Math 40510
Final Exam
May 15, 2021
4:15 - 6:15 PM

Name ______________________________

Make sure that your work is legible, and make sure that it is clearly marked where your answers are.

Show all work!

If a problem calls for a proof or explanation, you will not get full credit for a correct answer if you don't supply the proof or explanation. If you have some ideas for solving a problem but can't figure out how to finish it, be sure to show me what you do know!! If something isn't clear, ask me! If you need more space, feel free to use the blank page at the end.

The Honor Code is in effect for this examination. Please sign the honor pledge to signify that you have abided by the rules outlined here. Thanks. Good luck!!

Honor Pledge: ______________________________

1. (a) Recall that a ring is Noetherian if every ideal is finitely generated. We gave some equivalent conditions for a ring to be Noetherian. Name one. (By "name one" I mean explain what it means, not just give the name.)

   (b) Now assume that k is a field, and let $R = k[x_1, \ldots, x_n]$. Recall the following two facts:

   - a set V in k^n is an affine variety (by definition) if there exist $f_1, \ldots, f_s \in R$ such that
   $$V = \{P \in k^n \mid f_i(P) = 0 \text{ for } 1 \leq i \leq s\};$$
   - R is a Noetherian ring.

   You don't have to prove either of the above two bullets. Now let $I \subset R$ be an ideal and let
   $$\mathbb{V}(I) = \{P \in k^n \mid f(P) = 0 \text{ for all } f \in I\}.$$
   Prove that $\mathbb{V}(I)$ is an affine variety in the sense of the first bullet. You'll have to prove two inclusions, and you'll have to explain how you are using the fact that R is Noetherian.

2. (a) Give an example to show why the Affine Weak Nullstellensatz is not true without the assumption that the field is algebraically closed. Explain.

   (b) Assume k is algebraically closed. Give an example to show that the statement of the Affine Weak Nullstellensatz is not true if we just replace k^n by $\mathbb{P}^n_k$ in the statement. Explain.

3. (a) What is a prime ideal? (You don't have to explain "ideal," just "prime.")

   (b) What is a maximal ideal? (You don't have to explain "ideal," just "maximal.")

   (c) One of the above two ("prime ideal" and "maximal ideal") implies the other. Which way does the implication go? (No proof required.)

   (d) Give an example to show that the reverse implication in part (3c) is not true. (Be sure to specify which ring you're using.) No proof required.

   (e) When k is algebraically closed, what are the maximal ideals in $k[x_1, \ldots, x_n]$ (affine situation)? (No proof required – just quote the theorem.)

4. This problem concerns the notion of a homogeneous polynomial vanishing at a point of projective space. Let $k = \mathbb{R}$.

   (a) If $f = x^2 + y - z$ and $P = [1, 2, 3] \in \mathbb{P}^2_{\mathbb{R}}$, is the statement "$f(P) = 0$" well-defined? Explain.

   (b) Give an example of a polynomial $f \in \mathbb{R}[x, y, z]$ **that is not homogeneous**, but for which vanishing at the point $[1, 2, 3] \in \mathbb{P}^2$ **is** well-defined. Explain your answer and how you got it.

5. Let $R = k[x_1, \ldots, x_n]$, where k is a field, and let k^n be the corresponding affine space.

   (a) Let $V \subset k^n$ be any non-empty set. Show that $V \subseteq \mathbb{V}(\mathbb{I}(V))$.

   (b) Let $V \subset k^n$ be a non-empty affine variety. Show that $V \supseteq \mathbb{V}(\mathbb{I}(V))$. Be sure to explain where you use the assumption that V is a variety.

6. Let $I = \langle x^4 + 2x^2y^2 + y^4, x - 1\rangle \subset k[x, y]$. We will look at specific choices for k in this problem.

   (a) If $k = \mathbb{R}$, find $\mathbb{V}(I)$ in $\mathbb{R}^2$ and find $\mathbb{I}(\mathbb{V}(I))$.

$\mathbb{V}(I) =$
$\mathbb{I}(\mathbb{V}(I)) =$

   (b) If $k = \mathbb{C}$, find $\mathbb{V}(I)$ in $\mathbb{C}^2$ and find $\mathbb{I}(\mathbb{V}(I))$. (Note that one of the generators can be factored.) For convenience, feel free to write your answer for $\mathbb{I}(\mathbb{V}(I))$ as an intersection of ideals.

$\mathbb{V}(I) =$
$\mathbb{I}(\mathbb{V}(I)) =$

7. (a) Let
$$V = \{(t, t^2) \mid t \in \mathbb{R}\} \subset \mathbb{R}^2.$$
Prove that V is a parabola and find its equation. (You'll probably have to prove both inclusions, but your argument should be very short.)

(b) Let
$$X = \{(t, t^2) \mid t \in \mathbb{Z}^+\} \subset \mathbb{R}^2.$$
Note that $\mathbb{Z}^+$ represents the positive integers. Prove that the Zariski closure of X is the variety V of part (7a).

(c) Find the Zariski closure in $\mathbb{P}^2_{\mathbb{R}}$ of the affine variety in part (7a). Give the equation(s), but no proof required.

(d) Problem (7c) asks you to find a certain variety in $\mathbb{P}^2_{\mathbb{R}}$. How many points does this variety have on the line at infinity? Find all such points. Explain your answer.

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