Math 30710
Exam 2
April 19, 2023 **Name** ______________________________

Solutions

1. Let $\phi : G \to G'$ be a group homomorphism. In this problem I'd like you to recall the proofs of the following results that we proved in class. For any part you can use earlier parts of this problem.

   (a) (6 points) Let e, e' be the identity elements for G, G' respectively. Prove that $\phi(e) = e'$.

   Solution:
   $e' \cdot \phi(e) = \phi(e) = \phi(e \cdot e) = \phi(e) \cdot \phi(e)$ by the homomorphism property. By right cancellation we get $e' = \phi(e)$.

   (b) (6 points) Let $a \in G$. Prove that $\phi(a)^{-1} = \phi(a^{-1})$.

   Solution:
   $e' = \phi(e) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1})$ by the homomorphism property (where the first equality is by part (a)). So $\phi(a^{-1})$ is the inverse of $\phi(a)$, which means that $\phi(a)^{-1} = \phi(a^{-1})$.

   (c) (8 points) Recalling the notation

   $$\phi[G] = \{y \in G' \mid y = \phi(x) \text{ for some } x \in G\},$$

   prove that $\phi[G]$ is a subgroup of G'.

   Solution:

   - $\phi[G] \neq \emptyset$ because $e' = \phi(e) \in \phi[G]$.
   - $\phi[G]$ is closed under the binary operation of G': let $y_1, y_2 \in \phi[G]$. Then by definition there exist $x_1, x_2 \in G$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$. Then $y_1y_2 = \phi(x_1)\phi(x_2) = \phi(x_1x_2)$. Since G is a group, we know $x_1x_2 \in G$ so $y_1y_2 \in \phi[G]$.
   - $\phi[G]$ is closed under inverses: Let $y \in \phi[G]$, so $y = \phi(x)$ for some $x \in G$. Then by (b) we have $y^{-1} = \phi(x)^{-1} = \phi(x^{-1})$. Again since G is a group we have $x^{-1} \in G$, so $y^{-1} \in \phi[G]$.

2. Let $G = (\mathbb{Z}_6 \times \mathbb{Z}_8)$, and let $H = \langle (4,4) \rangle$ (the subgroup generated by the element $(4,4)$).

   (a) (6 points) Write out the elements of H. (No explanation needed.)

   Solution:

   $H = \{(4,4), (2,0), (0,4), (4,0), (2,4), (0,0)\}$.

   (b) (5 points) What is the order of G/H? (No explanation needed.)

   Solution:

   $|G/H| = |(\mathbb{Z}_6 \times \mathbb{Z}_8)/|H| = 48/6 = 8$.

(c) (6 points) According to the FTFGAG, what are the possible groups that G/H might be isomorphic to? (No explanation needed.)

Solution:

$\mathbb{Z}_8$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(d) (6 points) Find the order of the element $(1,1)+H$ in the quotient G/H. Explain your answer.

Solution:

We keep adding $(1,1)$ to itself until we get an element of H.

$$\begin{aligned} &(1,1) \text{ is not in } H \\ &(2,2) \text{ is not in } H \\ &(3,3) \text{ is not in } H \\ &(4,4) \text{ bf is in } H. \end{aligned}$$

So the order of $(1,1)+H$ is 4.

(e) (6 points) Which of the possibilities in problem 2c is ruled out by your answer to problem 2d? Explain.

Solution:

Only the last one, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, has no element of order 4.

3. For each of the following, say if it's true or false. Either way, give a (short!!) explanation.

(a) (5 points) The alternating group A_3 is cyclic.

Solution:

True. We know $|S_3| = 6$ and $|A_3|$ is half of that, or 3. Any group of order 3 is isomorphic to $\mathbb{Z}_3$, which is cyclic.

(b) (6 points) For $n > 3$ the symmetric group S_n is a simple group.

