# Math 40510, Algebraic Geometry

**Problem Set 1, due February 14, 2020**

**<u>Note</u>**: This is not the entire problem set. It is just the first set of problems I've assigned for Problem Set 1, last modified January 28, 2020.

1. Inside the affine space $\mathbb{R}^2$ we have the subset $\mathbb{Z}^2$ consisting of all the points $(a,b)$ where $a,b \in \mathbb{Z}$. Let $f \in \mathbb{R}[x,y]$ and assume that $f(a,b) = 0$ for all $(a,b) \in \mathbb{Z}^2$. Prove that $f$ must be the zero polynomial. (Note that we're assuming $f(a,b) = 0$ for all $(a,b) \in \mathbb{Z}^2$, not all $(a,b) \in \mathbb{R}^2$.)

2. Informally, we often "identify" the complex plane $\mathbb{C}$ with the plane $\mathbb{R}^2$. In algebraic geometry we have to be a bit more careful.

   Consider the affine spaces, $\mathbb{C}^1$ and $\mathbb{R}^2$. Consider the unit circle $C$ inside both of them. In $\mathbb{C}$ this is defined by $|z| = 1$, and in $\mathbb{R}^2$ it's defined by $x^2 + y^2 = 1$. Prove that as a subset of $\mathbb{C}^1$, $C$ is *not* a variety, while in $\mathbb{R}^2$ it *is* a variety. (Note that for the former, it is not enough to just say that it's not a variety because $|z|$ is not a polynomial.)

3. An ideal $I$ is said to be *radical* if the condition $f^m \in I$ necessarily implies $f \in I$.

   a) Prove that a prime ideal is always radical.

   b) Is the converse true (i.e. is a radical ideal always prime?)? If so, prove it. If not, give a counterexample and justify your claim that it's a counterexample.

   c) Prove that if $S$ is a set in an affine space $k^n$ then $\mathbb{I}(S)$ is a radical ideal.

4. Define a function $\phi : \mathbb{C}^2 \to \mathbb{C}^3$ by $\phi(a,b) = (a^2, ab, b^2)$, where $a,b \in \mathbb{C}$.

   a) Is $\phi$ surjective? Give a proof or disprove with a counterexample.

   b) Is $\phi$ injective? Give a proof or disprove with a counterexample.

   c) Find a point of $\mathbb{C}^3$ that has exactly one preimage point, and justify your answer.

   d) Let $R = \mathbb{C}[z_1, z_2, z_3]$. Prove that the image of $\phi$ is an affine variety in $\mathbb{C}^3$ by giving the defining polynomial(s) in $R$. Be sure to prove that the variety they define is *exactly* the image of $\phi$.

   e) Let $S = \mathbb{C}[x,y]$. Let $V = \mathbb{V}(x-y)$ inside $\mathbb{C}^2$. Prove that $\phi(V)$ is a subvariety of $\mathbb{C}^3$ by giving its defining polynomial(s). [It's not enough to just give the polynomial(s). You have to justify your claim that the polynomial(s) really do define $\phi(V)$.]

5. a) Explain why Problem #1 shows that $\mathbb{Z}^2$ is not an affine variety in $\mathbb{R}^2$. (This should only take a few lines.)

   b) If $X$ is a set in $\mathbb{R}^2$ (it may or may not be an affine variety), and $W$ is an affine variety that contains $X$, we'll say that "*$W$ is the smallest variety containing $X$*" if there is no <u>variety</u> $V$ such that
   $$X \subseteq V \subsetneq W.$$
   Referring again to Problem #1, what is the smallest affine variety in $\mathbb{R}^2$ that contains $\mathbb{Z}^2$? Carefully explain your answer.

   c) Recall that the twisted cubic curve in $\mathbb{R}^3$ is
   $$C = \{(t, t^2, t^3) \mid t \in \mathbb{R}\} = \mathbb{V}(y - x^2, z - x^3).$$
   Let
   $$X = \{(t, t^2, t^3) \mid t \in \mathbb{Z}\}.$$
   Find the smallest affine variety in $\mathbb{R}^3$ that contains $X$. Carefully explain your answer.

d) Let
$$X = \{(t, t^2, t^3) \mid 1 \le t \le 10 \ \ (t \in \mathbb{Z})\}.$$
Find the smallest affine variety in $\mathbb{R}^3$ that contains $X$. Carefully explain your answer.

**Remark.** The smallest variety containing a set $X$ is usually called the *Zariski closure* of $X$. We'll see why when we talk about the Zariski topology for affine space.

6. Assume that $\langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_t \rangle$ for some polynomials $f_1, \dots, f_s, g_1, \dots, g_t$.

   Prove that $\mathbb{V}(f_1, \dots, f_s) = \mathbb{V}(g_1, \dots, g_t)$. [Hint: in class we talked about what it means to say that $\langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_t \rangle$. You can use that.]

7. (Continuing problem #3.)

   a) Let $I$ be an ideal in $R = k[x_1, \dots, x_n]$. (We are *not* assuming that $I$ is a radical ideal in the language of problem #3.) Define
   $$\sqrt{I} = \{f \in R \mid f^r \in I \text{ for } \textit{some} \text{ integer } r \ge 0\}.$$
   Prove that $\sqrt{I}$ is again an ideal. (Note that if $f, g \in \sqrt{I}$ it means that $f^r \in I$ and $g^s \in I$ but not necessarily that $r = s$.)

   [If you weren't able to get part a), you can assume the statement in order to do parts b) and c).]

   b) Prove that $\sqrt{I}$ is a radical ideal in the sense of problem #3.

   c) Let $I = \langle x^2, y^3 \rangle \subset \mathbb{R}[x, y]$. Show that $\sqrt{I} = \langle x, y \rangle$. Be sure to prove your answer.