Math 40510, Algebraic Geometry

Problem Set 3, due Friday, May 7, 2021

<u>**Note**</u>: Answers that are sloppy, either from a mathematical point of view or because they are hard to read, will result in points being deducted even if they are technically correct.

Solutions

1. Find $\mathbb{V}(yz - x^2, yz - 4z^2 + x^2)$ as a subvariety of $\mathbb{P}^2_{\mathbb{R}}$ by hand. Explain your work. [Hint: the answer is a finite set of points in $\mathbb{P}^2_{\mathbb{R}}$.]

   Solution:
   If $yz - x^2 = 0$ and $yz - 4z^2 + x^2 = 0$ then also their sum $2yz - 4z^2 = 0$, i.e.

   $$z(y - 2z) = 0.$$

   So any solution must satisfy either $z = 0$ or $y - 2z = 0$. Consider the former.

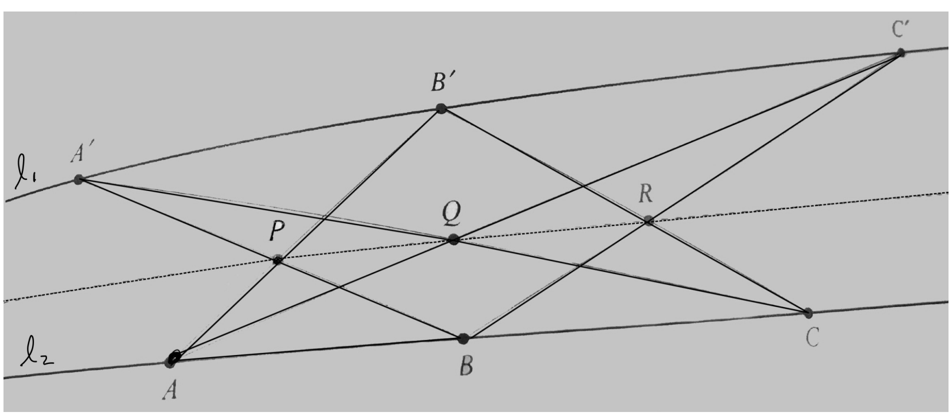
   Since $yz - x^2 = 0$, if $z = 0$ then we must have $x = 0$. Then y can be anything (check both defining polynomials!) so up to scalar multiple we get the (single) point $[0, 1, 0]$.

   Now suppose $y - 2z = 0$. This gives us $\mathbb{V}(2z^2 - x^2, -2z^2 + x^2) = \mathbb{V}(2z^2 - x^2)$. That means $2z^2 = x^2$, so $x = \pm\sqrt{2}z$. Since up to scalar multiple we can choose $z = 1$ (as long as $z \neq 0$, which we took care of above), this gives $y = 2$ and $x = \pm\sqrt{2}$. So our solution is

   $$\mathbb{V}(yz - x^2, yz - 4z^2 + x^2) = \{[0, 1, 0], [\sqrt{2}, 2, 1], [-\sqrt{2}, 2, 1]\}.$$

2. The classical theorem of Pappus says the following. Assume that we have two lines, namely ℓ_1 (containing the points A', B', C') and ℓ_2 (containing the points A, B, C) in $\mathbb{R}^2$ – see the picture below. Note that ℓ_1 and ℓ_2 are not assumed to be parallel. Let

   - P be the intersection of $\overline{AB'}$ and $\overline{A'B}$;
   - Q be the intersection of $\overline{AC'}$ and $\overline{A'C}$;
   - R be the intersection of $\overline{BC'}$ and $\overline{B'C}$.

