Math 30710
Practice Final Exam 2
December, 2017 **Name** ______________________________

This is a 2-hour exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are. **Show all work!** If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something is not clear, ASK ME!! Good luck!

1. Let φ be the Euler phi-function, namely $\varphi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n.

   (a) (5 points) Compute $\varphi(18)$ and put your answer in the provided space.

$\varphi(18) =$

   (b) (10 points) If p is a prime, compute $\varphi(p^2)$ and carefully explain your answer. [Note that this is asking for $\varphi(p^2)$, not $\varphi(p)$. A correct answer with no explanation will not get full credit.]

   (c) (5 points) State Euler's theorem (the generalization of Fermat's Little Theorem). Be sure to include all the hypotheses.

   (d) (10 points) Find the remainder of $7^{123,322}$ when divided by 11 and put your answer in the provided space.

Answer:

2. Consider the symmetric group S_6 and its subgroup, the alternating group A_6. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix}.$$

(a) (5 points) Give the orders of S_6 and of A_6 and put your answers in the provided space.

$\lvert S_6 \rvert =$	$\lvert A_6 \rvert =$

(b) (5 points) Write σ as a product of disjoint cycles.

(c) (5 points) Write σ as a product of transpositions.

(d) (5 points) Is $\sigma \in A_6$? Explain why or why not.

(e) (10 points) Find the order of σ and put your answer in the provided space.

Answer:

3. YOU CAN DISREGARD THIS PROBLEM! Consider the polynomial $f(x) = 2x^2 + x + 1$ in $\mathbb{Z}_7[x]$.
   (a) (10 points) How do you know that $f(x)$ is **not** irreducible over $\mathbb{Z}_7$ before you even try to factor it?

   (b) (10 points) Factor $f(x)$. (Remember that the field is $\mathbb{Z}_7$. Your work in (a) should help.)

4. (10 points) We know that a factor group of a cyclic group is cyclic. Is it also true that a factor group of a *non-cyclic* group is non-cyclic? If it's true, give a proof. If it's not true, give a counterexample.

5. Let F be the additive group of all continuous functions mapping $\mathbb{R}$ to $\mathbb{R}$. Let $\mathbb{R}$ be the additive group of real numbers. Let $\phi : F \to \mathbb{R}$ be given by

$$\phi(f) = \int_{-1}^{1} f(x)dx.$$

(a) (10 points) Prove that ϕ is a group homomorphism.

(b) (5 points) Give an example of a non-zero element in $\ker \phi$.

(c) (5 points) To what familiar group is $F/\ker \phi$ isomorphic? Justify your answer.

6. (10 points) Let G be a group of finite order n and let $g \in G$. Prove that $g^n = e$, where e is the identity element of G. [Hint: use Lagrange's theorem.]

7. Let $\phi : \mathbb{R}[x] \to \mathbb{R}$ be defined by $\phi(f) = f(3)$.

   (a) (10 points) Prove that ϕ is a **ring** homomorphism. (You can take for granted that $\mathbb{R}[x]$ and $\mathbb{R}$ are rings.)

   (b) (10 points) Give a geometric interpretation of $\ker\phi$ in terms of the graphs of the elements of $\mathbb{R}[x]$ (i.e. how can you tell from the graph of a polynomial $y = f(x)$ that $f \in \ker\phi$?).

8. (10 points) Let G be a group and let H be a subgroup of G (not necessarily normal). Let a, b be elements of g such that $aH = bH$. Prove that $Ha^{-1} = Hb^{-1}$.