Math 30710
Exam 2
April 19, 2023 **Name** ______________________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are. There's a blank page at the end in case you need it either to continue an answer or for scrap paper, but mark your answers clearly!!

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. Let $\phi : G \to G'$ be a group homomorphism. In this problem I'd like you to recall the proofs of the following results that we proved in class. For any part you can use earlier parts of this problem.
   (a) Let e, e' be the identity elements for G, G' respectively. Prove that $\phi(e) = e'$.

   (b) Let $a \in G$. Prove that $\phi(a)^{-1} = \phi(a^{-1})$.

   (c) Recalling the notation
   $$\phi[G] = \{y \in G' \mid y = \phi(x) \text{ for some } x \in G\},$$
   prove that $\phi[G]$ is a subgroup of G'.

2. Let $G = (\mathbb{Z}_6 \times \mathbb{Z}_8)$, and let $H = \langle (4,4) \rangle$ (the subgroup generated by the element $(4,4)$).

   (a) Write out the elements of H. (No explanation needed.)

   (b) What is the order of G/H? (No explanation needed.)

   (c) According to the FTFGAG, what are the possible groups that G/H might be isomorphic to? (No explanation needed.)

   (d) Find the order of the element $(1,1) + H$ in the quotient G/H. Explain your answer.

   (e) Which of the possibilities in problem 2c is ruled out by your answer to problem 2d? Explain.

3. For each of the following, say if it's true or false. Either way, give a (short!!) explanation.

   (a) The alternating group A_3 is cyclic.

   (b) For $n > 3$ the symmetric group S_n is a simple group.

4. Let $\phi : G \to G'$ be a group homomorphism. If N' is a normal subgroup of $\phi[G]$, show that $\phi^{-1}[N']$ is a normal subgroup of G. [You can use without proof the fact that $\phi^{-1}[N']$ is a subgroup of G. I just want to know why it's normal. You can also use without proof the equivalent ways we showed in class that a subgroup is normal, but be clear about what you are doing.]

5. Let G be a group (not necessarily abelian) and let H be a subgroup of G (not necessarily normal). Assume that $a, b \in G$ have the property that $aH = bH$. Prove that $Ha^{-1} = Hb^{-1}$. You can use any fact from class that gives conditions for when $aH = bH$.

6. In this problem we want to find an explicit homomorphism $\phi : \mathbb{Z}_6 \to S_5$.

   (a) Assume that we have some well-defined homomorphism $\phi : \mathbb{Z}_6 \to S_5$. Suppose that we find an element $\sigma \in S_5$ for which $\phi(1) = \sigma$. In terms of σ, what is $\phi(4)$?

   (b) Prove that if σ is the cycle $(1, 2, 3, 4, 5)$ then setting $\phi(1) = \sigma$ will not be well-defined.

   (c) Find a different σ for which $\phi(1) = \sigma$ is well-defined. Give a short argument for why this will be well-defined. (It does not have to be a detailed proof.)

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