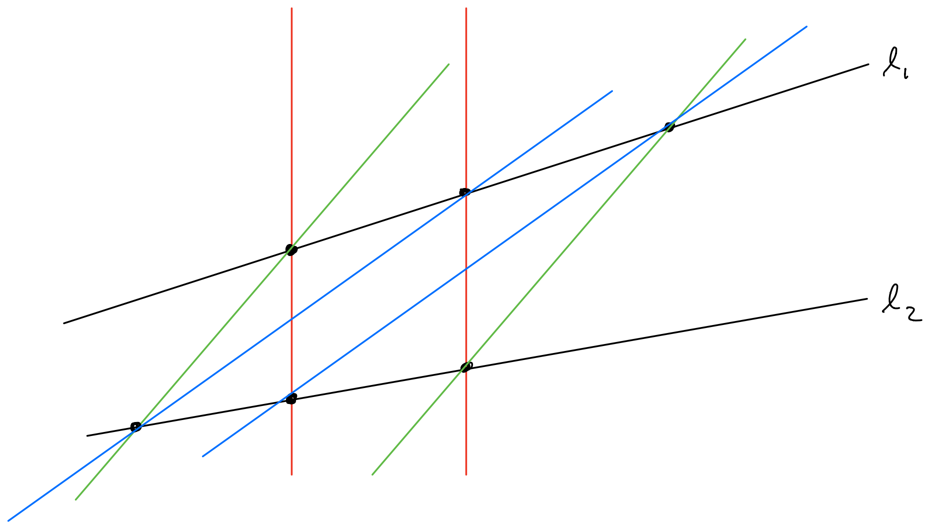


   Then the conclusion of Pappus's theorem is that P, Q, R must be collinear. This is just background and you don't have to prove this.

   Now suppose the picture is a little bit different. We again start with lines ℓ_1 and ℓ_2, (black in the picture below) which are not necessarily parallel. We have three points, A, B, C on one line and A', B', C' on the other. (You'll have to label the points as part of this problem.) Assume that

- the red lines are parallel,
- the green lines are parallel.

(See the picture below.)

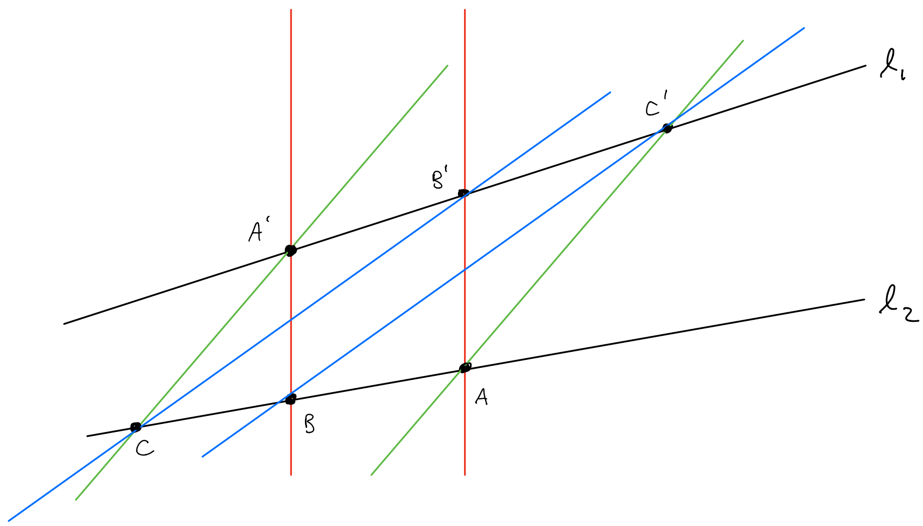


Prove that the blue lines are parallel (viewed in $\mathbb{R}^2$ of course). To do this, assume that Pappus's theorem is a theorem about points in $\mathbb{P}^2_{\mathbb{R}}$ rather than $\mathbb{R}^2$ (you don't have to justify this part), and make a careful study of what happens at infinity.

Note:

- you could answer this with one sentence in a way that would be technically correct, but I want your answer to really reflect the fact that you understand the geometry going on and the difference between the affine geometry and the projective geometry. So please put some thought and detail into your answer!
- Please answer this using Pappus's theorem. **No argument with similar triangles or such!!**
- Make sure your answer includes a labelled picture! The black dots need to be labelled A, A', B, B', C, C' in some suitable order, and you should refer to this labelled picture in your proof.

