Math 30710
Exam 1 Solutions
March 1, 2023 **Name** ______________________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are. There's a blank page at the end in case you need it either to continue an answer or for scrap paper, but mark your answers clearly!!

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. Short answer. For each of the following statements, if it's true then give a proof. If it's not true then explain why not (possibly by giving a counterexample, unless an explanation is more appropriate).

(a) (7 points) If G is a non-abelian group then every non-trivial subgroup of G is also non-abelian.

Answer:

False. For example, take $G = S_3$ and let σ be any element other than the identity. Then $\langle \sigma \rangle$ is a cyclic subgroup of S_3, and cyclic groups are always abelian.

(b) (7 points) If G is a multiplicative group then the identity element for G is the only element equal to its own square.

Answer:

True. Let x be an element for which $x \cdot x = x^2 = x = x \cdot e.$. By left cancelation, $x = e$.

(c) (7 points) If G is a group then the order of the identity element, e, is not well-defined because $e^k = e$ for all integers k.

Answer:

False. The order of the identity element is the *smallest* such k, namely 1.

(d) (7 points) D_4 is the smallest non-trivial subgroup of S_4.

Answer:

False.Take any element, σ, of order 2 (e.g. a flip). Then $\langle \sigma \rangle$ is a subgroup of S_4 of order 2, while D_4 has order 8.

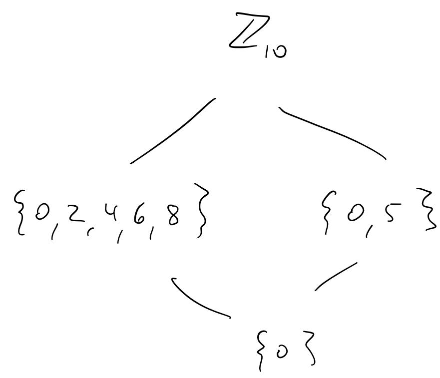
2. Let $G = (\mathbb{Z}_{10}, +)$.

   (a) (5 points) Write down the subgroup diagram for G.

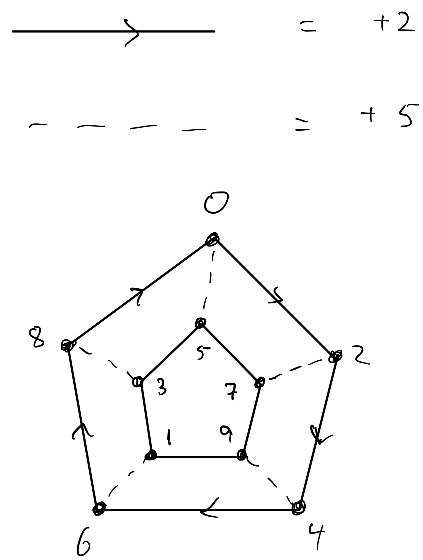
   (b) (7 points) Write the Cayley digraph for G using the generators 2 and 5. Be sure to indicate very carefully what each arrow represents. (Note that $5 + 5 = 0$ in $\mathbb{Z}_{10}$.)

   Answer:

   See below for both diagrams.



subgroup diagram



Cayley digraph

3. Let $G = (\mathbb{C}^*, \cdot)$ be the multiplicative group of non-zero complex numbers.

   (a) (7 points) Let H be a subgroup of G and assume that H is finite. Prove that every element of H has to lie on the unit circle.

   Answer:

   Let $\alpha \in H$ and suppose that α is not on the unit circle, i.e. $|\alpha| \neq 1$. Consider the positive powers $\alpha, \alpha^2, \alpha^3, \ldots$. These all have to be in H too, since H is closed under multiplication. If $|\alpha| > 1$ then this set spirals with absolute values getting bigger and bigger. If $0 < |\alpha| < 1$ then this set spirals toward zero. Either way, the set is infinite (no two powers can be the same since they have different absolute values). So H contains infinitely many elements. Contradiction.

   (b) (8 points) If $z = e^{i\theta}$ is a point on the unit circle, prove that the order of z under multiplication is finite if and only if some integer multiple of θ is a multiple of 2π, i.e. there is some positive integer n so that $n\theta$ is either 2π, 4π, etc.

