## Math 40510, Algebraic Geometry

## Problem Set 1, due March 5, 2021

Note: This is not the entire problem set. It is just the first set of problems I've assigned for Problem Set 1, last modified February 23, 2021. Answers that are sloppy, either from a mathematical point of view or because they are hard to read, will result in points being deducted even if they are technically correct.

1. In class we proved that if $k$ is an infinite field and $f \in k\left[x_{1}, \ldots, x_{n}\right]$ then the following are equivalent:

- $f$ is the zero polynomial.
- The evaluation function $f: k^{n} \rightarrow k$, defined by $f(P)=f\left(b_{1}, \ldots, b_{n}\right)$ for $P=\left(b_{1}, \ldots, b_{n}\right) \in k^{n}$, is the zero function. (I.e. $f$, evaluated at any point of $k^{n}$, vanishes.)

Here we'll explore what happens when $k$ is finite.
a) Let $k=\mathbb{Z}_{19}$. Give an example of a non-zero polynomial $f \in k[x]$ such that $f: k \rightarrow k$ defines the zero function.
b) Let $k=\mathbb{Z}_{5}$. Let $f \in k[x, y]$. Give an example of a non-zero polynomial $f$ of degree 5 , involving both variables, for which $f: k^{2} \rightarrow k$ is the zero function.
c) Let $k=\mathbb{Z}_{5}$. Let $f \in k[x, y]$. Assume that $f$ is either the zero polynomial or a polynomial of degree 4 . Prove that $f$ is the zero polynomial if and only if $f: k^{2} \rightarrow k$ is the zero function. [Pay attention to the fact that $f$ is a polynomial of two variables, not one variable!! I want a clear, well-argued proof.]
d) Parts b) and c) look somewhat contradictory at first glance. In b), a non-zero polynomial can define the zero function, while in c) you show that a non-zero polynomial can't define the zero function. Obviously the only difference is the assumption about the degree. Explain how the degree makes such a big difference here.
2. In class we defined the twisted cubic as $C=\mathbb{V}\left(y-x^{2}, z-x^{3}\right)=\left\{\left(t, t^{2}, t^{3}\right) \mid t \in \mathbb{R}\right\} \subset \mathbb{R}^{3}$ (a parametrization). We say that $X$ is a subvariety of $Y$ if $X$ is an affine variety and $X \subseteq Y$.
a) Prove that $\left\{\left(t, t^{2}, t^{3}\right\} \mid t=1,2,3, \ldots, 319\right\}$ is an affine subvariety of $C$. [Hint: you don't have to find explicit equations for this set!]
b) Prove that $A=\left\{\left(t, t^{2}, t^{3}\right) \mid t\right.$ is an even integer $\}$ is not an affine subvariety of $C$. More precisely, suppose $A$ were an affine variety. Then $A=\mathbb{V}\left(f_{1}, \ldots, f_{r}\right)$, where $f_{i} \in \mathbb{R}[x, y, z]$ for $1 \leq i \leq r$. Derive a contradiction from this. [Hint: your conclusion should not be that $f_{i}$ is the zero polynomial in $\mathbb{R}[x, y, z]$. Carefully explain what you can conclude.]
c) Let $\Lambda$ be any plane in $\mathbb{R}^{3}$. Prove that the intersection of $\Lambda$ with $C$ (i.e. $\Lambda \cap C$ ) consists of at most three points.
d) Find a specific plane $\Lambda$ with the property that $\Lambda \cap C$ consists of
(i) two points;
(ii) one point.

Be sure to explain your answers.
3. This problem is just to get your hands dirty a little bit with ideals and polynomials. Let $R=$ $\mathbb{R}[x, y, z]$. Let

$$
I=\left\langle y-x^{2}, z-x^{3}\right\rangle \text { and } J=\left\langle z-x y, y-x^{2}, y^{2}-x z\right\rangle .
$$

a) Without making any connections to the twisted cubic, prove directly that $I=J$. [You will probably want to prove the two inclusions.]
b) One of the given generators of $J$ is actually redundant, meaning that if you remove it, the ideal doesn't get smaller. Prove it.
4. Let $I$ and $J$ be ideals (in particular neither is empty) in $R=k\left[x_{1}, \ldots, x_{n}\right]$, where $k$ is a field. For each of the following, either prove that it is again an ideal or prove that it is not necessarily an ideal by giving a counterexample. Either way, make sure you provide enough details. Again, you are assuming that $I$ and $J$ are themselves already ideals.
a) $I \cap J$.
b) $I \cup J$.
c) $I \backslash J=\{f \in R \mid f \in I$ but $f \notin J\}$. [Hint: think before you start writing. Remember $J$ is an ideal.]
d) $I: J=\{f \in R \mid f \cdot g \in I$ for all $g \in J\}$.
5. In class we proved that

- If $V, W$ are sets in $k^{n}$ then

$$
V \subseteq W \Rightarrow \mathbb{I}(V) \supseteq \mathbb{I}(W)
$$

where, for a set $X$ in $k^{n}$,

$$
\mathbb{I}(X)=\left\{f \in k\left[x_{1}, \ldots, x_{n}\right] \mid f(P)=0 \text { for all } P \in X\right\}
$$

is defined exactly the same way that it is if $X$ is a variety.

- If $V$ is any set in $k^{n}$ and $W$ is a variety in $k^{n}$ then

$$
\mathbb{I}(V) \supseteq \mathbb{I}(W) \quad \Rightarrow \quad V \subseteq W .
$$

(So far this is a reminder of what we proved in class, not what you have to prove in this problem.)
This problem focuses on why we need the extra assumption, in the second bullet, that $W$ is a variety. Give an example of two sets, $V$ and $W$, such that

$$
\mathbb{I}(V) \supseteq \mathbb{I}(W) \quad \text { but } \quad V \text { is not a subset of } W \text {. }
$$

Be sure to justify your answer; don't just give an example without comment. [Hint: of course if $W$ is a variety then this is impossible, thanks to the second bullet above, so focus on examples where $W$ is not a variety. There's one in this problem set.]