Solution:
Here's one way. Label the points as follows:



The statement that the red lines are parallel in $\mathbb{R}^2$ means that $\overline{A'B}$ and $\overline{AB'}$ meet at a point P at infinity in $\mathbb{P}^2_{\mathbb{R}}$. The statement that the green lines are parallel means $\overline{A'C}$ and $\overline{AC'}$ also meet in a point, Q, at infinity. The line $\overline{PQ}$ spanned by P and Q is the line at infinity. Then Pappus says that the intersection point, R, of the blue lines, $\overline{B'C}$ and $\overline{BC'}$, has to lie on the same line, namely $\overline{PQ}$. This means that R also has to lie on the line at infinity. This precisely means that these two lines are parallel in $\mathbb{R}^2$.

3. Let $f(x,y) \in k[x,y]$ be a non-zero homogeneous polynomial. Show that there are finitely many points P of $\mathbb{P}^1_k$ (possibly none) where $f(P) = 0$ (in the sense discussed in class, meaning that f vanishes at P no matter which choice of coordinates we choose for P, which is ok since f is homogeneous).

   Solution:
   The fact that f is homogeneous allows us to talk about whether the vanishing of f at a point is well-defined or not. We have seen that we can identify $\mathbb{P}^1 \backslash \{[1,0]\}$ (where $[1,0]$ is the point at infinity) with k^1. If $f(x,y)$ vanishes at the point at infinity, this means $f(1,0) = 0$. But $[1,0]$ is just one point, so to see at how many points $P = [a,b] \in \mathbb{P}^1$ our polynomial f vanishes, we might as well assume $b \neq 0$. This means we can further assume (by scaling) that $b = 1$ (since $[a,b] = [\frac{a}{b}, 1]$). So $P = [a,1]$ is in the copy of k^1 in the identification just mentioned. The vanishing of $f(x,y)$ at $P = [a,1]$ corresponds to the vanishing of $f(x,1)$ at $x = a$. But $f(x,1)$ is a polynomial in one variable, and we have seen that it can have only finitely many roots. So even taking into account the possible vanishing at the point at infinity, $f \in k[x,y]$ has finitely many zeroes in $\mathbb{P}^1_k$.

4. We saw in class that if $f \in k[x_0, \ldots, x_n]$ then we can't view f as a function on $\mathbb{P}^n$ because for the same point $P = [a_0, \ldots, a_n]$, different representations of this point give different values when plugged into f. Let's explore what happens if we use rational functions instead of polynomials.

   Throughout this problem, assume that k is an infinite field. You can freely use the fact that over an infinite field, a non-constant polynomial (even in several variables) is never identically 0 at all points of $\mathbb{P}^n_k$. You don't have to prove this fact.

a) Let $f, g \in k[x_0, \ldots, x_n]$. Prove that if f and g are homogeneous of the same degree then f/g gives a well-defined function on $\mathbb{P}^n \backslash \mathbb{V}(g)$. Be sure to mention why we have to restrict to $\mathbb{P}^n \backslash \mathbb{V}(g)$ instead of all of $\mathbb{P}^n$.

Solution:
Let $P = [a_0, a_1, \ldots, a_n] \in \mathbb{P}^n \backslash \mathbb{V}(g)$. We want to show that

$$\frac{f}{g}(ta_0, \ldots, ta_n)$$

is the same value no matter what t is, as long as $t \neq 0$. Assume $\deg(f) = \deg(g) = d$ and that both are homogeneous. We know from class that

$$f(ta_0, \ldots, ta_n) = t^d f(a_0, \ldots, a_n) \quad \text{and} \quad g(ta_0, \ldots, ta_n) = t^d g(a_0, \ldots, a_n).$$

Hence

$$\frac{f}{g}(ta_0, \ldots, ta_n) = \frac{t^d f(a_0, \ldots, a_n)}{t^d g(a_0, \ldots, a_n)} = \frac{f(a_0, \ldots, a_n)}{g(a_0, \ldots, a_n)}$$

and this value is achieved no matter what t is, as long as $t \neq 0$. Since the denominator is not zero because $P \notin \mathbb{V}(g)$, we are done.

b) For simplicity, now assume that we have only two variables, x, y. Give an example to show that if f and g are homogeneous of different degrees then f/g is not well-defined as a function on $\mathbb{P}^1 \backslash \mathbb{V}(g)$. (Make sure you explain why it's not well-defined – it's enough to exhibit one point where you show it's not well-defined.)

