## Math 40510, Algebraic Geometry

## Problem Set 2, due Wednesday, April 7, 2021

Note: Answers that are sloppy, either from a mathematical point of view or because they are hard to read, will result in points being deducted even if they are technically correct.

## Solutions

1. In this problem, $k$ is a field but it is not necessarily algebraically closed. Let $R=k[x, y]$.
a) (6 points) Let $I=\left\langle x^{3}, y^{5}\right\rangle$. Prove the following fact about the radical of $I$.

$$
\sqrt{\left\langle x^{3}, y^{5}\right\rangle}=\langle x, y\rangle .
$$

(Please prove both inclusions.)

## Solution:

$\subseteq$ :
Let $f \in \sqrt{\left\langle x^{3}, y^{5}\right\rangle}$, so $f^{m} \in\left\langle x^{3}, y^{5}\right\rangle$ for some $m \geq 1$. We want to show that $f \in\langle x, y\rangle$. Suppose this were not true. Then $f$ has a non-zero constant term: $f=a+x h_{1}(x, y)+y h_{2}(x, y)$ where $0 \neq a \in k$. It follows that $f^{m}$ has $a^{m}$ as a non-zero constant term. Since no element of $\left\langle x^{3}, y^{5}\right\rangle$ has a non-zero constant term, this is a contradiction of the fact that $f^{m} \in\left\langle x^{3}, y^{5}\right\rangle$.

〇:
Let $f \in\langle x, y\rangle$. So

$$
f=a_{1,0} x+a_{0,1} y+a_{2,0} x^{2}+a_{1,1} x y+a_{0,2} y^{2}+\ldots .
$$

(Notice that the constant term is zero.) Then every term in $f^{8}$ has either a power of $x$ that is at least 3 , or a power of $y$ that is at least 5 (or both). So $f^{8} \in\left\langle x^{3}, y^{5}\right\rangle$ and $f \in \sqrt{\left\langle x^{3}, y^{5}\right\rangle}$.
b) (6 points) Give an example of polynomials $f$ and $g$ in $R$ so that

$$
\sqrt{\left\langle f^{3}, g^{5}\right\rangle} \neq\langle f, g\rangle .
$$

Be sure to justify your answer.

## Solution:

If we take $f=x^{2}$ and $g=y^{2}$ then $\sqrt{\left\langle f^{3}, g^{5}\right\rangle}$ is a radical ideal (by definition, since the radical of any ideal is always a radical ideal) while $\langle f, g\rangle$ is not (e.g. $x$ belongs to the radical but is not in $\langle f, g\rangle$ ). Hence they cannot be equal.
c) (5 points) Let $J=\left\langle x^{2}, x y, y^{2}\right\rangle$. Find $\mathbb{V}(J)$.

## Solution:

Let's show that $\mathbb{V}(J)=\{(0,0)\}$.
$\subseteq$ :
If $P=(a, b) \in \mathbb{V}(J)$ then in particular we must have that $x^{2}$ vanishes at $P$, so $a=0$. Similarly looking at $y^{2}$, we must have $b=0$. So $(a, b)=(0,0)$ and we have $\subseteq$.
?:
All three polynomials, $x^{2}, x y, y^{2}$ vanish at $(0,0)$.
d) (8 points) Let $I$ and $J$ be ideals. In Problem Set 1 we defined what's called the ideal quotient (I didn't mention this name at the time) to be

$$
I: J=\{f \in R \mid f \cdot J \subseteq I\}=\underset{1}{\{ } f \in R \mid f g \in I \text { for all } g \in J\}
$$

and you showed that this is an ideal. You don't have to re-prove that. Find $I: J$ where $I=\left\langle x^{2}, y^{2}\right\rangle$ and $J$ is the ideal given in part 1c. Specifically, find generators for this ideal. Be sure to explain your answer, proving both sides of your equality if necessary (i.e. you should start your answer with the assertion $I: J=\left\langle f_{1}, f_{2}, \ldots, f_{s}\right\rangle$ for specific $f_{1}, f_{2}, \ldots, f_{s}$ and prove both inclusions of this equality).