Answer:

$\Rightarrow$:
Assume that z has finite order under multiplication, so $z^n = 1$ for some n. That is, $(e^{i\theta})^n = 1$, i.e. $e^{in\theta} = 1$. So the direction for $e^{in\theta}$ is zero degrees, i.e. the angle $n\theta$ is a multiple of 2π.

$\Leftarrow$:
The assumption is that $n\theta$ is a multiple of 2π. So $(e^{i\theta})^n = 1$ (since it is on the unit circle with direction given by zero degrees), i.e. $z^n = 1$. Hence z has finite order.

4. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 11 & 2 & 7 & 10 & 9 & 8 & 1 & 6 & 4 & 12 & 5 \end{pmatrix}$.

   (a) (5 points) Write σ as a product of disjoint cycles.

   Answer:

   $\sigma = (1, 3, 2, 11, 12, 5, 10, 4, 7, 8)(6, 9)$

   (b) (5 points) Write σ as a product of transpositions and say if it is even or odd.

   Answer:

   $\sigma = (1, 8)(1, 7)(1, 4)(1, 10)(1, 5)(1, 12)(1, 11)(1, 2)(1, 3)(6, 9)$. It is even.

   It can also be written as $(1, 3)(3, 2)(2, 11)(11, 12)(12, 5)(5, 10)(10, 4)(4, 7)(7, 8)(6, 9)$.

5. Let

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}$$

   (a) (5 points) Find $\sigma_1\sigma_2$ and write your answer here:

$$\sigma_1\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 6 & 3 & 4 \end{pmatrix}$$

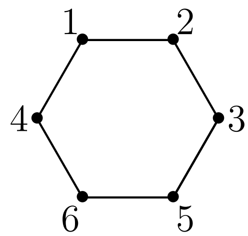
   You do not have to explain how you got your answer.

   (b) (5 points) Find σ_1^{-1} and write your answer here:

$$\sigma_1^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}$$

   You do not have to explain how you got your answer.

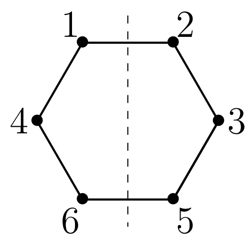
(c) (5 points) Suppose we number the corners of a regular hexagon in the usual (consecutive) way numbered clockwise:



Of the given permutations σ_1 and σ_2 in this problem, one is in D_6 and one is not. Which is in D_6? Describe it in terms of rotations and/or reflections. (It would probably help if you use the above picture, or one like it, in your answer.)

Answer:

σ_1 is the reflection about the vertical line bisecting the line segment joining 1 and 2.



6. (10 points) Let $G = \{e, a, b, c, d, f\}$ be a group and let $*$ be the binary operation on G. Assume e is the identity element for G. **Assume further that G is abelian.** Suppose the following is the group table for G.

$*$	e	a	b	c	d	f
e						
a		f	c	d	b	
b			f	e	a	
c				a	f	
d					e	
f						

Fill in the blank spaces so that you complete the group table. You'll have to use what you know about the group table of an abelian group. You do not need to explain your answer in this problem.

Answer:

We use the fact that the top row and leftmost column correspond to multiplication by e, that the array is symmetric about the main diagonal (because G is abelian), and that each element of G has to appear exactly once in each row and each column.

$*$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	f	c	d	b	e
b	b	c	f	e	a	d
c	c	d	e	a	f	b
d	d	b	a	f	e	c
f	f	e	d	b	c	a

7. (10 points) Prove that if $\phi : G \to G'$ is an isomorphism, and if G is abelian, then G' is abelian (i.e. the property of a group being abelian is preserved under isomorphism).

Answer:

Let $x', y' \in G'$. We want to show that $x'y' = y'x'$. Since ϕ is an isomorphism, in particular it is onto so there exist $x, y \in G$ such that $\phi(x) = x'$, $\phi(y) = y'$. Then

$$x'y' = \phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x) = y'x'.$$

Here the second equality and fourth equality are because ϕ has the homomorphism property. The third equality is because G is abelian.