Solution:
Let

$$f(x, y) = x^2 + y^2 \quad \text{and} \quad g(x, y) = x^3 + y^3.$$

Note f is homogeneous of degree 2 and g is homogeneous of degree 3. Let $P = [1, 1] = [2, 2] \in \mathbb{P}^1$. Since

$$\frac{f}{g}(1, 1) = \frac{1+1}{1+1} = 1 \quad \text{while} \quad \frac{f}{g}(2, 2) = \frac{4+4}{8+8} = \frac{1}{2},$$

we are done: we get different values for the same point, so it is not well-defined.

c) Again assuming that $f, g \in k[x, y]$, give an example of polynomials f, g that are not homogeneous but have the same degree, such that f/g is again not well-defined as a function on $\mathbb{P}^1 \backslash \mathbb{V}(g)$. (Make sure you explain why it's not well-defined – it's enough to exhibit one point where you show it's not well-defined.)

Solution:
Let $f(x, y) = x+y+1$ and $g(x, y) = 2x+y+1$. Neither is homogeneous, but both have degree 1. Let $P = [1, 1] = [2, 2] \in \mathbb{P}^1$. Then

$$\frac{f}{g}(1, 1) = \frac{1+1+1}{2+1+1} = \frac{3}{4} \quad \text{while} \quad \frac{f}{g}(2, 2) = \frac{2+2+1}{4+2+1} = \frac{5}{7},$$

so as in 4b we are done.

d) Let $f \in k[x_0, \ldots, x_n]$ be a non-constant polynomial (which may or may not be homogeneous). In this part of the problem we'll explore in a bit more detail what we discussed in class.

   (i) Prove that there exists a point $P = (a_0, \ldots, a_n) \in k^{n+1}$ such that $f(ta_0, \ldots, ta_n)$ is not constant as a function of t. [Warning: it's not enough to just say that this is immediate since we assumed that f is not constant at the start of the problem. For example, suppose

$f(x,y) = x^2 - 4y^2$. This is not constant as a polynomial in x, y, but if $P = (2,1)$ then $f(2t,t) = 0$ no matter what t is, and this a constant.]

Solution:
Write f as a sum of its homogeneous components:

$$f = f_d + f_{d-1} + \cdots + f_1 + f_0.$$

Let $P = (a_0, \ldots, a_n)$ be a point of k^{n+1} (which we have not yet shown can be found with the desired property). Then

$$f(ta_0, ta_1, \ldots, ta_n) = t^d f_d(a_0, \ldots, a_n) + t^{d-1} f_{d-1}(a_0, \ldots, a_n) + \cdots + t f_1(a_0, \ldots, a_n) + f_0.$$

Since k is infinite, as mentioned at the beginning of this problem there exist $a_0, \ldots, a_n$ such that $f_d(a_0, \ldots, a_n) \neq 0$, since f_d is a non-zero polynomial so it can't vanish at all points of k^{n+1}. With this specific choice of $a_0, , \ldots, a_n$, $f(ta_0, \ldots, ta_n)$ is thus a non-zero polynomial in the variable t that is not a constant since at least the coefficient of t^d is not zero.

(ii) Explain in a few words why this means that f does not give a well-defined function on $\mathbb{P}^n$ even if f is homogeneous.

Solution:
Suppose f did give a well-defined function on $\mathbb{P}^n$. Let $P = [a_0, \ldots, a_n]$ be the point you found in part (i). Then since $[a_0, \ldots, a_n] = [ta_0, \ldots, ta_n]$ for all $t \neq 0$, $f(ta_0, \ldots, ta_n)$ would have to have the same value no matter what t is (to make it well-defined), and so $f(ta_0, \ldots, ta_n)$ would be constant as a function of t, contradicting what we proved in (i).

5. For this problem recall that the twisted cubic, C, in $\mathbb{R}^3$ was determined, as an affine variety, by

$$C = \{(t, t^2, t^3) \mid t \in \mathbb{R}\} \quad \text{and} \quad \mathbb{I}(C) = \langle x_2 - x_1^2 \ , \ x_3 - x_1^3 \rangle.$$

We also saw that to extend this to $\mathbb{P}^3_{\mathbb{R}}$ we need to add just one point at infinity. Let's elaborate on this a bit.

