Math 30710
Practice Exam 2
Exam October 23, 2020 **Name** ______________________________

WARNING!! Our exam is a bit off the schedule of my usual second exam, which normally has some questions about rings. So I've merged two practice exams and removed ring questions. Also, I've put in a permutation/cycle question that I left off our first exam. So this is longer than the exam you'll find on Friday. I hope it gives you an idea of the kinds of questions you might find.

This is a 1-hour exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. Groups of order 24.

   (a) Give an example of a non-abelian group of order 24. (You don't have to justify your answer.)

   Solution:
   S_4.

   (b) Up to isomorphism, what are all possible *abelian* groups of order 24? [Hint: there are three.]

   Solution:
   Since $24 = 2^3 \cdot 3$, the possibilities are

   $$\mathbb{Z}_8 \times \mathbb{Z}_3, \quad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3.$$

   (c) For each of your answers in (b), find an element of order 6.

   Solution:

   $$(4,1), \quad (0,1,1), \quad (1,1,1,1)$$

   (respectively) are some examples.

2. In class we showed that A_4 has no subgroup of order 6. Use this fact (without reproving it) to prove that if $\phi : A_4 \to \mathbb{Z}_2$ is a homomorphism then ϕ must be trivial (i.e. $\phi(x) = 0$ for all $x \in A_4$).

   Solution:
   We know that $\phi[G]$ is either $\{0\}$ or $\mathbb{Z}_2$, and we know $|A_4| = 12$. We also know that $G/\ker\phi \cong \phi[G]$. If $\pi[G]$ were $\mathbb{Z}_2$ then $|\ker\phi| = \frac{12}{2} = 6$, which we know is not possible. So $\phi[G] = \{0\}$ as desired.

3. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 9 & 5 & 4 & 10 & 3 & 1 & 6 & 7 & 8 \end{pmatrix}$$

(a) Write σ as a product of disjoint cycles.

Solution:

$$(1, 2, 9, 7)(3, 5, 10, 8, 6)$$

(b) Write σ as a product of transpositions.

One Solution:

$$(1, 2)(2, 9)(9, 7)(3, 5)(5, 10)(10, 8)(8, 6).$$

(c) Is σ an element of A_{10}? Why or why not?

Solution:
No, because it is the product of an odd number of transpositions.

(d) Is σ an element of D_{10} (the 10^{th} dihedral group, i.e. the group of symmetries of a regular 10-gon)? Explain your answer. (It does not have to be a rigorous proof.)

Solution:
No. A symmetry of a regular 10-gon can't cycle four vertices one way and five another and keep one fixed. In fact, in order to keep 4 fixed it would need to keep 9 fixed too (the one directly opposite it) and be a reflection about this diagonal.

(e) Find σ^{-1}.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 1 & 6 & 4 & 3 & 8 & 9 & 10 & 2 & 5 \end{pmatrix}$$

4. Let G be a group, H a subgroup, and $g \in G$. Show that gHg^{-1} is also a subgroup of G.

Answer:

To show that a subset H of a group G is a subgroup, it's enough to show that it is closed under the group operation and closed under inverses. For the first, if ghg^{-1} and $gh'g^{-1}$ are in gHg^{-1} then $(ghg^{-1})(gh'g^{-1}) = g(hh')g^{-1} \in gHg^{-1}$ since H being a subgroup implies $hh' \in H$. For the second, since H is a subgroup, it is closed under inverses, so if $h \in H$ then $h^{-1} \in H$. Then note that $(ghg^{-1})(gh^{-1}g^{-1}) = e$ so $(ghg^{-1})^{-1} = gh^{-1}g^{-1} \in gHg^{-1}$.

5. Let $G = \mathbb{Z}_{10} \times \mathbb{Z}_{12}$. Note that $|G| = 120 = 2^3 \cdot 3 \cdot 5$. Let $H \subset G$ be the subgroup $\langle (2, 4) \rangle$.

(a) Find $|H|$. (Note: The problem says "Find $|H|$", **not** "Find H". Right now you're looking for a number, not a list of elements.) Briefly explain your answer.

Solution:
Since 2 has order 5 in $\mathbb{Z}_{10}$ and 4 has order 3 in $\mathbb{Z}_{12}$, and $LCM(5, 3) = 15$, the order of $(2, 4)$ (which is equal to $|H|$) is 15.

(b) Having figured out $|H|$, find $|G/H|$. (Again we are asking for an integer, not a set.)

Solution:

$$\frac{120}{15} = 8.$$

(c) Using the answer to (b), what are the possible groups that G/H could be isomorphic to, according to the FTFGAG?

Solution:

$$\mathbb{Z}_8,\ \mathbb{Z}_4 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

(d) Find the elements of H. Make sure your answer agrees with what you put in (a).

Solution:
If we add $(2,4)$ to itself repeatedly, we obtain the sequence

$$(2,4),\ (4,8),\ (6,0),\ (8,4),\ (0,8),\ (2,0),\ (4,4),\ (6,8),\ (8,0),\ (0,4),\ (2,8),\ (4,0),\ (6,4),\ (8,8),\ (0,0).$$

(e) In G/H find the order of $(1,1)+H$. (Your answer to (d) will be useful.) Explain your answer – a correct answer without a justification will not receive full credit.