Solution:
We claim that $I: J=\langle x, y\rangle$. We'll prove both inclusions.
$\subseteq:$
Let $f \in I: J=\left\langle x^{2}, y^{2}\right\rangle:\left\langle x^{2}, x y, y^{2}\right\rangle$. If $f$ were not in $\langle x, y\rangle$ then it would have a non-zero constant term: $f=a+x h_{1}(x, y)+y h_{2}(x, y)$ where $0 \neq a \in k$. Since $f \in I: J$, in particular $f \cdot x y \in I$. But $f \cdot x y$ is a sum of terms that includes axy, and all other terms of $f \cdot x y$ have powers of either $x$ or $y$ (or both) that are at least two, so all such terms are in $\left\langle x^{2}, y^{2}\right\rangle$. Since an ideal is closed under addition and subtraction, this forces $a x y$ to be in $I=\left\langle x^{2}, y^{2}\right\rangle$, which is not true. Contradiction.
?:
It's enough to notice that

$$
\begin{aligned}
& x \cdot x^{2}=x^{3} \in I \\
& x \cdot x y=x^{2} y \in I \\
& x \cdot y^{2}=x y^{2} \in I \\
& y \cdot x^{2}=x^{2} y \in I \\
& y \cdot x y=x y^{2} \in I \\
& y \cdot y^{2}=y^{3} \in I .
\end{aligned}
$$

2. Let $R=\mathbb{R}[x, y]$. In this problem you can use the fact that $R$ is a Unique Factorization Domain (UFD): this means that it is an integral domain, and it has the property that every non-zero element $f$ can be written as a product of irreducible polynomials in a unique way, except for scalar multiples (e.g. $\left(2 x^{3}+4 y\right)(x+5 y)$ is not considered to be different from $\left(x^{3}+2 y\right)(2 x+10 y)$ or $\left.2\left(x^{3}+2 y\right)(x+5 y)\right)$.
a) (8 points) Prove that $\left\langle x^{2}+y^{2}\right\rangle$ is a radical ideal.

## Solution:

Notice that $x^{2}+y^{2}$ is irreducible: indeed, we've seen that the vanishing locus of $x^{2}+y^{2}$ is $\{(0,0)\}$, while if $x^{2}+y^{2}$ were to factor as a product of linear polynomials then the vanishing locus would be a union of two lines.
Let $f \in \sqrt{\left\langle x^{2}+y^{2}\right\rangle}$, so $f^{m} \in\left\langle x^{2}+y^{2}\right\rangle$ for some $m \geq 1$. We want to show that $f$ is already in $\left\langle x^{2}+y^{2}\right\rangle$. We know that $\left\langle x^{2}+y^{2}\right\rangle$ is generated by $x^{2}+y^{2}$, so

$$
f^{m}=\underbrace{f \cdot f \cdots \cdots f}_{m \text { times }}=\left(x^{2}+y^{2}\right) G
$$

for some $G \in \mathbb{R}[x, y]$. By unique factorization, $\left(x^{2}+y^{2}\right)$ divides $f^{m}$. (Since it is a factor on the right, it must be a factor on the left.) Since it is irreducible, it must divide $f$ (you can't divide parts of $x^{2}+y^{2}$ into different copies of $f$ in the product). Since $f$ is thus divisible by $x^{2}+y^{2}$, we get $f \in\left\langle x^{2}+y^{2}\right\rangle$.
Note: we could have done this without using irreducibility of $x^{2}+y^{2}$. For example, if it had been $x^{2}-y^{2}=(x+y)(x-y)$, we'd have

$$
f^{m}=\underbrace{f \cdot f \cdots \cdots f}_{m \text { times }}=(x+y)(x-y) G
$$

and we'd have that $(x+y)$ divides one copy of $f$ on the left and $(x-y)$ does too. But on the left it's just a product of copies of the same $f$, so both $(x+y)$ and $(x-y)$ divide $f$, so their product does too.
b) ( 6 points) Prove that $\left\langle x^{2}+2 x y+y^{2}\right\rangle$ is not a radical ideal.

## Solution:

$x^{2}+2 x y+y^{2}=(x+y)^{2}$ so $(x+y) \in \sqrt{\left\langle x^{2}+2 x y+y^{2}\right\rangle}$, while $(x+y) \notin\left\langle x^{2}+2 x y+y^{2}\right\rangle$ (its degree is too small).
3. In this problem we'll work over the field $\mathbb{R}$. Let $V$ be the parabola defined by $y=x^{2}$ in $\mathbb{R}^{2}$ and let $W$ be the tangent line to $V$ at the point $(1,1)$.
a) (5 points) Of course $V=\mathbb{V}\left(y-x^{2}\right)$. Find a polynomial $\ell$ of degree 1 so that $W=\mathbb{V}(\ell)$.

