Math 30710
Second Practice Exam 1 Solutions
March 1, 2023 **Name** ______________________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. For each of the following statements, give a **counterexample** (no explanation needed beyond the counterexample itself).

   (a) (5 points) Every finite abelian group is cyclic.

   Answer:

   The Klein group is finite and abelian but is not cyclic.

   (b) (5 points) If G is a group and H is a non-empty subset of G that is closed under inverses then H is a subgroup of G.

   Answer:

   Take $G = (\mathbb{Z}, +)$ and $H =$ all non-zero integers. Then H is closed under inverses but does not contain the identity element.

   (c) (5 points) If G is an abelian group and a, b are elements of G such that $a^3 = b^3$ then $a = b$. (You can convert this statement to additive notation if you prefer that.)

   Answer:

   Let $G = U_3$, the cube roots of unity. Let $a = e^{\frac{2\pi i}{3}}$ and $b = e^{\frac{4\pi i}{3}}$. Then $a \neq b$ but $a^3 = b^3 = 1$.

2. (5 points) Consider the two binary structures (**not necessarily groups**) $(\mathbb{Z}, +)$ and $(\mathbb{Z}, *)$, where $*$ is a binary operation that you will be finding in this problem.

   Let $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, *)$ be defined by $\phi(x) = x + 1$. In order for ϕ to have the homomorphism property (i.e. $\phi(x + y) = \phi(x) * \phi(y)$), how should $*$ be defined? Please put your answer in the box and explain briefly in the space to the right of the box.

   answer:

   For $a, b \in \mathbb{Z}$, define $a * b = a + b - 1$.

   We want to have $\phi(x+y) = \phi(x)*\phi(y)$, i.e. $x+y+1 = (x + 1) * (y + 1)$. To make this work, make $*$ add the two integers and subtract 1.

3. The following is the group table for the dihedral group D_4. For this problem it is not so important to know what precise permutation each element represents; it is enough to use the group table to work out your answers.

	e	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
e	e	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	e	δ_1	δ_2	μ_2	μ_1
ρ_2	ρ_2	ρ_3	e	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	e	ρ_1	ρ_2	δ_2	δ_1	μ_1	μ_2
μ_1	μ_1	δ_2	μ_2	δ_1	e	ρ_2	ρ_3	ρ_1
μ_2	μ_2	δ_1	μ_1	δ_2	ρ_2	e	ρ_1	ρ_3
δ_1	δ_1	μ_1	δ_2	μ_2	ρ_1	ρ_3	e	ρ_2
δ_2	δ_2	μ_2	δ_1	μ_1	ρ_3	ρ_1	ρ_2	e

(a) (10 points) List all the **cyclic** subgroups of D_4 (in terms of the generators). Include the list of elements in the subgroups, and don't repeat any subgroup. For example, if the group had been $(\mathbb{Z}_8, +)$ you would write (as one of your cyclic subgroups)

$$\langle 2 \rangle = \langle 6 \rangle = \{2, 4, 6, 0\}$$

rather than listing $\langle 2 \rangle$ and $\langle 6 \rangle$ as separate subgroups; if you did list them separately you would lose points. (Hint: Do this just using the group table without visualizing the motions of the square.)

Answer:

$\langle e \rangle = \{e\}$
$\langle \rho_1 \rangle = \{\rho_1, \rho_2, \rho_3, e\} = \langle \rho_3 \rangle$
$\langle \rho_2 \rangle = \{\rho_2, e\}$
$\langle \mu_1 \rangle = \{\mu_1, e\}$
$\langle \mu_2 \rangle = \{\mu_2, e\}$
$\langle \delta_1 \rangle = \{\delta_1, e\}$
$\langle \delta_2 \rangle = \{\delta_2, e\}$

(b) (10 points) Find a subgroup of D_4 that is isomorphic to the Klein 4-group V. **You do not have to produce the isomorphism explicitly**. Just giving a list of elements (using the table as a guide) will be enough. Give a short explanation of how you chose your elements.

Answer:

One answer is $\{\mu_1, \mu_2, \rho_2, e\}$. Notice that each has order 2 and the product of any two (other than e) is equal to the third.

4. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 3 & 1 & 6 \end{pmatrix}.$$

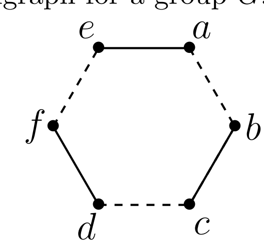
(a) (5 points) Fill in the second row: **Answer:** $\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 5 & 4 & 6 \end{pmatrix}$

(b) (5 points) Fill in the second row: **Answer:** $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 1 & 3 & 6 \end{pmatrix}$

(c) (5 points) Find the order of σ and explain your answer.

