Math 40510, Algebraic Geometry

Problem Set 2, due March 20, 2020

1. Throughout this problem, we will let $R = \mathbb{R}[x, y, z]$ and let C be the twisted cubic curve in $\mathbb{R}^3$:
$$C = \mathbb{V}(y - x^2, z - x^3) = \{(t, t^2, t^3) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}.$$
Using the parametrization should be helpful in this problem, as will thinking geometrically.

a) Let ℓ be a linear polynomial: $\ell = ax + by + cz + d$ where $a, b, c, d \in \mathbb{R}$. Recall that as long as ℓ is not a constant, i.e. as long as at least one of a, b, c is non-zero, $\mathbb{V}(\ell)$ is a plane in $\mathbb{R}^3$, and $\mathbb{V}(y - x^2, z - x^3, \ell)$ represents the intersection of C with this plane (you don't have to prove either of these facts).

In this problem be sure to explain each answer. Find specific values of a, b, c, d (with at least one of a, b, c non-zero) so that

(i) (3 points) $\mathbb{V}(y - x^2, z - x^3, \ell)$ is empty (**remember at least one of a, b, c has to be non-zero**);

Solution:
Take $\ell = y + 1$. Then
$$C \cap \mathbb{V}(\ell) = \{(t, t^2, t^3) \in \mathbb{R}^3 \mid t^2 + 1 = 0\} = \emptyset.$$

(ii) (3 points) $\mathbb{V}(y - x^2, z - x^3, \ell)$ consists of one point;

Solution:
Take $\ell = x$. Then
$$C \cap \mathbb{V}(\ell) = \{(t, t^2, t^3) \in \mathbb{R}^3 \mid t = 0\} = \{(0, 0, 0)\}.$$

(iii) (3 points) $\mathbb{V}(y - x^2, z - x^3, \ell)$ consists of two distinct points;

Solution:
Take $\ell = y - 1$. Then
$$C \cap \mathbb{V}(\ell) = \{(t, t^2, t^3) \in \mathbb{R}^3 \mid t^2 - 1 = 0\} = \{(1, 1, 1), (-1, 1, -1)\}.$$

(iv) (3 points) $\mathbb{V}(y - x^2, z - x^3, \ell)$ consists of three distinct points.

Solution:
Take $\ell = 2x - 3y + z$. Then
$$\begin{aligned} C \cap \mathbb{V}(\ell) &= \{(t, t^2, t^3) \in \mathbb{R}^3 \mid 2t - 3t^2 + t^3 = 0\} \\ &= \{(t, t^2, t^3) \in \mathbb{R}^3 \mid t(t-1)(t-2) = 0\} \\ &= \{(0, 0, 0), (1, 1, 1), (2, 4, 8)\}. \end{aligned}$$

b) (5 points) Prove that $\mathbb{V}(y - x^2, z - x^3, \ell)$ cannot consist of four or more distinct points.

Solution:
Let $\ell = ax + by + cz + d$, so
$$C \cap \mathbb{V}(\ell) = \{(t, t^2, t^3) \in \mathbb{R}^3 \mid at + bt^2 + ct^3 + d = 0\}.$$
Remember that a, b, c, d are just real numbers, not polynomials. Then $at + bt^2 + ct^3 + d$ is a polynomial of degree at most 3 in the variable t, no matter what a, b, c, d are, so it can have at most three roots. That is, there are at most three such points.

c) (5 points) Recall that in $\mathbb{R}^3$, if P and Q are two **distinct** points then there is a unique line, $\overline{PQ}$, joining P and Q. (You don't have to prove this fact.) If P, Q both happen to be points on C, $\overline{QP}$ is called a *secant line* of C.

Prove that a secant line to C can't meet C in a third point (i.e. it can't be a *trisecant* line).

