Math 30710
Exam 2 Solutions
October 23, 2020 **Name** ______________________________

This is a 50-minute exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 9 & 6 & 4 & 10 & 3 & 1 & 5 & 7 & 8 \end{pmatrix}$$

(a) (3 points) Write σ as a product of disjoint cycles.

Solution:

$$(1,2,9,7)(3,6)(5,10,8).$$

(b) (3 points) Find the order of σ and briefly explain your answer.

Solution:
The order is the least common multiple of 4, 2 and 3, i.e. it is 12.

(c) (3 points) Write σ as a product of transpositions.

Solution:
One way to do it is $(1,7)(1,9)(1,2)(3,6)(5,8)(5,10)$.

(d) (3 points) Is σ an element of A_{10}? Why or why not?

Solution:
Yes it is, since it is a product of 6 transpositions.

(e) (3 points) Is σ an element of D_{10} (the 10^{th} dihedral group, i.e. the group of symmetries of a regular 10-gon)? Explain your answer.

Solution:
No it is not. σ has order 12, which does not divide the order of D_{10} (which is 20).

(f) (3 points) Find σ^{-1}.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 1 & 6 & 4 & 8 & 3 & 9 & 10 & 2 & 5 \end{pmatrix}$$

2. Let $G = \mathbb{R}/\mathbb{Z}$.

   (a) (3 points) Find an element of G of order 2. Briefly explain your answer.

   Solution:
   The element $\frac{1}{2} + \mathbb{Z}$ has order 2 since

   $$\left(\frac{1}{2} + \mathbb{Z}\right) + \left(\frac{1}{2} + \mathbb{Z}\right) = 1 + \mathbb{Z} = 0 + \mathbb{Z}$$

   in $\mathbb{R}/\mathbb{Z}$ (since $1 - 0 \in \mathbb{Z}$).

   (b) (3 points) For each positive integer n, show that G has an element of order n.

   Solution:
   $\frac{1}{n} + \mathbb{Z}$ has order n since if you add $\frac{1}{n} + \mathbb{Z}$ to itself m times, the first time you get $0 + \mathbb{Z}$ is when $m = n$.

   (c) (3 points) Referring to part (b), **now assume** $n > 2$. Is there always a unique such element of order n, or is the element sometimes unique and sometimes not unique (depending on n), or is there always more than one such element? Explain your answer.

   Solution:
   Always more than one. $\frac{1}{n} + \mathbb{Z}$ and $-\frac{1}{n} + \mathbb{Z}$ both have order n. Since $n > 2$, these are not the same element of $\mathbb{R}/\mathbb{Z}$ since

   $$\frac{1}{n} - \left(-\frac{1}{n}\right) = \frac{2}{n} \notin \mathbb{Z}.$$

   (d) (3 points) Find an element of G of infinite order. Briefly explain your answer.

   Solution:
   $\pi + \mathbb{Z}$ has infinite order, since no integer multiple of π is again an integer.

3. Let $G = \mathbb{Z}_9 \times \mathbb{Z}_8$ and let $H = \langle (3, 2) \rangle$.

   (a) (4 points) List the elements of H.

   Solution:

   $$H = \{(0,0), (3,2), (6,4), (0,6), (3,0), (6,2), (0,4), (3,6), (6,0), (0,2), (3,4), (6,6)\}.$$

   (b) (3 points) Find $|G/H|$.

   Solution:
   $72/12 = 6$.

   (c) (4 points) To what common group is G/H isomorphic? Explain your answer.

   Solution:
   Since G is abelian, the FTFGAG says that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. Since 2 and 3 are relatively prime, this is isomorphic to $\mathbb{Z}_6$.

   (d) (4 points) Find the order of $(1, 2) + H$ in G/H.

   Solution:

All we have to do is add (1,2) to itself until we reach an element of H. Note that the answer had better divide 6. We have

$$(1,2),(2,4),3,6).$$

Since $(3,6) \in H$, the order is 3.

4. As a reminder, the FTFGAG says the following (in particular):

> If G is a finite abelian group then G is isomorphic to a group of the form
> $$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \times \mathbb{Z}_{(p_n)^{r_n}}$$
> where the p_i are primes (not necessarily distinct), and the r_i are positive integers (not necessarily distinct). Apart from changing the order, this decomposition is unique up to isomorphism.

