Math 40510, Algebraic Geometry

Problem Set 3, due April 24, 2020

Note: This problem set is not yet in final form.

1. Extend Example 3 from March 30. Specifically:

a) (5 points) In class we looked at the polynomial mapping

$$\phi : \mathbb{R}^3 \to \mathbb{R}^2$$

defined by $\phi = (f_1, f_2)$, where

$$f_1(x_1, x_2, x_3) = x_1 \quad \text{and} \quad f_2(x_1, x_2, x_3) = x_2.$$

If $(a, b) \in \mathbb{R}^2$ is any point, find the preimage of this point under ϕ. (This is also called a *fiber* of ϕ, denoted $\phi^{-1}(a, b)$.) In particular, describe this fiber geometrically.

Solution:
The set of points mapping to (a, b) is $\phi^{-1}(a, b) = \{(a, b, t) \mid t \in \mathbb{R}\}$. This is the vertical line through the point $(a, b, 0)$ in the (x, y)-plane.

b) (5 points) What does your work in part 1a) show about injectivity of ϕ (i.e. is it or isn't it injective?) and what does it show about surjectivity of ϕ (is it or is it not surjective?)? Give a short explanation.

Solution:
ϕ is not injective since infinitely many points map to (a, b) for any (a, b). On the other hand, since (a, b) can be chosen arbitrarily in $\mathbb{R}^2$, ϕ is surjective.

c) (6 points) Now let

$$V = \mathbb{V}(x_2 - x_1^2, x_3 - x_1^3) \subset \mathbb{R}^3$$

(the twisted cubic), using the ring $R = \mathbb{R}[x_1, x_2, x_3]$ for $\mathbb{R}^3$. Let

$$W = \mathbb{V}(y - x^2) \subset \mathbb{R}^2,$$

using the ring $S = \mathbb{R}[x, y]$ for $\mathbb{R}^2$.

Show that in fact $\phi(V) = W$, so we can also consider ϕ as a polynomial mapping $\phi : V \to W$.

Solution:
Since $V = \{(t, t^2, t^3) \mid t \in \mathbb{R}\} \subset \mathbb{R}^3$, we see that $\phi(V)$ is the set of points

$$\phi(V) = \{(t, t^2) \mid t \in \mathbb{R}\} \subset \mathbb{R}^2.$$

This is a parametrization for $W = \mathbb{V}(y - x^2)$. Thus we have

$$\begin{array}{cccc} \phi : & \{(t, t^2, t^3) \mid t \in \mathbb{R}\} & \longrightarrow & \{(t, t^2) \mid t \in \mathbb{R}\} \\ & || & & || \\ & V & & W \end{array} \tag{1}$$

d) (6 points) For the mapping in part 1c), show that ϕ is 1-1 and onto.

Solution:
Each point of W is of the form (a, a^2) for some $a \in \mathbb{R}$. Using (1), by looking at the first coordinate it is clear that (a, a^2) has one and only one preimage point, namely (a, a^2, a^3). Thus ϕ is surjective and injective.

e) (6 points) Find a polynomial mapping $\psi : W \to V$ that is the inverse of ϕ. In particular, you'll have to find polynomials $g_1(x,y)$, $g_2(x,y)$ and $g_3(x,y)$ such that

$$\psi = (g_1, g_2, g_3) : W \to V$$

satisfies $\psi \circ \phi$ is the identity on V and $\phi \circ \psi$ is the identity on W.

Solution:
Take $g_1(x,y) = x$, $g_2(x,y) = y$, and $g_3(x,y) = xy$. Then

$$(\psi \circ \phi)(a, a^2, a^3) = \psi(a, a^2) = (a, a^2, a \cdot a^2) = (a, a^2, a^3)$$

and

$$(\phi \circ \psi)(a, a^2) = \phi(a, a^2, a \cdot a^2) = (a, a^2).$$

2. Let $f = y^2 - (x^2 - 9)(16 - x^2)$ and let $V = \mathbb{V}(f) \subset \mathbb{R}^2$. Define polynomial functions

$$\phi_1 : V \to \mathbb{R} \quad \text{where} \quad \phi_1(x,y) = x - y$$

and

$$\phi_2 : V \to \mathbb{R} \quad \text{where} \quad \phi_2(x,y) = x.$$

Keep in mind: in this problem you'll be looking at these two polynomial functions. In each case you'll choose a point c in the target $\mathbb{R}$ and you'll be investigating how many points in V are in the fiber $\phi^{-1}(c)$, i.e. how many points map to c. Now the specific questions.

a) (7 points) Start with the polynomial function ϕ_1. If $c \in \mathbb{R}$, show that a fiber $\phi_1^{-1}(c)$ cannot have more than four points of V. [Hint: express $\phi^{-1}(c)$ as a variety.]

