Math 30710
Practice Exam 2
Exam October 23, 2020 **Name** ____________________

WARNING!! Our exam is a bit off the schedule of my usual second exam, which normally has some questions about rings. So I've merged two practice exams and removed ring questions. Also, I've put in a permutation/cycle question that I left off our first exam. So this is longer than the exam you'll find on Friday. I hope it gives you an idea of the kinds of questions you might find.

This is a 1-hour exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. Groups of order 24.
   (a) Give an example of a non-abelian group of order 24. (You don't have to justify your answer.)

   (b) Up to isomorphism, what are all possible *abelian* groups of order 24? [Hint: there are three.]

   (c) For each of your answers in (b), find an element of order 6.

2. In class we showed that A_4 has no subgroup of order 6. Use this fact (without reproving it) to prove that if $\phi : A_4 \to \mathbb{Z}_2$ is a homomorphism then ϕ must be trivial (i.e. $\phi(x) = 0$ for all $x \in A_4$).

3. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 9 & 5 & 4 & 10 & 3 & 1 & 6 & 7 & 8 \end{pmatrix}$$

(a) Write σ as a product of disjoint cycles.

(b) Write σ as a product of transpositions.

(c) Is σ an element of A_{10}? Why or why not?

(d) Is σ an element of D_{10} (the 10^{th} dihedral group, i.e. the group of symmetries of a regular 10-gon)? Explain your answer. (It does not have to be a rigorous proof.)

(e) Find σ^{-1}.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ & & & & & & & & & \end{pmatrix}$$

4. Let G be a group, H a subgroup, and $g \in G$. Show that gHg^{-1} is also a subgroup of G.

5. Let $G = \mathbb{Z}_{10} \times \mathbb{Z}_{12}$. Note that $|G| = 120 = 2^3 \cdot 3 \cdot 5$. Let $H \subset G$ be the subgroup $\langle (2,4) \rangle$.

(a) Find $|H|$. (Note: The problem says "Find $|H|$", **not** "Find H". Right now you're looking for a number, not a list of elements.) Briefly explain your answer.

(b) Having figured out $|H|$, find $|G/H|$. (Again we are asking for an integer, not a set.)

(c) Using the answer to (b), what are the possible groups that G/H could be isomorphic to, according to the FTFGAG?

(d) Find the elements of H. Make sure your answer agrees with what you put in (a).

(e) In G/H find the order of $(1,1) + H$. (Your answer to (d) will be useful.) Explain your answer – a correct answer without a justification will not receive full credit.

(f) Conclude that G/H is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Explain.

6. Let $G = \mathbb{Z}_6 \times \mathbb{Z}_9$ and let H be the cyclic subgroup $\langle (2,2) \rangle$.
   (a) Explicitly write the elements of H.

   (b) Find $|G|$, $|H|$ and $|G/H|$.

   (c) According to the Fundamental Theorem of Finitely Generated Abelian Groups, to what group is G/H isomorphic? Explain your answer. (Hint: from the information contained in (a) and (b), you can already answer this part without any further computation.)

7. In this problem we want to look at group homomorphisms.
   (a) Let $G = \langle a \rangle$ be a **cyclic** group. If $\phi : G \to G'$ is a group homomorphism, explain how you can find $\phi(x)$ for any $x \in G$ as long as you know $\phi(a)$.

   (b) What are the possible **group** homomorphisms $\phi : \mathbb{Z}_6 \to \mathbb{Z}_6$? Just tell me what the possible values for $\phi(1)$ are. Explain your answer.

8. Let $G = \mathbb{Z}_8 \times \mathbb{Z}_{12}$ and let

$$H = \{(6,6),(4,0),(2,6),(0,6),(6,0),(4,6),(2,0),(0,0)\},$$

so $|G| = 96$, $|H| = 8$ and $|G/H| = 12$. You do **not** have to prove that H is actually a subgroup, or justify these numbers – take it as a fact.

(a) According to the Fundamental Theorem of Finitely Generated Abelian Groups, what are the *two* possibilities for G/H up to isomorphism?

(b) I don't want you wasting time with a tedious calculation, so let's just see what you **would** do to determine which of the two possibilities in (a) is the correct one. Without actually making any computations, what isomorphism invariant would you look for to distinguish between the two possibilities? (Your answer should not take more than a line or two.)

(c) Compute the order of the element $(3,5) + H$ in G/H, showing your work.

9. **Short** proofs. Each of the following proofs should only take a couple of lines.

(a) Let G be a group and let H be a subgroup. Let $a \in G$ and let $h \in H$. Prove that we have an equality of cosets $aH = (ah)H$.

(b) Let $G = \langle a \rangle$ be a cyclic group. Let H be a subgroup. Explain why H is normal, and prove that G/H is again cyclic.

(c) Assume that R is a ring with unity. If a and b are non-zero elements of R such that $ab = 0$, prove that a is not a unit.

(Extra sheet.)