## Solution:

Since $\frac{d}{d x}\left(x^{2}\right)=2 x$, the line we're looking for has slope 2 and passes through ( 1,1 ). So its equation is

$$
y-1=2(x-1), \quad \text { i.e. } \quad y=2 x-1 .
$$

Hence $W=\mathbb{V}(y-2 x+1)$.
b) (6 points) Let $I=\left\langle y-x^{2}, \ell\right\rangle$ (where $\ell$ is your answer to (3a)). Prove that $I$ is not a radical ideal.

## Solution:

We have
$I=\left\langle y-x^{2}, y-2 x+1\right\rangle=\left\langle y-x^{2},-x^{2}+2 x-1\right\rangle=\left\langle y-x^{2}, x^{2}-2 x+1\right\rangle=\left\langle y-x^{2},(x-1)^{2}\right\rangle$.
(For the second equality, keep the first generator and replace the second by subtracting the second from the first.) This shows that $x-1 \in \sqrt{I}$. Looking at the last choice of generators, it's clear that $x-1 \notin I$.
c) ( 8 points) Let $V$ be the circle of radius 1 centered at $(0,1)$ and let $W$ be the circle of radius 2 centered at $(0,2)$. Let

$$
J=\mathbb{I}(V)+\mathbb{I}(W) .
$$

Prove that $J$ is not a radical ideal, and find the radical of $J$. [Hint: you can use without proof the fact that both $\mathbb{I}(V)$ and $\mathbb{I}(W)$ are principal, i.e. are generated by a single polynomial.]

## Solution:

We have

$$
\left.\mathbb{I}(V)=\left\langle x^{2}+(y-1)^{2}-1\right\rangle, \quad \mathbb{I}(W)=\left\langle x^{2}+(y-2)^{2}-4\right)\right\rangle
$$

so

$$
\begin{aligned}
J & =\mathbb{I}(V)+\mathbb{I}(W) \\
& =\left\langle x^{2}+(y-1)^{2}-1, x^{2}+(y-2)^{2}-4\right\rangle \\
& =\left\langle x^{2}+y^{2}-2 y, x^{2}+y^{2}-4 y\right\rangle \\
& =\left\langle x^{2}+y^{2}-2 y,\left(x^{2}+y^{2}-2 y\right)-\left(x^{2}+y^{2}-4 y\right)\right\rangle \\
& =\left\langle x^{2}+y^{2}-2 y, 2 y\right\rangle \\
& =\left\langle x^{2}+y^{2}-2 y, y\right\rangle \\
& =\left\langle x^{2}, y\right\rangle .
\end{aligned}
$$

Then clearly $x \in \sqrt{J}$ and $x \notin J$, and equally clearly $\sqrt{J}=\langle x, y\rangle$ (this is like problem 1a but a bit easier).
4. For some ideals it happens to be the case that ideal quotients have a nice property. Let's look at an example, and then prove something a bit more general. Throughout this problem assume that $k$ is an infinite field. (The choice of the field shouldn't enter into your arguments.)
a) (8 points) If $I$ is an ideal and $F \in R$ (a polynomial ring), show that

$$
I:\langle F\rangle=\{G \in R \mid F G \in I\}
$$

In other words, $\langle F\rangle$ contains infinitely many elements, but to see if the set $G \cdot\langle F\rangle$ is contained in $I$, it's enough to check if the single element $G F$ is in $I$. This can be used in the remaining parts of this problem.

## Solution:

$\subseteq$ :
Let $G \in I:\langle F\rangle$. So $G \cdot\langle F\rangle \subseteq I$, i.e. $G(A F) \in I$ for all $A \in R$. In particular, take $A=1$. Then $G F \in I$, and we are done.
?:
Let $G$ be a polynomial such that $F G \in I$. We want to show $G \cdot\langle F\rangle \subseteq I$. Every element of $\langle F\rangle$ is of the form $F A$ for some $A \in R$, so we want to show that $G(F A) \in I$ for all $A \in R$. But since we already know that $F G \in I$, and $I$ is an ideal, we also get $A(F G)=G(F A) \in I$ and we are done.