Solution:

$$\phi^{-1}(c) = \{(a,b) \in V \mid a - b = c\} = \mathbb{V}(y^2 - (x^2 - 9)(16 - x^2)\ ,\ x - y - c).$$

Since in particular we have $x - y - c = 0$, i.e. $y = x - c$, we get that we have to solve the equation

$$(x - c)^2 - (x^2 - 9)(16 - x^2) = 0.$$

This is a equation of degree 4, so there are at most 4 roots.

b) (7 points) Now and in the next part, we'll turn to the polynomial function ϕ_2. If $c \in \mathbb{R}$, find the maximum number of points in a fiber $\phi_2^{-1}(c)$. Explain your answer.

Solution:
Now we have

$$\phi^{-1}(c) = \{(a,b) \in V \mid a = c\} = \mathbb{V}(y^2 - (x^2 - 9)(16 - x^2), x - c).$$

In particular, since $x - c = 0$, we get that the number of points in the fiber is the number of solutions of the equation

$$y^2 - (c^2 - 9)(16 - c^2).$$

This is an equation of degree 2 in y (remember c is fixed), so it has at most two roots, hence the fiber has at most $N = 2$ points.

c) (7 points) For each integer m such that $0 \leq m \leq N$ (where N is the number you got in part 2b), describe the set of points c in $\mathbb{R}$ for which the fiber $\phi_2^{-1}(c)$ has m points.

Solution:
For a real number A, the equation $y^2 - A = 0$ (i.e. $y^2 = A$) has two solutions for $A > 0$, one solution for $A = 0$ and no solutions for $A < 0$. In our case we have $A = (c^2 - 9)(16 - c^2)$. So we want to know when $(c^2 - 9)(16 - c^2)$ is positive, when it's 0 and when it's negative.

- **Two solutions:** $(c^2-9)(16-c^2)>0$ means we want both factors to be positive or we want both factors to be negative.

  Both factors positive means $|c|>3$ and $|c|<4$, i.e. $3<|c|<4$. Both factors negative means $|c|<3$ and $|c|>4$; these can't both be true. So there are two roots if and only if $3<|c|<4$. That is, the number of elements of $\phi^{-1}(c)$ is 2 if and only if $3<|c|<4$. In other words, c is in the following union of open intervals: $(-4,-3)\cup(3,4)$.

- **One solution:** $(c^2-9)(16-c^2)=0$ means either $|c|=3$ or $|c|=4$. So there is exactly one element in $\phi^{-1}(c)$ if and only if $c\in\{-4,-3,3,4\}$.

- **No solutions:** $(c^2-9)(16-c^2)<0$ means we want one factor to be positive and one negative. This is true for all c not covered in the above two cases, i.e. for $|c|<3$ or $|c|>4$, i.e. for $c\in(-\infty,-4)\cup(-3,3)\cup(4,\infty)$.

3. In this problem we will look at the variety $V=\mathbb{V}(x^2+y^2)\subset k^2$, where $x^2+y^2\in k[x,y]$, for different fields k. Our question will whether V is irreducible or not. In each case, if it is irreducible, explain why; if it is not irreducible, write it in the form $V=V_1\cup V_2$ as in the definition of irreducibility. (In the latter case make sure your decomposition satisfies the necessary properties.)

   You can use without proof the fact that if ℓ is linear (i.e. ℓ is of the form $\ell=ax+by+c$ for some $a,b,c\in k$) then $\mathbb{V}(\ell)$ is irreducible.

   a) (6 points) $k=\mathbb{C}$.

   Solution:
   $x^2+y^2=(x+iy)(x-iy)$, so $\mathbb{V}(x^2+y^2)=\mathbb{V}(x+iy)\cup\mathbb{V}(x-iy)$, and $\mathbb{V}(x^2+y^2)$ is not equal to either of $\mathbb{V}(x+iy)$ or $\mathbb{V}(x-iy)$, so V is reducible.

   b) (6 points) $k=\mathbb{R}$.

   Solution
   $\mathbb{V}(x^2+y^2)$ is the single point $(0,0)$ in $\mathbb{R}^2$, so V is irreducible.

   c) (6 points) $k=\mathbb{Z}_2$ (the integers modulo 2).

