Math 40510, Algebraic Geometry

Problem Set 3, due April 24, 2020

Note: This problem set is in final form.

1. Extend Example 3 from March 30. Specifically:

   a) (5 points) In class we looked at the polynomial mapping
   $$\phi : \mathbb{R}^3 \to \mathbb{R}^2$$
   defined by $\phi = (f_1, f_2)$, where
   $$f_1(x_1, x_2, x_3) = x_1 \quad \text{and} \quad f_2(x_1, x_2, x_3) = x_2.$$
   If $(a, b) \in \mathbb{R}^2$ is any point, find the preimage of this point under ϕ. (This is also called a *fiber* of ϕ, denoted $\phi^{-1}(a, b)$.) In particular, describe this fiber geometrically.

   b) (5 points) What does your work in part 1a) show about injectivity of ϕ (i.e. is it or isn't it injective?) and what does it show about surjectivity of ϕ (is it or is it not surjective?)? Give a short explanation.

   c) (6 points) Now let
   $$V = \mathbb{V}(x_2 - x_1^2, x_3 - x_1^3) \subset \mathbb{R}^3$$
   (the twisted cubic), using the ring $R = \mathbb{R}[x_1, x_2, x_3]$ for $\mathbb{R}^3$. Let
   $$W = \mathbb{V}(y - x^2) \subset \mathbb{R}^2,$$
   using the ring $S = \mathbb{R}[x, y]$ for $\mathbb{R}^2$.

   Show that in fact $\phi(V) = W$, so we can also consider ϕ as a polynomial mapping $\phi : V \to W$.

   d) (6 points) For the mapping in part 1c), show that ϕ is 1-1 and onto.

   e) (6 points) Find a polynomial mapping $\psi : W \to V$ that is the inverse of ϕ. In particular, you'll have to find polynomials $g_1(x, y)$, $g_2(x, y)$ and $g_3(x, y)$ such that
   $$\psi = (g_1, g_2, g_3) : W \to V$$
   satisfies $\psi \circ \phi$ is the identity on V and $\phi \circ \psi$ is the identity on W.

2. Let $f = y^2 - (x^2 - 9)(16 - x^2)$ and let $V = \mathbb{V}(f) \subset \mathbb{R}^2$. Define polynomial functions
   $$\phi_1 : V \to \mathbb{R} \quad \text{where} \quad \phi_1(x, y) = x - y$$
   and
   $$\phi_2 : V \to \mathbb{R} \quad \text{where} \quad \phi_2(x, y) = x.$$

   Keep in mind: in this problem you'll be looking at these two polynomial functions. In each case you'll choose a point c in the target $\mathbb{R}$ and you'll be investigating how many points in V are in the fiber $\phi^{-1}(c)$, i.e. how many points map to c. Now the specific questions.

   a) (7 points) Start with the polynomial function ϕ_1. If $c \in \mathbb{R}$, show that a fiber $\phi_1^{-1}(c)$ cannot have more than four points of V. [Hint: express $\phi^{-1}(c)$ as a variety.]

   b) (7 points) Now and in the next part, we'll turn to the polynomial function ϕ_2. If $c \in \mathbb{R}$, find the maximum number of points in a fiber $\phi_2^{-1}(c)$. Explain your answer.

   c) (7 points) For each integer m such that $0 \le m \le N$ (where N is the number you got in part 2b), describe the set of points c in $\mathbb{R}$ for which the fiber $\phi^{-1}(c)$ has m points.

3. In this problem we will look at the variety $V = \mathbb{V}(x^2+y^2) \subset k^2$, where $x^2+y^2 \in k[x,y]$, for different fields k. Our question will whether V is irreducible or not. In each case, if it is irreducible, explain why; if it is not irreducible, write it in the form $V = V_1 \cup V_2$ as in the definition of irreducibility. (In the latter case make sure your decomposition satisfies the necessary properties.)

   You can use without proof the fact that if ℓ is linear (i.e. ℓ is of the form $\ell = ax+by+c$ for some $a, b, c \in k$) then $\mathbb{V}(\ell)$ is irreducible.

   a) (6 points) $k = \mathbb{C}$.

   b) (6 points) $k = \mathbb{R}$.

   c) (6 points) $k = \mathbb{Z}_2$ (the integers modulo 2).

   d) (6 points) $k = \mathbb{Z}_5$.

4. Let $R = \mathbb{R}[x,y,z]$ and let C be the curve in $\mathbb{P}^2_{\mathbb{R}}$ defined by $C = \mathbb{V}(x^2 - 2xz + y^2)$. Let

   $$U_0 = \{[a,b,c] \in \mathbb{P}^2 \mid a \neq 0\}, \quad U_1 = \{[a,b,c] \in \mathbb{P}^2 \mid b \neq 0\}, \quad U_2 = \{[a,b,c] \in \mathbb{P}^2 \mid c \neq 0\}$$

   in $\mathbb{P}^2_{\mathbb{R}}$. Recall that for $i = 0,1,2$ we can identify U_i with $\mathbb{R}^2$, and that the complement of U_i is "a projective line, $\mathbb{P}^1_{\mathbb{R}}$, at infinity."

   a) (7 points) For each of $i = 0,1,2$ find the equation(s) for the variety $C_i = C \cap U_i$ in $\mathbb{R}^2$. Be sure to use the variables y, z for U_0, the variables x, z for U_1 and the variables x, y for U_2.

   b) (7 points) In each of the three cases, find the specific point(s) where C meets the line at infinity. (Remember that to specify a point in $\mathbb{P}^2$ you need three coordinates, and remember that the three cases have different "lines at infinity." You might want to think about the equation, in each case, of the line at infinity.)

   c) (9 points) Now remember your high school math about conic sections. For each of the curves C_0, C_1 and C_2 from part 4a, **sketch the curve** and say which kinds of conic sections they are.

      Hints:

      - If you need a refresher about conic sections, look up "conic section" in wikipedia and notice the picture to the right of "Definition." In particular, you have circles, ellipses, parabolas and hyperbolas. Remember that a circle is a kind of ellipse.

      - Feel free to use some calculus if you want, to make your sketch. It is also not illegal to use a computer graphing program, but you can do this by hand too.

      - Pay attention to the equations of the asymptote lines, if any.

   d) (4 points) For this part, no proof is needed and I don't want any equations.

      The first three parts of this problem illustrate, via a specific example, a general fact that I want you to discover. This part does not involve a specific example, but it's motivated by the example in this problem. Your answer should reflect what's going on in the previous parts of this problem.

      Speculate about the connection between parts 4b and 4c. Specifically, suppose you have an irreducible conic (degree 2) curve C in $\mathbb{P}^2_{\mathbb{R}}$ (like the curve C we used above). Let ℓ be a line in $\mathbb{P}^2_{\mathbb{R}}$ that you want to consider as the line at infinity, and let $U = \mathbb{P}^2_{\mathbb{R}} \setminus \ell$, which you can identify with $\mathbb{R}^2$. What is it about the relation between C and ℓ that determines whether $C \cap U$ will be an ellipse (including circles as a special case), a parabola or a hyperbola? I'm just looking for the geometry of what you think is going on. No proof needed.