We will use this fact for all the remaining parts of this problem without further comment.
b) (6 points) Let $R=k[x, y, z]$. Let $I=\left\langle x^{2}, y^{2}\right\rangle$. Prove that

$$
I:\langle z\rangle=I .
$$

This should be approached purely algebraically. Don't use the geometry-algebra dictionary.
Solution:
〇:
If $F \in I$ then $F z \in I$ (since $I$ is an ideal) so $F \in I:\langle z\rangle$.
$\subseteq$ :
Assume $F \in I:\langle z\rangle$, so $F z \in I$. Since $F z \in I$ we get

$$
F z=A x^{2}+B y^{2}
$$

for some $A, B \in R$.
We want to show $F \in I$. Let's break up $F$. Without loss of generality there exist $C, D \in R$ such that

$$
\begin{equation*}
F=C x^{2}+D y^{2}+M \tag{1}
\end{equation*}
$$

where

$$
M=\left(\text { sum of terms none of which is divisible by either } x^{2} \text { or } y^{2}\right) .
$$

We'd like to show $M=0$. When we multiply any term in $M$ by $z$, it still is not divisible by either $x^{2}$ or $y^{2}$. We have

$$
\begin{equation*}
F z=C x^{2} z+D y^{2} z+M z \tag{2}
\end{equation*}
$$

Combining the two expressions for $F z$ we get

$$
A x^{2}+B y^{2}=C x^{2} z+D y^{2} z+M z .
$$

Hence

$$
\begin{equation*}
(A-C z) x^{2}+(B-D z) y^{2}=M z \tag{3}
\end{equation*}
$$

Case 1: Assume $A=C z$ and $B=D z$.
Then from (3) we see $M z=0$, so $M=0$ since $R$ is an integral domain. Done.
Case 2: Assume the lefthand side of (3) is zero for some other reason (e.g. $A-C z=-y^{2}, B-$ $D z=x^{2}$ ).

So the righthand side of (3) is also zero, i.e. $M z=0$. But $R$ is an integral domain, and $z$ is a variable, so $M=0$.

Case 3: Assume the lefthand side of (3) is not zero.
This means also $M z \neq 0$. And remember that no term of $M z$ is divisible by either $x^{2}$ or $y^{2}$. But then no linear combination (over $k$ ) of terms of $M z$ can pick up an $x^{2}$ or a $y^{2}$ as factors, which means that $M z \notin I$. Now going back to (2), we get

$$
F z-C x^{2} z-D y^{2} z=M z
$$

From what we've seen, the lefthand side is in $I$ but the righthand side is not, which is impossible.
c) (6 points) Again let $R=k[x, y, z]$ and let $I=\left\langle x^{2}, y^{2}\right\rangle$. Find

$$
I:\langle x\rangle .
$$

Specifically, note that this is not equal to $I$. [Again, this should be approached purely algebraically. Don't use the geometry-algebra dictionary. Be sure to explain your work; just giving the answer isn't enough.]

## Solution:

We'll claim that $I:\langle x\rangle=\left\langle x, y^{2}\right\rangle$.
?:
This direction is clear.
$\subseteq$ :
Set $F \in I:\langle x\rangle$. So $x F \in I=\left\langle x^{2}, y^{2}\right\rangle$. Write

$$
F=A+x B+x^{2} C+y D+y^{2} E
$$

where $A, B, C, D, E \in R$. Assume that these are chosen so that (in this order)

- any term of $F$ divisible by $x^{2}$ is included in $x^{2} C$,
- any term divisible by $x$ but not $x^{2}$ is included in $x B$,
- any term not divisible by $x$ but divisible by $y^{2}$ is included in $y^{2} E$, and
- any term not divisible by $x$, divisible by $y$, but not divisible by $y^{2}$ is included in $y D$.

Thus

$$
A \in k[z], B \in k[y, z], C \in R, D \in k[z], E \in k[y, z]
$$

Notice that

$$
x F=x A+x^{2}(B+x C)+x y D+y^{2}(x E) .
$$

We have assumed that $x F \in\left\langle x^{2}, y^{2}\right\rangle=I$. So

$$
x A+x y D=x F-x^{2}(B+x C)-y^{2}(x E) \in I
$$

In order for $x A+x y D$ to be in $I$, we must have $A=D=0$ (since $A, D \in k[z]$ ). Thus from (4), we get $F=x B+x^{2} C+y^{2} E \in\left\langle x, y^{2}\right\rangle$.
d) (8 points) Now let $R=k\left[x_{1}, \ldots, x_{n}\right]$.

- Let $I=\left\langle f_{2}, \ldots, f_{t}\right\rangle$ (for some $t \geq 2$ );
- Let $J=\left\langle f_{1}, f_{2}, \ldots, f_{t}\right\rangle$. [We're just adding one more generator to $I$.]
- Let $F$ be some polynomial.
- ASSUME that $I:\langle F\rangle=I$. [Don't mix up $I$ and $J$ here!]
[Part 4b) gives an example to show you that this can happen sometimes, and part 4c shows that sometimes it doesn't happen. Parts 4b) and 4c) are not otherwise connected to this problem.]
Prove that $\left\langle F f_{1}, f_{2}, \ldots, f_{t}\right\rangle:\langle F\rangle=J$. Be sure to indicate where the assumption of the fourth bullet is used.