Solution:

False. A_n has index 2 so it is a normal subgroup of S_n, so S_n is not a simple group.

4. (8 points) Let $\phi : G \to G'$ be a group homomorphism. If N' is a normal subgroup of $\phi[G]$, show that $\phi^{-1}[N']$ is a normal subgroup of G. [You can use without proof the fact that $\phi^{-1}[N']$ is a subgroup of G. I just want to know why it's normal. You can also use without proof the equivalent ways we showed in class that a subgroup is normal, but be clear about what you are doing.]

Solution:

Let $x \in G$ be any element of G. Let $g \in \phi^{-1}[N']$ be any element of $\phi^{-1}[N']$. It's enough to prove that $xgx^{-1} \in \phi^{-1}[N']$. So we have to prove that $\phi(xgx^{-1}) \in N'$. Notice that $\phi(g) \in N'$ since $g \in \phi^{-1}[N']$. Note also that $\phi(x) \in \phi[G]$. Then

$$\phi(xgx^{-1}) = \phi(x)\phi(g)\phi(x^{-1}) = \phi(x)\phi(g)(\phi(x))^{-1}$$

(using problem #1). Now since N' is normal in $\phi[G]$ and since $\phi(g) \in N'$ and $\phi(x) \in \phi[G]$, we get $\phi(x)\phi(g)(\phi(x))^{-1} \in N'$ (again using our criterion for when a subgroup is normal), so $\phi(xgx^{-1}) \in N'$ and we're done.

5. (8 points) Let G be a group (not necessarily abelian) and let H be a subgroup of G (not necessarily normal). Assume that $a, b \in G$ have the property that $aH = bH$. Prove that $Ha^{-1} = Hb^{-1}$. You can use any fact from class that gives conditions for when $aH = bH$.

   Solution:

   The fact that $aH = bH$ gives us that $b \in aH$ so we can write $b = ah$ for some $h \in H$. Then
$$b^{-1} = (ah)^{-1} = h^{-1}a^{-1} \in Ha^{-1}.$$
   This means that b^{-1} and a^{-1} determine the same right coset of H, i.e. $Hb^{-1} = Ha^{-1}$.

6. In this problem we want to find an explicit homomorphism $\phi : \mathbb{Z}_6 \to S_5$.

   (a) (8 points) Assume that we have some well-defined homomorphism $\phi : \mathbb{Z}_6 \to S_5$. Suppose that we find an element $\sigma \in S_5$ for which $\phi(1) = \sigma$. In terms of σ, what is $\phi(4)$?

   Solution:

   $\phi(4) = \phi(1 + 1 + 1 + 1) = \phi(1)\phi(1)\phi(1)\phi(1) = \sigma^4$.

   (b) (8 points) Prove that if σ is the cycle $(1, 2, 3, 4, 5)$ then setting $\phi(1) = \sigma$ will not be well-defined.

   Solution:

   Since σ is a 5-cycle, it has order 5. This means that $\phi(5) = \sigma^5 = e$ (as in part (a) of this problem). But then
$$\phi(0) = \phi(1 + 1 + 1 + 1 + 1 + 1) = \sigma^6 = \sigma^5 \cdot \sigma = e \cdot \sigma = \sigma \neq e.$$
   But for a group homomorphism, the identity has to map to the identity (see problem #1) so this is impossible.

   (c) (8 points) Find a different σ for which $\phi(1) = \sigma$ is well-defined. Give a short argument for why this will be well-defined. (It does not have to be a detailed proof.)

   Solution:

   We need a $\sigma \in S_5$ whose order divides 6. This will avoid the issue discussed in part (b). Any 2-cycle or any 3-cycle would work. But also $\sigma = (1, 2)(3, 4, 5)$ has order 6 (LCM of 2 and 3), so setting $\phi(1) = \sigma$ in this case not only gives a well-defined homomorphism but also gives an injective homomorphism (which was not part of the problem but it's nice extra information).