Solution:
This follows from b). If three points of C lie on a line λ, let P be another point of C not on λ. Then λ and P span a plane, and this plane contains those four points of C, which we saw is impossible.

d) (5 points) Let the lines $\overline{AB}$ and $\overline{PQ}$ be **distinct** secant lines to the twisted cubic curve C (in particular, A, B, P, Q are points of C). Prove that if the lines $\overline{AB}$ and $\overline{PQ}$ meet in one point, then one of A, B has to be equal to one of P, Q. You can use standard facts from high school geometry without proof.

Solution:
If A, B, P, Q are not all distinct, we know that $A \neq B$ and $P \neq Q$ so one of A, B has to be equal to one of P, Q and we are done. So assume they are all distinct, and we'll get a contradiction.

Since the lines meet in some point E, then those lines span a plane, say Λ. By b), the plane Λ meets C in at most 3 points. We first claim that E has to lie on C. If E does not lie on C, and if A, B, P, Q are all distinct points as we've assumed, then Λ meets C in four points, which is impossible by b).

So E lies on C. We saw in part c) that C has no trisecant lines, so we have to have E is equal to one of A, B and one of P, Q, giving the conclusion.

2. (6 points) In $\mathbb{R}^2$, let $V = \mathbb{V}(y - x^2)$ (a parabola). Mimic the proof of the example in section 4 of chapter 1 of Cox-Little-O'Shea (page 33 in the 4th edition, pages 33–34 of the 3rd edition) to show that $\mathbb{I}(V) = \langle y - x^2 \rangle$.

Solution:
It's clear that $\mathbb{I}(V) \supseteq \langle y - x^2 \rangle$ so we just have to prove the reverse inclusion. Notice that we have a parametrization

$$V = \{(t, t^2) \mid t \in \mathbb{R}\}.$$

Claim: *Every polynomial $f \in \mathbb{R}[x, y]$ can be written as*

$$f = h(y - x^2) + r$$

where $h \in \mathbb{R}[x, y]$ and r is a polynomial in the variable x alone.

To prove this we follow CLO. First prove it for a monomial $x^\alpha y^\beta$.

$$\begin{aligned} x^\alpha y^\beta &= x^\alpha (x^2 + (y - x^2))^\beta \\ &= x^\alpha (x^{2\beta} + \text{ terms that involve } y - x^2) \\ &= h \cdot (y - x^2) + x^{\alpha + 2\beta} \end{aligned}$$

for some polynomial h. But an arbitrary polynomial is an $\mathbb{R}$-linear combination of monomials, so we are done with the claim.

Now let $f \in \mathbb{I}(V)$. We want to show that $f = (y - x^2) \cdot h$ for some $h \in \mathbb{R}[x, y]$. By the claim, we can write

$$f = h \cdot (y - x^2) + r$$

where $h \in \mathbb{R}[x, y]$ and r is a polynomial in the variable x alone. Of course f vanishes at every point of V by definition of $\mathbb{I}(V)$. Thanks to the parametrization above, we get

$$0 = f(t, t^2) = 0 + r(t)$$

for all $t \in \mathbb{R}$. Hence r is the zero polynomial so $f = h \cdot (y - x^2)$ and we are done.

3. (5 points) Let V and W be varieties in $\mathbb{C}^n$ such that $V \cap W = \emptyset$. Prove that there exist $f \in \mathbb{I}(V)$ and $g \in \mathbb{I}(W)$ such that $f + g = 1$.

   Solution:
   We know that $\mathbb{V}(\mathbb{I}(V)) = V$, $\mathbb{V}(\mathbb{I}(W)) = W$ and $\mathbb{V}(\mathbb{I}(V) + \mathbb{I}(W)) = V \cap W = \emptyset$. Since $\mathbb{C}$ is algebraically closed, the Weak Nullstellensatz says that $\mathbb{I}(V) + \mathbb{I}(W) = \langle 1 \rangle = R$. The result follows immediately

4. For this problem let k be a field, which we do **not** necessarily assume is algebraically closed. In part c) we will further assume that $k = \mathbb{R}$.