Let

$$I = \langle x_0x_2 - x_1^2,\ x_0^2x_3 - x_1^3,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2, \rangle \subseteq \mathbb{R}[x_0, x_1, x_2, x_3] = R.$$

This is a homogeneous ideal, so $\mathbb{V}(I)$ defines a subvariety of $\mathbb{P}^3_{\mathbb{R}}$. For convenience, let

$$f_1 = x_0x_2 - x_1^2, \qquad f_2 = x_0^2x_3 - x_1^3, \qquad f_3 = x_0x_3 - x_1x_2, \qquad f_4 = x_1x_3 - x_2^2.$$

a) One of the generators f_1, f_2, f_3, f_4 of I is already a linear combination of the others, so it is redundant (i.e. it can be removed without changing the ideal). Find which one is redundant, and show how it is a linear combination of the other three (with coefficients in R).

Solution:
Notice that f_2 has degree 3 while the rest have degree 2, so you would expect that f_2 is the redundant one. And in fact,

$$x_0^2x_3 - x_1^3 = x_0(x_0x_3 - x_1x_2) + x_1(x_0x_2 - x_1^2), \quad \text{i.e. } f_2 = x_0 \cdot f_3 + x_1 \cdot f_1.$$

b) Once you remove the redundant generator in part 5a), you're left with three generators. Give the dehomogenizations of these three with respect to x_0, and show how one of the three dehomogenizations is again a linear combination of the other two (with coefficients in R).

Solution:

The dehomogenizations are

$$g_1 = x_2 - x_1^2, \qquad g_2 = x_3 - x_1^3, \qquad g_4 = x_1x_3 - x_2^2.$$

Then we compute

$$x_1x_3 - x_2^2 = x_1(x_3 - x_1^3) - (x_2 + x_1^2)(x_2 - x_1^2), \quad \text{i.e.} \quad g_4 = x_1 \cdot g_2 - (x_2 + x_1^2) \cdot g_1.$$

c) Let $V = \mathbb{V}(I)$ (where again, I is the ideal given above). Using the ideas from class, show that $V \cap U_0 = C$ and use the equations to find the single point at infinity (remembering that the plane at infinity is given by $x_0 = 0$). Show your work to find the point at infinity, but for the whole problem feel free to quote anything from class.

Solution:
We saw that for a variety $V = \mathbb{V}(f_1, \ldots, f_s) \subset \mathbb{P}^n$, we have

$$V \cap U_0 = \mathbb{V}(f_1(1, x_1, \ldots, x_n), \ldots, f_s(1, x_1, \ldots, x_n))$$

so the above dehomogenizations in 5b) give that

$$V \cap U_0 = \mathbb{V}(g_1, g_2, g_4) = \mathbb{V}(g_1, g_2) = C.$$

(We can ignore g_4 because it is a linear combination of g_1 and g_2.)
To find what's going on at infinity, we look at

$$I + \langle x_0 \rangle = \langle x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2, x_0 \rangle = \langle x_1^2, x_1x_2, x_1x_3 - x_2^2, x_0 \rangle.$$

The locus where $x_1^2 = 0$ is the same as the locus where $x_1 = 0$, so then $\mathbb{V}(I, x_0)$ is obtained setting $x_0 = 0,\ x_1 = 0$ and $x_2 = 0$, so we get the point $[0, 0, 0, 1]$ as we saw in class by a different method.

6. In this problem you can use (without proof) the following facts to help you.

- if I is a homogeneous ideal, and if we define

$$[I]_d = \{\text{homogeneous polynomials of degree } d \text{ in } I\},$$

then $[I]_d$ is a vector space over the field k.

- the number of monomials of degree d in R is $\binom{d+2}{2}$. (So for example if $d = 1$ there are $\binom{1+2}{2} = 3$, namely x, y, z while if $d = 2$ there are $\binom{2+2}{2} = 6$, namely $x^2, xy, xz, y^2, yz, z^2$.)
- If I is an ideal generated by monomials, then I is a homogeneous ideal, and you can always find a basis for $[I]_d$ consisting only of monomials.