   Solution:
   Parts c) and d) were a bit careless of me. I was thinking from the point of view of whether x^2+y^2 was irreducible. But of course k^2 is finite union of points in either case, so as long as $\mathbb{V}(x^2+y^2)$ consists of more than one point, it will be reducible. If you gave either answer, you did not receive any points off. Below I'll just give my original answers.

   Over $\mathbb{Z}_2$, $(x^2+y^2)=(x+y)^2$ (since $2=0$), so

   $$\mathbb{V}(x^2+y^2)=\mathbb{V}((x+y)^2)=\mathbb{V}(x+y)$$

   is a line (hence irreducuble).

   d) (6 points) $k=\mathbb{Z}_5$.

   Solution:
   Over $\mathbb{Z}_5$, $(x^2+y^2)=(x+2y)(x+3y)$ (since $2\cdot 3=1$ and $2+3=0$). So

   $$\mathbb{V}(x^2+y^2)=\mathbb{V}(x+2y)\cup\mathbb{V}(x+3y)$$

   and again V is reducible.

4. Let $R = \mathbb{R}[x, y, z]$ and let C be the curve in $\mathbb{P}^2_{\mathbb{R}}$ defined by $C = \mathbb{V}(x^2 - 2xz + y^2)$. Let

$$U_0 = \{[a, b, c] \in \mathbb{P}^2 \mid a \neq 0\}, \quad U_1 = \{[a, b, c] \in \mathbb{P}^2 \mid b \neq 0\}, \quad U_2 = \{[a, b, c] \in \mathbb{P}^2 \mid c \neq 0\}$$

in $\mathbb{P}^2_{\mathbb{R}}$. Recall that for $i = 0, 1, 2$ we can identify U_i with $\mathbb{R}^2$, and that the complement of U_i is "a projective line, $\mathbb{P}^1_{\mathbb{R}}$, at infinity."

a) (7 points) For each of $i = 0, 1, 2$ find the equation(s) for the variety $C_i = C \cap U_i$ in $\mathbb{R}^2$. Be sure to use the variables y, z for U_0, the variables x, z for U_1 and the variables x, y for U_2.

Solution:
U_0 is the set of points where $x \neq 0$, so the complement is $\mathbb{V}(x)$ (i.e. the set where $x = 0$).
U_1 is the set of points where $y \neq 0$, so the complement is $\mathbb{V}(y)$ (i.e. the set where $y = 0$).
U_2 is the set of points where $z \neq 0$, so the complement is $\mathbb{V}(z)$ (i.e. the set where $z = 0$).

Since a projective point doesn't change when the coordinates are multiplied by a non-zero element of k, we might as well set $x = 1$ for U_0, $y = 1$ for U_1 and $z = 1$ for U_2. Thus:

- The equation for C_0 is $1 - 2z + y^2$, i.e. $z = \frac{1}{2}y^2 + \frac{1}{2}$.
- The equation for C_1 is $x^2 - 2xz + 1 = 0$.
- The equation for C_2 is $x^2 - 2x + y^2 = 0$, i.e. $x^2 - 2x + 1 + y^2 = 1$, i.e. $(x - 1)^2 + y^2 = 1$.

b) (7 points) In each of the three cases, find the specific point(s) where C meets the line at infinity. (Remember that to specify a point in $\mathbb{P}^2$ you need three coordinates, and remember that the three cases have different "lines at infinity." You might want to think about the equation, in each case, of the line at infinity.)

Solution:

- In the first case, the line at infinity is $\mathbb{V}(x)$ so we have
$$\mathbb{V}(x^2 - 2xz + y^2) \cap \mathbb{V}(x) = \mathbb{V}(x^2 - 2xz + y^2, x) = \mathbb{V}(y^2, x) = \mathbb{V}(x, y),$$
a single point, namely $[0, 0, 1]$.

- In the second case, the line at infinity is $\mathbb{V}(y)$ so we have
$$\mathbb{V}(x^2 - 2xz + y^2) \cap \mathbb{V}(y) = \mathbb{V}(x^2 - 2xz + y^2, y) = \mathbb{V}(x^2 - 2xz, y) = \mathbb{V}(x(x - 2z), y).$$
So there are two points of intersection: when $x = y = 0$ we have the point $[0, 0, 1]$, and when $y = 0$ and $x - 2z = 0$ we have $[2, 0, 1]$.