(a) (8 points) For each of the following groups, write it in the form given in the FTFGAG:

$$\mathbb{Z}_8 \times \mathbb{Z}_6 \times \mathbb{Z}_{40} \cong \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8 \times \mathbb{Z}_5.$$

$$\mathbb{Z}_4 \times \mathbb{Z}_{20} \times \mathbb{Z}_{24} \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_8 \times \mathbb{Z}_3.$$

(b) (6 points) Using your answer to (a) and using the FTFGAG, are the two groups

$$\mathbb{Z}_8 \times \mathbb{Z}_6 \times \mathbb{Z}_{40} \quad \text{and} \quad \mathbb{Z}_4 \times \mathbb{Z}_{20} \times \mathbb{Z}_{24}$$

isomorphic? Briefly explain your answer.

Solution:
No, these latter two forms are not the same up to reordering of the factors.

5. (10 points) Let $\phi : G \to G'$ be a group isomorphism. Since it's one-to-one and onto, we know that as a *function*, ϕ has an inverse, ϕ^{-1}. You don't have to prove this. Prove that in fact the function ϕ^{-1} has the homomorphism property. That is, if $y_1, y_2 \in G'$ then prove $\phi^{-1}(y_1 y_2) = \phi^{-1}(y_1)\phi^{-1}(y_2)$.

Solution:
Since ϕ is onto (surjective) and $y_1, y_2 \in G'$, there exist $x_1, x_2 \in G$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$. Then

$$\begin{array}{rcll} \phi^{-1}(y_1 y_2) & = & \phi^{-1}(\phi(x_1)\phi(x_2)) & \\ & = & \phi^{-1}(\phi(x_1 x_2)) & \text{since } \phi \text{ is a homomorphism} \\ & = & x_1 x_2 & \text{by definition of inverse functions} \\ & = & \phi^{-1}(y_1)\phi^{-1}(y_2). & \end{array}$$

6. (6 points) Let G be a simple non-abelian group. Find the center of G and explain your answer.

Solution:
Recall that the center of a group is always a normal subgroup. Since G is simple, this means that it has no proper normal subgroup, so the center is either $\{e\}$ or all of G. Since G is not abelian but the center is, this means the center is $\{e\}$.

7. (10 points) Let $\phi : G \to G'$ be a group homomorphism. Assume that G' is abelian. (We don't know if G is abelian or not.) Prove that for all $x, y \in G$, we have $xyx^{-1}y^{-1} \in \ker\phi$. It's a short proof, but be sure to justify every step. (If you use a fact from class, just quote it, don't reprove it!)

   Solution:
   We have to show that $\phi(xyx^{-1}y^{-1}) = e'$.

$$\begin{array}{rcll} \phi(xyx^{-1}y^{-1}) & = & \phi(x)\phi(y)\phi(x^{-1})\phi(y^{-1}) & \text{since } \phi \text{ is a homomorphism} \\ & = & \phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1} & \text{proved in class} \\ & = & \phi(x)\phi(x)^{-1}\phi(y)\phi(y)^{-1} & \text{since } G' \text{ is abelian} \\ & = & e' & \text{by definition of inverses.} \end{array}$$

8. Let G be a group.

   (a) (8 points) If N_1 and N_2 are normal subgroups of G, show that $N_1 \cap N_2$ is also normal in G.

   Solution:
   From what we showed in class, it is enough to show that $axa^{-1} \in N_1 \cap N_2$ for all $a \in G$ and all $x \in N_1 \cap N_2$. Since $x \in N_1$ and N_1 is normal, we get $axa^{-1} \in N_1$. Same for N_2. Thus $axa^{-1} \in N_1 \cap N_2$ as desired.

   (b) (7 points) If H is any subgroup of G and N is a normal subgroup of G then show that $H \cap N$ is a normal subgroup of H.

   Solution:
   We've seen that the intersection of two subgroups is again a subgroup, so we just have to check the normality condition.

   Let $a \in H$ and $x \in H \cap N$. We want to show that $axa^{-1} \in H \cap N$. Since a and x are both in H, certainly $axa^{-1} \in H$. Since N is normal in G, and since $a \in G$ and $x \in N$, we get $axa^{-1} \in N$. So combining, we have $axa^{-1} \in H \cap N$.

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