Answer:
In the same way as (a) you can compute $\sigma^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 1 & 3 & 6 \end{pmatrix}$ and σ^4 is the identity. Thus the order of σ is 4.

5. Consider the following Cayley digraph for a group G:



where we will assume that e is the identity element. (Remember that a, b, c, d, e, f are elements of G.) Notice that none of the edges has an arrow, and that a and f generate G. (You may recognize this digraph from class, but I'd like your answers to be just in terms of this picture.)

(a) (5 points) Find ab (i.e. which of a, b, c, d, e, f is ab equal to?) and justify your answer.

Answer:

Note that multiplication by a is represented by the solid line and multiplication by f is represented by the dashed line. So we have the equation $af = b$. Then $ab = a(af) = (a^2)f = ef = f$.

(b) (5 points) Prove that G is not abelian by exhibiting two elements that do not commute (and justifying your answer).

Answer:

$af = b$ but $fa = d$.

(c) (5 points) Find an element of order 3 in G and explain your answer.

Answer:

d has order 3. In fact, since we just saw $d = fa$, we have

$$d^3 = d(fa)^2 = (df)afa = cafa = bfa = a^2 = e$$

but you can check $d^2 \neq e$.

6. Let n be an integer ≥ 2. Let G be a group with n elements. In this problem we will use multiplicative notation. Assume that the following statement is true:

   If $x \in G$ is any element **other than** *e (the identity element), and k is an integer in the range $1 \leq k \leq n-1$, then it is never the case that $x^k = e$.*

(a) (5 points) Choose any $a \in G$, $a \neq e$, and consider the set $A = \{a, a^2, \ldots, a^{n-1}\}$. Prove that these $n-1$ elements are all distinct.

Answer:

If not we have $a^i = a^j$ for some i, j $(i \neq j)$ in the range between 1 and $n-1$. Without loss of generality assume $i < j$. Then $j - i$ is also between 1 and $n-1$, and we have

$$a^{j-i} = a^j \cdot a^{-i} = e.$$

This contradicts our assumption.

(b) (10 points) Using the result of (6a) (even if you weren't able to prove it), prove that G is cyclic.

Answer:

The set A contains $n-1$ distinct elements (by (6a)), none of which is e. So $A \cup \{e\}$ is a subset of G with n elements. Since G itself has n elements, we get $G = A \cup \{e\} = \langle a \rangle$.

(c) (5 points) From your knowledge about subgroups of cyclic groups, prove that in fact n must be prime. (For this part you can take for granted that G is cyclic even if you didn't get part (6b).)

Answer:

We know that if a cyclic group G has order n then it contains a subgroup H of any order k that divides n. But the subgroup of a cyclic group is cyclic, so H has an element of order k. But this would mean that G has some element of order k. On the other hand, the assumption was that for **any** element of G other than e, that element does not have an order that is less than n. So there can't be a k dividing n in that range, i.e. n is prime.

7. (10 points) Let $G = GL_2(\mathbb{R})$, the **multiplicative** group of 2×2 invertible matrices with real entries. (You don't have to prove that G itself is a group. We talked about it in class. And remember "invertible" means the determinant is not zero.) I'll remind you of some facts from linear algebra that you can use in this problem:

- $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$
- If A and B are 2×2 matrices then $\det(AB) = \det(A) \cdot \det(B)$.
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$

Let H be the subset of G consisting of 2×2 matrices whose determinant is 1 or -1. Prove that H is a subgroup of G.

Answer:

We have to show:

- H is not empty. But in fact it contains the 2×2 identity matrix, so this is clear.
- H is closed under the group multiplication. This is clear from the second bullet above, since $(1)(1) = 1, (1)(-1) - 1, (-1)(-1) = 1$.
- H is closed under inverses. We know the inverse from the third bullet, so

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\right) = \left(\frac{d}{ad-bc}\right)\left(\frac{a}{ad-bc}\right) - \left(\frac{-b}{ad-bc}\right)\left(\frac{-c}{ad-bc}\right) = \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc} = \pm 1$$

(where the penultimate equality is obtained by cancelling $ad - bc$ from numerator and denominator, which is ok since it is not zero).

Extra credit (5 points):

State Cayley's theorem and give a very short synopsis of the proof.

Answer:

Every group is isomorphic to a subgroup of some symmetric group.
idea of proof:

Let G be a group and let S_G be the symmetric group on the elements of G (thought of just as a set). For any $x \in G$ let $\lambda_x : G \to G$ be given by left multiplication: $\lambda_x(g) = xg$ for any $g \in G$. Then λ_x is a permutation (1-1 onto function) from G to itself, so it's an element of S_G. The function ϕ that sends x to λ_x is a homomorphism from G to S_G, and it is 1-1. So G is isomorphic to its image, which is a subgroup of S_G.

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