   a) (5 points) Let I and J be ideals in $k[x_1, \ldots, x_n]$. Suppose we happen to know that there exist $f \in I$ and $g \in J$ such that $f + g = 1$. Prove that

   $$\mathbb{V}(I) \cap \mathbb{V}(J) = \emptyset;$$

   Solution:
   We know that

   $$\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(I + J) = \mathbb{V}(\langle 1 \rangle) = \emptyset.$$

   b) (5 points) Let I and J be ideals in $k[x_1, \ldots, x_n]$. Suppose we happen to know that there exist $f \in I$ and $g \in J$ such that $f + g = 1$. Prove that

   $$IJ = I \cap J.$$

   Solution:
   We saw in class that it's always true that $IJ \subseteq I \cap J$ so we only have to prove the reverse inclusion. Let $h \in I \cap J$. Then we have

   $$f + g = 1 \Longrightarrow fh + gh = h.$$

   But since $h \in I \cap J$, h is in each of I and J. Then $fh \in IJ$ and $gh \in IJ$ so their sum is in IJ since IJ is an ideal, i.e. $h \in IJ$.

   c) (5 points) Give (with proof) an example of two varieties V, W in $\mathbb{R}^2$ such that $V \cap W = \emptyset$ but there is *no* $f \in \mathbb{I}(V)$ and $g \in \mathbb{I}(W)$ such that $f + g = 1$. [Hint: this wouldn't be true over $\mathbb{C}$ thanks to problem #3, so you should take advantage of some property of $\mathbb{R}$. One solution involves the result of problem 2, which you can use whether or not you were able to solve it.]

   Solution:
   Let $V = \mathbb{V}(y - x^2)$ (parabola) and let $W = \mathbb{V}(y + 1)$ (the horizontal line defined by $y = -1$). Then $V \cap W = \mathbb{V}(y - x^2, y + 1)$. To find the common zeros, if $y + 1 = 0$ we have $y = -1$, so then we need the vanishing of $-1 - x^2 = -(1 + x^2)$, which over $\mathbb{R}$ is empty.

   We have $\mathbb{I}(V) = \langle y - x^2 \rangle$ from problem 2. Now claim that $\mathbb{I}(W) = \langle y + 1 \rangle$. As before, $\supseteq$ is clear. If $f \in \mathbb{I}(W)$, write

$$\begin{aligned} f &= g_m(x)y^m + g_{m-1}(x)y^{m-1} + \cdots + g_1(x)y + g_0(x) \\ &= g_m(x)((y+1)-1)^m + g_{m-1}(x)((y+1)-1)^{m-1} + \cdots + g_1(x)((y+1)-1) + g_0(x) \end{aligned}$$

In this way, as before, we can write f as

$$f = (y+1)h + r$$

where $h \in \mathbb{R}[x,y]$ and $r \in \mathbb{R}[x]$. Since $f(t,-1) = 0$ for all $t \in \mathbb{R}$ (because $f \in \mathbb{I}(W)$ and W is the horizontal line $y = -1$), again we get r is the zero polynomial so $\mathbb{I}(W) = \langle y+1 \rangle$.

Now we can verify that there is *no* $f \in \mathbb{I}(V)$ and $g \in \mathbb{I}(W)$ such that $f + g = 1$. Suppose there were (looking for a contradiction). This would mean that

$$1 = h_1 \cdot (y - x^2) + h_2 \cdot (y+1)$$

for some $h_1, h_2 \in \mathbb{R}[x,y]$. Write the first as $h_1(x,y)$ to stress that it is a polynomial in x and y. If this equation is true as polynmomials, it remains true if we set $y = -1$. Thus

$$1 = -h_1(x,-1)(1+x^2).$$

But the degree of the product of two polynomials in $\mathbb{R}[x]$ is the sum of the degrees of the individual polynomials (look at the leading terms), so this is impossible since the polynomial on the left is a constant (degree $= 0$) and the one on the right has degree at least 2.

5. Fun with colon ideals. In this problem, all ideals are in $R = k[x_1, \ldots, x_n]$ where k is some field. Prove the following facts. (All of these are relatively short proofs, no more than 10 lines.)

   a) (5 points) If $J \subseteq K$ then $I : J \supseteq I : K$.

