Math 40510
Final Exam
May 9, 2018 **Name** ______________________________

This is a 2-hour exam. Books and notes are not allowed. Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, there is a blank sheet at the back.

The Honor Code is in effect for this examination, including keeping your own exam under cover. Good luck!!

1. Show how the Ascending Chain Condition (ACC) for ideals implies the Descending Chain Condition (DCC) for varieties. (I'm only asking for this one implication, not the reverse implication. You can use the inclusion-reversing property without proof.)

2. Zariski closures.

   (a) According to google, there are approximately 10^{80} atoms in the universe. Let S be a set of 10^{100} points evenly spaced between 0 and 1 on the x-axis in the affine space $\mathbb{R}^1$. What is the Zariski closure of S? Briefly explain your answer.

   (b) Let $S = \{(t,t) \mid t \in \mathbb{Q} \text{ and } 0 \leq t \leq 1\} \subset \mathbb{R}^2$. Find the Zariski closure of S in $\mathbb{R}^2$. Briefly explain your answer.

3. Fill in both boxes. Let k be a field. Let $V \subset k^n$ be an affine variety and let $\mathbb{I}(V) \subset R = k[x_1, \ldots, x_n]$ be its ideal. Then

V is irreducible if and only if $\mathbb{I}(V)$ is [ ]

if and only if $k[V] = R/\mathbb{I}(V)$ is [ ] .

4. Weak Nullstellensatz.

   (a) State the Affine Weak Nullstellensatz.

   (b) Give an example to show that the affine Weak Nullstellensatz is false if you don't assume that k is algebraically closed. (No proof needed.)

   (c) State the Projective Weak Nullstellensatz. (Just give two of the equivalent conditions.)

5. Let $V = \mathbb{V}(f_1, \ldots, f_s) \subset k^n$ be an affine variety. Prove that $\mathbb{V}(\mathbb{I}(V)) = V$.

6. If I and J are homogeneous ideals in $k[x_0, \ldots, x_n]$, prove that the following are also homogeneous ideals. (You don't have to prove that they are ideals; just that they are homogeneous.) [Hint: there are two equivalent conditions for an ideal to be homogeneous.]

   (a) $I \cap J$

   (b) IJ

7. Let k be any field. Let $R = k[x_1, \ldots, x_n]$. This problem does not assume that any ideal is homogeneous.

   (a) Prove that any prime ideal in R is radical.

   (b) Give an example of a radical ideal in R that is not prime. (No proof needed.)

   (c) If $I \subset k[x_1, \ldots, x_n]$ is a prime ideal and k is algebraically closed, prove that $\mathbb{I}(\mathbb{V}(I)) = I$.

8. If $I \subset k[x_1, \ldots, x_n]$ is any ideal, prove that $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$ in k^n.

9. Let $k = \mathbb{R}$ (which of course is *not* algebraically closed). Give an example of an ideal $I \subset \mathbb{R}[x, y]$ such that
   - $\mathbb{I}(\mathbb{V}(I))$ is not equal to $\sqrt{I}$, *and*
   - $\mathbb{V}(I) \neq \emptyset$.

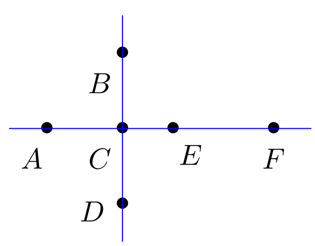
   [Hint: for example $\mathbb{V}(I)$ could be a single point.]

$I =$

$\mathbb{V}(I) =$

$\mathbb{I}(\mathbb{V}(I)) =$

10. This problem is the reverse of the one in Problem Set 3. The following is a picture of a set of points in $(\mathbb{P}^2)^\vee$. Sketch the configuration of lines in $\mathbb{P}^2$ *dual* to these points, and label them ℓ_A through ℓ_F. The important thing is to indicate wherever three or more lines meet in a point. (You don't necessarily have to show were only two lines meet.)



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