Solution:
We'll add (1,1) to itself until we reach an element of H. We obtain the sequence

$$(1,1), (2,2), (3,3), (4,4)$$

so $(1,1) + H$ has order 4.

(f) Conclude that G/H is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Explain.

Solution:
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has no element of order 4 – all elements have order 2.

6. Let $G = \mathbb{Z}_6 \times \mathbb{Z}_9$ and let H be the cyclic subgroup $\langle(2,2)\rangle$.

(a) Explicitly write the elements of H.

Answer:

$$\langle(2,2)\rangle = \{(2,2), (4,4), (0,6), (2,8), (4,1), (0,3), (2,5), (4,7), (0,0)\}.$$

(b) Find $|G|$, $|H|$ and $|G/H|$.
Answer:

$$|G| = 54,\ |H| = 9,\ |G/H| = 6.$$

(c) According to the Fundamental Theorem of Finitely Generated Abelian Groups, to what group is G/H isomorphic? Explain your answer. (Hint: from the information contained in (a) and (b), you can already answer this part without any further computation.)

Answer:
The FTFGAG gives us that $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, which is also isomorphic to $\mathbb{Z}_6$.

7. In this problem we want to look at group homomorphisms.

   (a) Let $G = \langle a \rangle$ be a **cyclic** group. If $\phi : G \to G'$ is a group homomorphism, explain how you can find $\phi(x)$ for any $x \in G$ as long as you know $\phi(a)$.

   Answer:

   If $x \in G$ then $x = a^n$ for some n, and $\phi(x) = \phi(a^n) = \phi(a)^n$, so you know $\phi(x)$ if you know $\phi(a)$.

   (b) What are the possible **group** homomorphisms $\phi : \mathbb{Z}_6 \to \mathbb{Z}_6$? Just tell me what the possible values for $\phi(1)$ are. Explain your answer.

   Answer:

   Since

   $$0 = \phi(0) = \phi(1+1+1+1+1+1) = \phi(1)+\phi(1)+\phi(1)+\phi(1)+\phi(1)+\phi(1),$$

   the order of $\phi(1)$ has to divide 6. Since $\phi(1)$ is in $\mathbb{Z}_6$, we again get that its order has to divide 6, which is not a new condition. So $\phi(1)$ can be anything: 0, 1, 2, 3, 4 or 5.

8. Let $G = \mathbb{Z}_8 \times \mathbb{Z}_{12}$ and let

$$H = \{(6,6),(4,0),(2,6),(0,6),(6,0),(4,6),(2,0),(0,0)\},$$

so $|G| = 96$, $|H| = 8$ and $|G/H| = 12$. You do **not** have to prove that H is actually a subgroup, or justify these numbers – take it as a fact.

   (a) According to the Fundamental Theorem of Finitely Generated Abelian Groups, what are the *two* possibilities for G/H up to isomorphism?

   Answer:

   $\mathbb{Z}_3 \times \mathbb{Z}_4$ or $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

   (b) I don't want you wasting time with a tedious calculation, so let's just see what you **would** do to determine which of the two possibilities in (a) is the correct one. Without actually making any computations, what isomorphism invariant would you look for to distinguish between the two possibilities? (Your answer should not take more than a line or two.)

   Answer:

   Look for an element of order 4, or an element of order 12. If either exists, the answer is $\mathbb{Z}_3 \times \mathbb{Z}_4$. If neither exists, it's $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

   (c) Compute the order of the element $(3,5)+H$ in G/H, showing your work.

   Answer:

   We want to count how many times we can add $(3,5)$ to itself before we get to an element in H.

   $$\begin{aligned}
   1 \cdot [(3,5)+H] &= (3,5)+H \\
   2 \cdot [(3,5)+H] &= (6,10)+H \\
   3 \cdot [(3,5)+H] &= (1,3)+H \\
   4 \cdot [(3,5)+H] &= (4,8)+H \\
   5 \cdot [(3,5)+H] &= (7,1)+H \\
   6 \cdot [(3,5)+H] &= (2,6)+H = (0,0)+H
   \end{aligned}$$

   so the order is 6.

9. **Short** proofs. Each of the following proofs should only take a couple of lines.

   (a) Let G be a group and let H be a subgroup. Let $a \in G$ and let $h \in H$. Prove that we have an equality of cosets $aH = (ah)H$.

   Answer:

   We know that we have an equality of cosets $aH = bH$ if and only if $a^{-1}b \in H$. So here, since $a^{-1}(ah) = h \in H$, we have the desired equality.

   (b) Let $G = \langle a \rangle$ be a cyclic group. Let H be a subgroup. Explain why H is normal, and prove that G/H is again cyclic.

   Answer:

   H is normal because every cyclic group is abelian, and any subgroup of an abelian group is normal. We claim $G/H = \langle aH \rangle$. Let $bH \in G/H$. Then $b \in G$, so $b = a^n$ for some n. Thus $bH = a^n H = (aH)^n$.