   Solution:
   Let $f \in I : K$. We want to show that $f \in I : J$. Let $g \in J$. We want to show that $fg \in I$. Since $J \subseteq K$, we also have $g \in K$, so since $f \in I : K$ we get $fg \in I$ as desired.

   b) (5 points) If I is radical then $I : J$ is also radical.

   Solution:
   Let $f \in R$ be a polynomial such that $f^m \in I : J$ for some integer $m \geq 1$. We want to show that $f \in I : J$. That is, we want to show that $fg \in I$ for each $g \in J$.

   So let $g \in J$. Since $f^m \in I : J$, we know $f^m g \in I$. Hence we also have $f^m g^m = (fg)^m \in I$. But I is a radical ideal, so $fg \in I$, which was what we wanted to show.

   c) (5 points) $J \subseteq \sqrt{I}$ if and only if $I : J^\infty = k[x_1, \ldots, x_n]$.

   Solution:
   For any ideal K, the statement that $K = k[x_1, \ldots, x_n]$ is equivalent to the statement that $1 \in K$. So apply this to the ideal $K = I : J^\infty$.

$$\begin{aligned} I : J^\infty = k[x_1, \ldots, x_n] &\iff 1 \in I : J^\infty \\ &\iff \text{for each } g \in J,\ 1 \cdot g^m \in I \text{ for some } m \geq 1 \\ &\iff \text{for each } g \in J,\ g^m \in I \text{ for some } m \geq 1 \\ &\iff \text{for each } g \in J,\ g \in \sqrt{I} \\ &\iff J \subseteq \sqrt{I}. \end{aligned}$$

   d) (5 points) Let I$\subseteq k[x_1, \ldots, x_n]$ be any ideal. Let $J = I^2$. Find

$$I : J^\infty$$

   and explain your answer. [Hint: look at the other parts of this problem. There is a one-line answer.]

   Solution:

We have $J = I^2 \subseteq I \subseteq \sqrt{I}$ so by the previous problem, $I : J^\infty = k[x_1, \ldots, c_n]$.

e) (5 points) $(I \cap J) : K = (I : K) \cap (J : K)$.

Solution:
We'll prove both inclusions at the same time.

$$\begin{aligned} f \in (I \cap J) : K &\iff \text{ for each } g \in K,\ fg \in I \cap J \\ &\iff \text{ for each } g \in K,\ fg \in I \text{ and } fg \in J \\ &\iff f \in I : K \text{ and } f \in J : K \\ &\iff f \in (I : K) \cap (J : K). \end{aligned}$$

f) (5 points) $(I \cap J) : K^\infty = (I : K^\infty) \cap (J : K^\infty)$.

Solution:
You have to be a little careful with this one. It's not easy to prove both directions at once so we'll prove both inclusions separately.
$\subseteq$:
Let $f \in (I \cap J) : K^\infty$. Let $g \in K$. So we know that $fg^m \in I \cap J$ for some $m \geq 1$. Hence $fg^m \in I$ and $fg^m \in J$, so $f \in (I : K^\infty) \cap (J : K^\infty)$.

$\supseteq$:
Let $f \in (I : K^\infty) \cap (J : K^\infty)$. Let $g \in K$. So $fg^{\ell_1} \in I$ for some $\ell_1 \geq 1$ and $fg^{\ell_2} \in J$ for some $\ell_2 \geq 1$. Let $m = \max\{\ell_1, \ell_2\}$. Then $fg^m \in I$ and $fg^m \in J$, so $fg^m \in I \cap J$ and hence $f \in (I \cap J) : K^\infty$.

g) (5 points) $I : (J + K) = (I : J) \cap (I : K)$.

Solution:
$\subseteq$:
Let $f \in I : (J + K)$. This means that for each $g \in J + K$ we have $fg \in I$. Since $J \subseteq J + K$ and $K \subseteq J + K$, in particular if $g \in J$ then $fg \in I$, and if $g \in K$ then $fg \in I$. This means $f \in (I : J) \cap (I : K)$.

