicative group of positive real numbers. Define $\phi : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ by $\phi(x) = 3x$. Is ϕ a group homomorphism? If so, prove it. If not, explain carefully why not.

Solution:

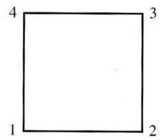
No. $\phi(xy) = 3xy$, while $\phi(x)\phi(y) = (3x)(3y) = 9xy$.

(b) (5 points) Recall that $\mathbb{R}$ is the additive group of real numbers. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(x) = 3x$. Is ϕ a group homomorphism? If so, prove it. If not, explain carefully why not.

Solution:
Yes. Since it's an additive group, we'll use additive notation.

$$\phi(x + y) = 3(x + y) = 3x + 3y = \phi(x) + \phi(y).$$

3. The following is the group table for the dihedral group D_4.



8.11 Figure

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \qquad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix},$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

8.12 Table

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_1	δ_2	μ_2	μ_1
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_2	δ_1	μ_1	μ_2
μ_1	μ_1	δ_2	μ_2	δ_1	ρ_0	ρ_2	ρ_3	ρ_1
μ_2	μ_2	δ_1	μ_1	δ_2	ρ_2	ρ_0	ρ_1	ρ_3
δ_1	δ_1	μ_1	δ_2	μ_2	ρ_1	ρ_3	ρ_0	ρ_2
δ_2	δ_2	μ_2	δ_1	μ_1	ρ_3	ρ_1	ρ_2	ρ_0

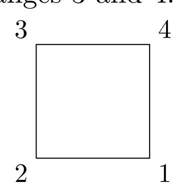
(a) (5 points) If the square begins in the position shown above, find its ending position after applying the permutation $\rho_2\mu_2$. To give your answer, just label the corners of the following square.

Answer:

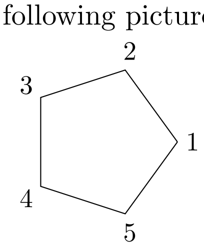
Note that

$$\rho_2\mu_2 = \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

so it interchanges 1 and 2, and interchanges 3 and 4.



(b) (10 points) We know that the dihedral group D_5 has 10 elements. Write them in the following spaces as permutations, based on the following picture.



Solution:

The first is the identity, the next four are rotations, and the last five are flips.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}$$

4. Let $\phi : G \to G'$ be a group homomorphism. Let e, e' be the respective identity elements.

   (a) (5 points) Prove that $\phi(e) = e'$.

   Solution:

   We know $\phi(e) = \phi(ee) = \phi(e)\phi(e)$ so canceling $\phi(e)$ from both sides we get $e' = \phi(e)$.

   (b) (10 points) The following statement (given as a lemma in class) was important in the proof of Cayley's theorem. Prove this statement.

   The image, $\phi[G]$, of G under ϕ is a subgroup of G'.

   [Hint: remember your checklist to prove that a subset of a group is actually a subgroup.]

   Solution:

   - $\phi[G]$ is not empty since $e \in G$ and so $e' = \phi(e) \in \phi[G]$.
   - $\phi[G]$ is closed under the binary operation of G':

     Let $y_1, y_2 \in \phi[G]$. Then $y_1 = \phi(x_1)$ and $y_2 = \phi(x_2)$ for some $x_1, x_2 \in G$ by definition of $\phi[G]$. It follows that $y_1y_2 = \phi(x_1)\phi(x_2) = \phi(x_1x_2)$. Since $x_1x_2 \in G$ (because G is closed under its binary operation), we get $y_1y_2 \in \phi[G]$ by definition.
   - $\phi[G]$ is closed under inverses:

     Let $y \in \phi[G]$. Then $y = \phi(x)$ for some $x \in G$. We want to show that $y^{-1} \in \phi[G]$. We'll claim that $y^{-1} = \phi(x^{-1})$, which proves that $y^{-1} \in \phi[G]$ since $x^{-1} \in G$. Using part (a) we have

     $$e' = \phi(e) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1}).$$

     Thus $\phi(x^{-1})$ is the inverse of $\phi(x) = y$, i.e. $\phi(x^{-1}) = y^{-1}$ as claimed.

   Therefore $\phi[G]$ is a subgroup of G'.

5. Let $\phi : \mathbb{Z}_6 \to \mathbb{Z}_8$ be a group homomorphism (where as usual these are additive groups). In this problem, remember (as a hint) that for both $\mathbb{Z}_6$ and $\mathbb{Z}_8$, the element 1 (in the respective groups) generates the whole group.

   (a) (5 points) If $\phi(1) = 4$, find $\phi(3)$.

   Solution:

   $\phi(3) = \phi(1 + 1 + 1) = \phi(1) + \phi(1) + \phi(1) = 4 + 4 + 4 = 4.$

   (b) (5 points) Explain why it is impossible to have $\phi(1) = 2$.

Solution:

If $\phi(1) = 2$ then $\phi(0) = \phi(1+1+1+1+1+1) = 2+2+2+2+2+2 = 4$. But we know that $\phi(0)$ has to equal 0 since the identity element goes to the identity element in a homomorphism.

(c) (10 points) If $\phi(1) = 4$, find $\phi[\mathbb{Z}_6]$.

Solution:

$\phi(0) = 0$, $\phi(1) = 4$, $\phi(2) = 4+4 = 0$, etc. The only image points are 0 and 4, so $\phi[G] = \{0, 4\}$.

6. (10 points) Let G be a finite group and let $g \in G$. Prove that there is a positive integer n for which $g^n = e$, where e is the identity element of G. [Hint: it might be easiest to prove it by contradiction.]

Solution:

Consider
$$\{g, g^2, g^3, \dots\}.$$
Suppose that none of these powers is equal to e. Since G is finite, eventually two of these powers have to be equal to each other. Say $g^m = g^n$ with $m < n$. Then $g^{n-m} = e$. Contradiction.

7. In this problem we'll talk about permutations of the set $\mathbb{R}$.

(a) (5 points) What do we mean by "a permutation of $\mathbb{R}$" (i.e. give the definition)?

Solution:

A permutation of $\mathbb{R}$ is a one-to-one, onto function $f : \mathbb{R} \to \mathbb{R}$.

(b) (10 points) In this part you can state facts you know from calculus without proof (but make sure you state anything that you use for your answer). Let n be a positive integer. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$ is a permutation of $\mathbb{R}$ if and only if n is odd. [Hint: since $f(x) = x$ is the identity, which is the trivial permutation, assume $n \geq 2$.]

Solution:

Let $f(x) = x^n$, $n \geq 2$. If n is even then f is an even function, meaning $f(x) = f(-x)$. Thus f is not one-to-one, so it's not a permutation. (You could also argue that $f(x) \geq 0$ for all x so it's not onto.)

Now assume n is odd. Since $f'(x) = nx^{n-1}$ and $n \geq 3$, we have $f'(x) > 0$ for all x except $x = 0$. So f is strictly increasing, hence one-to-one. Since we know from calculus (or by inspection) that $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$, it is also onto. Therefore it's a permutation.