- In the third case, the line at infinity is $\mathbb{V}(z)$ so we have
$$\mathbb{V}(x^2 - 2xz + y^2) \cap \mathbb{V}(z) = \mathbb{V}(x^2 - 2xz + y^2, z) = \mathbb{V}(x^2 + y^2).$$
But $x^2 + y^2 = 0$ over $\mathbb{R}$ if and only if both x and y are 0. But this would give $x = y = z = 0$, and $[0, 0, 0]$ is not a point in $\mathbb{P}^2$. Hence there is no point of intersection with the line at infinity in this case.

c) (9 points) Now remember your high school math about conic sections. For each of the curves C_0, C_1 and C_2 from part 4a, **sketch the curve** and say which kinds of conic sections they are.

Hints:

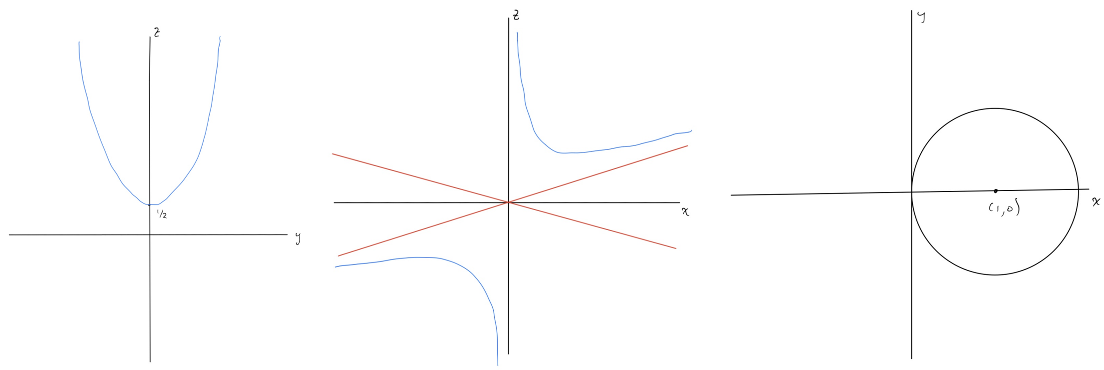
- If you need a refresher about conic sections, look up "conic section" in wikipedia and notice the picture to the right of "Definition." In particular, you have circles, ellipses, parabolas and hyperbolas. Remember that a circle is a kind of ellipse.

- Feel free to use some calculus if you want, to make your sketch. It is also not illegal to use a computer graphing program, but you can do this by hand too.

- Pay attention to the equations of the asymptote lines, if any.

Solution:

- The equation for C_0 is $z = \frac{1}{2}y^2 + \frac{1}{2}$, so C_0 is a parabola and the graph is the leftmost one below.
- The equation for C_1 is $x^2 - 2xz + 1 = 0$. Solving for z we get $z = \frac{x^2+1}{2x}$. As x goes to ∞ we see that the lines $z = \pm\frac{1}{2}x$ are asymptotes, and using calculus (or a graphing calculator or other computer software) we get the middle graph below. (The red lines are the asymptotes, not part of the graph.)
- The equation for C_2 is $(x-1)^2 + y^2 = 1$, which is a circle centered at (1,0). Its graph is on the right below.



d) (4 points) For this part, no proof is needed and I don't want any equations.

The first three parts of this problem illustrate, via a specific example, a general fact that I want you to discover. This part does not involve a specific example, but it's motivated by the example in this problem. Your answer should reflect what's going on in the previous parts of this problem.

Speculate about the connection between parts 4b and 4c. Specifically, suppose you have an irreducible conic (degree 2) curve C in $\mathbb{P}^2_{\mathbb{R}}$ (like the curve C we used above). Let ℓ be a line in $\mathbb{P}^2_{\mathbb{R}}$ that you want to consider as the line at infinity, and let $U = \mathbb{P}^2_{\mathbb{R}} \setminus \ell$, which you can identify with $\mathbb{R}^2$. What is it about the relation between C and ℓ that determines whether $C \cap U$ will be an ellipse (including circles as a special case), a parabola or a hyperbola? I'm just looking for the geometry of what you think is going on. No proof needed.

Solution:
When ℓ is tangent to C (i.e. meets C in one point), you get a parabola, which meets the line at infinity at one point. (Notice that the slope of the parabola as x gets big tends to infinity on both sides, so the curve meets the line at infinity at one point.) When ℓ meets C in two points (which is the usual situation) then you get a hyperbola, and the asymptote lines "point to" the relevant points at infinite corresponding to the intersection of ℓ with C. When ℓ is disjoint from C, then C has no points at infinity and is an ellipse in this affine part. This is why the same C gives rise to different kinds of conic sections when you choose different lines to be the line at infinity.