$\supseteq$:
Let $f \in (I : J) \cap (I : K)$. Let $g + h \in J + K$, where $g \in J$ and $h \in K$. Then

$$f(g + h) = fg + fh \in I$$

so $f \in I : (J + K)$.

6. In this problem, let $R = k[x, y, z, w]$, where k is a field. Let

$$I = \langle x, y \rangle^3 \cap \langle z, w \rangle^3$$

and let

$$J = \langle x, y \rangle^2.$$

You can use results from previous problems. (Recall $\langle x, y \rangle^2 = \langle x^2, xy, y^2 \rangle$.)

a) (6 points) Find $I : J$ and explain your answer.

Solution:
By problem 5 e),

(1) $$I : J = [\langle x, y \rangle^3 \cap \langle z, w \rangle^3] : \langle x, y \rangle^2 = [\langle x, y \rangle^3 : \langle x, y \rangle^2] \cap [\langle z, w \rangle^3 : \langle z, w \rangle^2]$$

<u>Claim 1</u>: $\langle x, y \rangle^3 : \langle x, y \rangle^2 = \langle x, y \rangle$.

We are claiming that

$$\langle x^2, x^2y, xy^2, y^3\rangle : \langle x^2, xy, y^2\rangle = \langle x, y\rangle.$$

$\supseteq$:
It's clear that x and y individually are in this ideal quotient, so the ideal that they generate is too.

$\subseteq$:
Let $f \in \langle x, y\rangle^3 : \langle x, y\rangle^2$. Write

$$f = a(x, y, z, w) \cdot x + b(x, y, z, w) \cdot y + c(z, w).$$

We want to show that $c(z, w)$ is the zero polynomial, so $f \in \langle x, y\rangle$.

Since we know that $x, y \in \langle x, y\rangle^3 : \langle x, y\rangle^2$ and $f \in \langle x, y\rangle^3 : \langle x, y\rangle^2$ by assumption, we get

$$f - (ax + by) = c(z, w) \in \langle x, y\rangle^3 : \langle x, y\rangle^2$$

by basic properties of an ideal. But it's clear that $c(z, w) \cdot x^2$ is not in $\langle x^3, x^2y, xy^2, y^3\rangle$, except when $c(z, w)$ is the zero polynomial, so we are done with Claim 1.

<u>Claim 2</u>: $\langle z, w\rangle^3 : \langle x, y\rangle^2 = \langle z, w\rangle^3$.

$\supseteq$:
As before, this inclusion is clear.

$\subseteq$:
Use part g) of problem #5 (generalized to the sum of three ideals, with the same argument). You can check that

$$\begin{array}{rcl} \langle z, y\rangle^3 : \langle x^2\rangle & = & \langle z, w\rangle^3 \\ \langle z, y\rangle^3 : \langle xy\rangle & = & \langle z, w\rangle^3 \\ \langle z, y\rangle^3 : \langle y^2\rangle & = & \langle z, w\rangle^3 \end{array}$$

so since the intersection of the three ideals on the right is clearly $\langle z, w\rangle^3$, this proves Claim 2.

From equation (1) and the two claims, we get

$$I : J = \langle x, y\rangle \cap \langle z, w\rangle^3.$$

b) (6 points) Assume that k is algebraically closed, and find $\mathbb{V}(I : J^\infty)$. [Hint: you'll find it much easier to use a theorem from class or from the book than to compute $I : J^\infty$ directly.]

Solution:
In class we proved that

$$\mathbb{V}(I : J^\infty) = \overline{\mathbb{V}(I) \backslash \mathbb{V}(J)}.$$

Since intersections of ideals correspond to unions of varieties, we have

$$\mathbb{V}(I) = \mathbb{V}(x, y) \cup \mathbb{V}(z, w) \quad \text{and} \quad \mathbb{V}(J) = \mathbb{V}(x, y).$$

Thus $\mathbb{V}(I : J^\infty) = \mathbb{V}(z, w)$.