**Math 30710**
**Exam 2**
October 23, 2020 **Name** ______________________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

# Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 9 & 6 & 4 & 10 & 3 & 1 & 5 & 7 & 8 \end{pmatrix}$$

(a) (3 points) Write $\sigma$ as a product of disjoint cycles.

(b) (3 points) Find the order of $\sigma$ and briefly explain your answer.

(c) (3 points) Write $\sigma$ as a product of transpositions.

(d) (3 points) Is $\sigma$ an element of $A_{10}$? Why or why not?

(e) (3 points) Is $\sigma$ an element of $D_{10}$ (the $10^{th}$ dihedral group, i.e. the group of symmetries of a regular 10-gon)? Explain your answer.

(f) (3 points) Find $\sigma^{-1}$.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ & & & & & & & & & \end{pmatrix}$$

2. Let $G = \mathbb{R}/\mathbb{Z}$.

   (a) (3 points) Find an element of $G$ of order 2. Briefly explain your answer.

   (b) (3 points) For each positive integer $n$, show that $G$ has an element of order $n$.

   (c) (3 points) Referring to part (b), **now assume** $n > 2$. Is there always a unique such element of order $n$, or is the element sometimes unique and sometimes not unique (depending on $n$), or is there always more than one such element? Explain your answer.

   (d) (3 points) Find an element of $G$ of infinite order. Briefly explain your answer.

3. Let $G = \mathbb{Z}_9 \times \mathbb{Z}_8$ and let $H = \langle (3, 2) \rangle$.

   (a) (4 points) List the elements of $H$.

   (b) (3 points) Find $|G/H|$.

   (c) (4 points) To what common group is $G/H$ isomorphic? Explain your answer.

   (d) (4 points) Find the order of $(1, 2) + H$ in $G/H$.

4. As a reminder, the FTFGAG says the following (in particular):

> If $G$ is a finite abelian group then $G$ is isomorphic to a group of the form
> $$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \times \mathbb{Z}_{(p_n)^{r_n}}$$
> where the $p_i$ are primes (not necessarily distinct), and the $r_i$ are positive integers (not necessarily distinct). Apart from changing the order, this decomposition is unique up to isomorphism.

(a) (8 points) For each of the following groups, write it in the form given in the FTFGAG:

$\mathbb{Z}_8 \times \mathbb{Z}_6 \times \mathbb{Z}_{40} \cong$ ____________________________

$\mathbb{Z}_4 \times \mathbb{Z}_{20} \times \mathbb{Z}_{24} \cong$ ____________________________

(b) (6 points) Using your answer to (a) and using the FTFGAG, are the two groups

$$\mathbb{Z}_8 \times \mathbb{Z}_6 \times \mathbb{Z}_{40} \quad \text{and} \quad \mathbb{Z}_4 \times \mathbb{Z}_{20} \times \mathbb{Z}_{24}$$

isomorphic? Briefly explain your answer.

5. (10 points) Let $\phi : G \to G'$ be a group isomorphism. Since it's one-to-one and onto, we know that as a *function*, $\phi$ has an inverse, $\phi^{-1}$. You don't have to prove this. Prove that in fact the function $\phi^{-1}$ has the homomorphism property. That is, if $y_1, y_2 \in G'$ then prove $\phi^{-1}(y_1 y_2) = \phi^{-1}(y_1)\phi^{-1}(y_2)$.

6. (6 points) Let $G$ be a simple non-abelian group. Find the center of $G$ and explain your answer.

7. (10 points) Let $\phi : G \to G'$ be a group homomorphism. Assume that $G'$ is abelian. (We don't know if $G$ is abelian or not.) Prove that for all $x, y \in G$, we have $xyx^{-1}y^{-1} \in \ker \phi$. It's a short proof, but be sure to justify every step. (If you use a fact from class, just quote it, don't reprove it!)

8. Let $G$ be a group.

   (a) (8 points) If $N_1$ and $N_2$ are normal subgroups of $G$, show that $N_1 \cap N_2$ is also normal in $G$.

   (b) (7 points) If $H$ is any subgroup of $G$ and $N$ is a normal subgroup of $G$ then show that $H \cap N$ is a normal subgroup of $H$.

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