For this problem let

$$I = \langle\ x^2,\ xy,\ y^2\ \rangle \subset R = k[x, y, z].$$

(Note that the generators involve only x and y, but the ring has 3 variables.)

a) A basis for $[I]_2$ is clearly given by x^2, xy and y^2, so $[I]_2$ is a 3-dimensional vector space. (You don't have to prove this.) Find a basis for $[I]_3$ and for $[I]_4$. [Hint: don't forget the third variable, and remember that you have to remove repeated terms to find a basis!] As a result, what is the dimension of $[I]_3$ and of $[I]_4$?

Solution:

First look in degree 3. We multiply each generator by x, y and z, but then check for double-counting.

$$\begin{array}{lll} x \cdot x^2 = x^3 & x \cdot xy = x^2y & x \cdot y^2 = xy^2 \\ y \cdot x^2 = x^2y & y \cdot xy = xy^2 & y \cdot y^2 = y^3 \\ z \cdot x^2 = x^2z & z \cdot xy = xyz & z \cdot y^2 = y^2z \end{array}$$

The double-counted elements are x^2y and xy^2. Thus the basis is

$$\{x^3, xy^2, x^2y, y^3, x^2z, xyz, y^2z\}$$

and

$$\dim[I]_3 = 3 \cdot 3 - 2 \cdot 1 = 7.$$

Now look in degree 4. We multiply each generator by x^2, xy, xz, y^2, yz and z^2, but then check for double-counting.

$$\begin{array}{lll} x^2 \cdot x^2 = x^4 & x^2 \cdot xy = x^3y & x^2 \cdot y^2 = x^2y^2 \\ xy \cdot x^2 = x^3y & xy \cdot xy = x^2y^2 & xy \cdot y^2 = xy^3 \\ xz \cdot x^2 = x^3z & xz \cdot xy = x^2yz & xz \cdot y^2 = xy^2z \\ y^2 \cdot x^2 = x^2y^2 & y^2 \cdot xy = xy^3 & y^2 \cdot y^2 = y^4 \\ yz \cdot x^2 = x^2yz & yz \cdot xy = xy^2z & yz \cdot y^2 = y^3z \\ z^2 \cdot x^2 = x^2z^2 & z^2 \cdot xy = xyz^2 & z^2 \cdot y^2 = y^2z^2 \end{array}$$

The double-counted elements are x^3y, x^2yz, xy^3, xy^2z, and x^2y^2 is triple counted. So a basis is given by

$$\{x^4, x^3y, x^2y^2, x^2y^2, xy^3, x^3z, x^2yz, xy^2z, y^4, y^3z, xyz^2, y^2z^2\}.$$

Thus

$$\dim[I]_4 = 3 \cdot 6 - 2 \cdot 3 = 12.$$

b) Based on patterns you see in 6a), find a formula for $\dim[I]_d$ for any $d \geq 2$. Explain your answer. As long as you see the right pattern, I won't be too fussy about proving it. [Hint: you'll need some binomial coefficients. Focus on a pattern for how much you have to subtract because of over-counting.]

Solution:

$$3 \cdot \binom{d}{2} - 2 \cdot \binom{d-1}{2}.$$

c) We maintain the notation that $I = \langle\ x^2,\ xy,\ y^2\ \rangle$. For any degree d, define a function

$$h(d) = \binom{d+2}{2} - \dim[I]_d.$$

(Fun fact: this function is named after our old buddy Hilbert.) Prove that $h(d) = 3$ for all $d \geq 2$. You'll need your answer to part 6b).

Solution:

$$
\begin{aligned}
\binom{d+2}{2} - \left[3 \cdot \binom{d}{2} - 2 \cdot \binom{d-1}{2}\right] &= \frac{(d+2)(d+1)}{2} - \left[\frac{3d(d-1)}{2} - \frac{2(d-1)(d-2)}{2}\right] \\
&= \frac{d^2 + 3d + 2 - \left[3d^2 - 3d - 2d^2 + 6d - 4\right]}{2} \\
&= \frac{6}{2} \\
&= 3.
\end{aligned}
$$