Math 40510, Algebraic Geometry

Problem Set 2, due Wednesday, April 7, 2021

Note: This is not the entire problem set. It is just the first set of problems I've assigned for Problem Set 1, last modified March 26, 2021. Answers that are sloppy, either from a mathematical point of view or because they are hard to read, will result in points being deducted even if they are technically correct.

1. In this problem, k is a field but it is not necessarily algebraically closed. Let $R = k[x, y]$.
   a) Let $I = \langle x^3, y^5 \rangle$. Prove the following fact about the radical of I.
   $$\sqrt{\langle x^3, y^5 \rangle} = \langle x, y \rangle.$$
   (Please prove both inclusions.)
   b) Give an example of polynomials f and g in R so that
   $$\sqrt{\langle f^3, g^5 \rangle} \neq \langle f, g \rangle.$$
   Be sure to justify your answer.
   c) Let $J = \langle x^2, xy, y^2 \rangle$. Find $\mathbb{V}(J)$.
   d) Let I and J be ideals. In Problem Set 1 we defined what's called the *ideal quotient* (I didn't mention this name at the time) to be
   $$I : J = \{f \in R \mid f \cdot J \subset I\} = \{f \in R \mid fg \in I \text{ for all } g \in J\}$$
   and you showed that this is an ideal. You don't have to re-prove that. Find $I : J$ where $I = \langle x^2, y^2 \rangle$ and J is the ideal given in part 1c. Specifically, find generators for this ideal. Be sure to explain your answer, proving both sides of your equality if necessary (i.e. you should start your answer with the assertion $I : J = \langle f_1, f_2, \ldots, f_s \rangle$ for specific $f_1, f_2, \ldots, f_s$ and prove both inclusions of this equality).

2. Let $R = \mathbb{R}[x, y]$. In this problem you can use the fact that R is a Unique Factorization Domain (UFD): this means that it is an integral domain, and it has the property that every non-zero element f can be written as a product of irreducible polynomials in a unique way, except for scalar multiples (e.g. $(2x^3+4y)(x+5y)$ is not considered to be different from $(x^3+2y)(2x+10y)$ or $2(x^3+2y)(x+5y)$).
   a) Prove that $\langle x^2 + y^2 \rangle$ is a radical ideal.
   b) Prove that $\langle x^2 + 2xy + y^2 \rangle$ is not a radical ideal.

3. In this problem we'll work over the field $\mathbb{R}$. Let V be the parabola defined by $y = x^2$ in $\mathbb{R}^2$ and let W be the tangent line to V at the point $(1, 1)$.
   a) Of course $V = \mathbb{V}(y - x^2)$. Find a polynomial ℓ of degree 1 so that $W = \mathbb{V}(\ell)$.
   b) Let $I = \langle y - x^2, \ell \rangle$ (where ℓ is your answer to (3a)). Prove that I is not a radical ideal.
   c) Let V be the circle of radius 1 centered at $(0, 1)$ and let W be the circle of radius 2 centered at $(0, 2)$. Let
   $$J = \mathbb{I}(V) + \mathbb{I}(W).$$
   Prove that J is not a radical ideal, and find the radical of J. [Hint: you can use without proof the fact that both $\mathbb{I}(V)$ and $\mathbb{I}(W)$ are principal, i.e. are generated by a single polynomial.]

4. For some ideals it happens to be the case that ideal quotients have a nice property. Let's look at an example, and then prove something a bit more general. Throughout this problem assume that k is an infinite field. (The choice of the field shouldn't enter into your arguments.)

   a) If I is an ideal and $F \in R$ (a polynomial ring), show that
   $$I : \langle F \rangle = \{G \in R \mid FG \in I\rangle\}.$$
   In other words, $\langle F \rangle$ contains infinitely many elements, but to see if the set $G \cdot \langle F \rangle$ is contained in I, it's enough to check if the single element GF is in I. This can be used in the remaining parts of this problem.

   b) Let $R = k[x, y, z]$. Let $I = \langle x^2, y^2 \rangle$. Prove that
   $$I : \langle z \rangle = I.$$
   This should be approached purely algebraically. Don't use the geometry-algebra dictionary.

   c) Again let $R = k[x, y, z]$ and let $I = \langle x^2, y^2 \rangle$. Find
   $$I : \langle x \rangle.$$
   Specifically, note that this is not equal to I. [Again, this should be approached purely algebraically. Don't use the geometry-algebra dictionary. Be sure to explain your work; just giving the answer isn't enough.]

   d) Now let $R = k[x_1, \ldots, x_n]$.

   - Let $I = \langle f_2, \ldots, f_t \rangle$ (for some $t \geq 2$);
   - Let $J = \langle f_1, f_2, \ldots, f_t \rangle$. [We're just adding one more generator to I.]
   - Let F be some polynomial.
   - ASSUME that $I : \langle F \rangle = I$. [Don't mix up I and J here!]

     [Part 4b) gives an example to show you that this can happen sometimes, and part 4c shows that sometimes it doesn't happen. Parts 4b) and 4c) are not otherwise connected to this problem.]

   Prove that $\langle F f_1, f_2, \ldots, f_t \rangle : \langle F \rangle = J$. Be sure to indicate where the assumption of the fourth bullet is used.

5. Assume that k is algebraically closed. Let $R = k[x, y, z]$. Also let
   $$I = \langle x^4, x^2y^2, y^4 \rangle \quad \text{and} \quad J = \langle x, y^2 \rangle.$$

   a) Prove that $\mathbb{V}(I) = \mathbb{V}(J)$ in k^3 and find this variety explicitly; (very) briefly explain your answer.

   b) Notice the following facts:

   - $I : J = \langle x^3, xy^3, x^2y^2, y^5 \rangle$.
   - $\mathbb{V}(I : J) = \mathbb{V}(I) = \mathbb{V}(J)$;
   - $\overline{\mathbb{V}(I) \backslash \mathbb{V}(J)} = \emptyset$;
   - $\mathbb{I}(\emptyset) = \langle 1 \rangle$ since k is algebraically closed.

   You don't have to prove any of the above bullet points, but they motivate the following:

   Find the smallest integer $N \geq 1$ so that $I : J^N = \langle 1 \rangle$. Explain your answer: specifically, why does your N work and why is it the smallest? [Hint: J^N has $N + 1$ generators in this case.]