## Solution:

?:
Let $G=a_{1} f_{1}+a_{2} f_{2}+\cdots+a_{t} f_{t} \in J=\left\langle f_{1}, f_{2}, \ldots, f_{t}\right\rangle$, where $a_{1}, \ldots, a_{t} \in R$. Then

$$
F G=a_{1}\left(F f_{1}\right)+\left(a_{2} F\right) f_{2}+\cdots+\left(a_{t} F\right) f_{t} \in\left\langle F f_{1}, f_{2}, \ldots, f_{t}\right\rangle
$$

$\subseteq:$
Let $G \in\left\langle F f_{1}, f_{2}, \ldots, f_{t}\right\rangle:\langle F\rangle$, so $F G \in\left\langle F f_{1}, f_{2}, \ldots, f_{t}\right\rangle$. We want to show that $G \in J=$ $\left\langle f_{1}, f_{2}, \ldots, f_{t}\right\rangle$. We know that for some $a_{1}, a_{2}, \ldots, a_{t} \in R$,

$$
F G=a_{1}\left(F f_{1}\right)+a_{2} f_{2}+\cdots+a_{t} f_{t}
$$

so

$$
F\left(G-a_{1} f_{1}\right)=a_{2} f_{2}+\cdots+a_{t} f_{t} \in I
$$

This means $G-a_{1} f_{1} \in I:\langle F\rangle$, and we assumed that this was equal to $I$. So $G-a_{1} f_{1} \in I$ and we have

$$
G-a_{1} f_{1}=b_{2} f_{2}+\cdots+b_{t} f_{t}
$$

i.e.

$$
G=a_{1} f_{1}+b_{2} f_{2}+\cdots+b_{t} f_{t} \in J
$$

and we are done.
5. Assume that $k$ is algebraically closed. Let $R=k[x, y, z]$. Also let

$$
I=\left\langle x^{4}, x^{2} y^{2}, y^{4}\right\rangle \quad \text { and } \quad J=\left\langle x, y^{2}\right\rangle .
$$

a) (6 points) Prove that $\mathbb{V}(I)=\mathbb{V}(J)$ in $k^{3}$ and find this variety explicitly; (very) briefly explain your answer.

## Solution:

Both $\mathbb{V}(I)$ and $\mathbb{V}(J)$ are equal to $\mathbb{V}(x, y)$, which is the $z$-axis.
b) (8 points) Notice the following facts:

- $I: J=\left\langle x^{3}, x y^{2}, y^{4}\right\rangle$.
- $\mathbb{V}(I: J)=\mathbb{V}(I)=\mathbb{V}(J)$;
- $\overline{\mathbb{V}(I) \backslash \mathbb{V}(J)}=\emptyset$;
- $\mathbb{I}(\emptyset)=\langle 1\rangle$ since $k$ is algebraically closed.

You don't have to prove any of the above bullet points, but they motivate the following:
Find the smallest integer $N \geq 1$ so that $I: J^{N}=\langle 1\rangle$. Explain your answer: specifically, why does your $N$ work and why is it the smallest? [Hint: $J^{N}$ has $N+1$ generators in this case.]

## Solution:

Clearly $I: J^{N} \subseteq\langle 1\rangle=R$ for all $N \geq 1$, so we only have to worry about $\supseteq$. That is, for what $N$ is it true that $1 \cdot J^{N} \subset I$ ? That is, for what $N$ is $J^{N} \subseteq I$ ? Let's check each $N$ until it is true.

- $N=1$. $J^{1}=J=\left\langle x, y^{2}\right\rangle$ and neither generator is in $I$. So $N=1$ doesn't work.
- $N=2 . J^{2}=\left\langle x^{2}, x y^{2}, y^{4}\right\rangle$ and $x^{2} \notin I$.
- $N=3$. $J^{3}=\left\langle x^{3}, x^{2} y^{2}, x y^{4}, y^{6}\right\rangle$ and $x^{3} \notin I$.
- $N=4$. $J^{4}=\left\langle x^{4}, x^{3} y^{2}, x^{2} y^{4}, x y^{6}, y^{8}\right\rangle$. All the generators are in $I$, so the answer is $N=4$.

