Math 40510, Algebraic Geometry

Problem Set 2, due March 20, 2020

Note: At this point this is the entire problem set.

1. Throughout this problem, we will let $R = \mathbb{R}[x, y, z]$ and let C be the twisted cubic curve in $\mathbb{R}^3$:
$$C = \mathbb{V}(y - x^2, z - x^3) = \{(t, t^2, t^3) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}.$$
Using the parametrization should be helpful in this problem, as will thinking geometrically.

   a) Let ℓ be a linear polynomial: $\ell = ax + by + cz + d$ where $a, b, c, d \in \mathbb{R}$. Recall that as long as ℓ is not a constant, i.e. as long as at least one of a, b, c is non-zero, $\mathbb{V}(\ell)$ is a plane in $\mathbb{R}^3$, and $\mathbb{V}(y - x^2, z - x^3, \ell)$ represents the intersection of C with this plane (you don't have to prove either of these facts).

   In this problem be sure to explain each answer. Find specific values of a, b, c, d (with at least one of a, b, c non-zero) so that

      (i) $\mathbb{V}(y - x^2, z - x^3, \ell)$ is empty (**remember at least one of** a, b, c **has to be non-zero**);
      (ii) $\mathbb{V}(y - x^2, z - x^3, \ell)$ consists of one point;
      (iii) $\mathbb{V}(y - x^2, z - x^3, \ell)$ consists of two distinct points;
      (iv) $\mathbb{V}(y - x^2, z - x^3, \ell)$ consists of three distinct points.

   b) Prove that $\mathbb{V}(y - x^2, z - x^3, \ell)$ cannot consist of four or more distinct points.

   c) Recall that in $\mathbb{R}^3$, if P and Q are two **distinct** points then there is a unique line, $\overline{PQ}$, joining P and Q. (You don't have to prove this fact.) If P, Q both happen to be points on C, $\overline{QP}$ is called a *secant line* of C.

   Prove that a secant line to C can't meet C in a third point (i.e. it can't be a *trisecant* line).

   d) Let the lines $\overline{AB}$ and $\overline{PQ}$ be **distinct** secant lines to the twisted cubic curve C (in particular, A, B, P, Q are points of C). Prove that if the lines $\overline{AB}$ and $\overline{PQ}$ meet in one point, then one of A, B has to be equal to one of P, Q. You can use standard facts from high school geometry without proof.

2. In $\mathbb{R}^2$, let $V = \mathbb{V}(y - x^2)$ (a parabola). Mimic the proof of the example in section 4 of chapter 1 of Cox-Little-O'Shea (page 33 in the 4th edition, pages 33–34 of the 3rd edition) to show that $\mathbb{I}(V) = \langle y - x^2 \rangle$.

3. Let V and W be varieties in $\mathbb{C}^n$ such that $V \cap W = \emptyset$. Prove that there exist $f \in \mathbb{I}(V)$ and $g \in \mathbb{I}(W)$ such that $f + g = 1$.

4. For this problem let k be a field, which we do **not** necessarily assume is algebraically closed. In part c) we will further assume that $k = \mathbb{R}$.

   a) Let I and J be ideals in $k[x_1, \ldots, x_n]$. Suppose we happen to know that there exist $f \in I$ and $g \in J$ such that $f + g = 1$. Prove that
$$\mathbb{V}(I) \cap \mathbb{V}(J) = \emptyset;$$

   b) Let I and J be ideals in $k[x_1, \ldots, x_n]$. Suppose we happen to know that there exist $f \in I$ and $g \in J$ such that $f + g = 1$. Prove that
$$IJ = I \cap J.$$

c) Give (with proof) an example of two varieties V, W in $\mathbb{R}^2$ such that $V \cap W = \emptyset$ but there is *no* $f \in \mathbb{I}(V)$ and $g \in \mathbb{I}(W)$ such that $f + g = 1$. [Hint: this wouldn't be true over $\mathbb{C}$, so you should take advantage of some property of $\mathbb{R}$. One solution involves the result of problem 2, which you can use whether or not you were able to solve it.]

5. Fun with colon ideals. In this problem, all ideals are in $R = k[x_1, \dots, x_n]$ where k is some field. Prove the following facts.

   a) If $J \subseteq K$ then $I : J \supseteq I : K$.

   b) If I is radical then $I : J$ is also radical.

   c) $J \subseteq \sqrt{I}$ if and only if $I : J^\infty = k[x_1, \dots, x_n]$.

   d) Let $I \subseteq k[x_1, \dots, x_n]$ be any ideal. Let $J = I^2$. Find
   $$I : J^\infty$$
   and explain your answer. [Hint: look at the other parts of this problem.]

   e) $(I \cap J) : K = (I : K) \cap (J : K)$.

   f) $(I \cap J) : K^\infty = (I : K^\infty) \cap (J : K^\infty)$.

   g) $I : (J + K) = (I : J) \cap (I : K)$.

6. In this problem, let $R = k[x, y, z, w]$, where k is a field. Let
   $$I = \langle x, y\rangle^3 \cap \langle z, w\rangle^3$$
   and let
   $$J = \langle x, y\rangle^2.$$
   You can use results from previous problems. (Recall $\langle x, y\rangle^2 = \langle x^2, xy, y^2\rangle$.)

   a) Find $I : J$ and explain your answer.

   b) Assume that k is algebraically closed, and find $\mathbb{V}(I : J^\infty)$. [Hint: you'll find it much easier to use a theorem from class or from the book than to compute $I : J^\infty$ directly.]