Math 40510, Algebraic Geometry

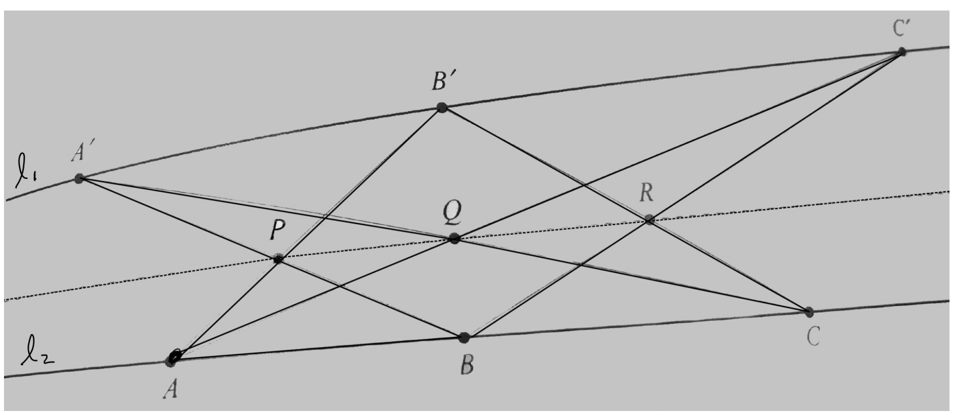
Problem Set 3, due Friday, May 7, 2021

Note: This is the entire problem set. Answers that are sloppy, either from a mathematical point of view or because they are hard to read, will result in points being deducted even if they are technically correct.

1. Find $\mathbb{V}(yz - x^2, yz - 4z^2 + x^2)$ as a subvariety of $\mathbb{P}^2_{\mathbb{R}}$ by hand. Explain your work. [Hint: the answer is a finite set of points in $\mathbb{P}^2_{\mathbb{R}}$].

2. The classical theorem of Pappus says the following. Assume that we have two lines, namely ℓ_1 (containing the points A', B', C') and ℓ_2 (containing the points A, B, C) in $\mathbb{R}^2$ – see the picture below. Note that ℓ_1 and ℓ_2 are not assumed to be parallel. Let

   - P be the intersection of $\overline{AB'}$ and $\overline{A'B}$;
   - Q be the intersection of $\overline{AC'}$ and $\overline{A'C}$;
   - R be the intersection of $\overline{BC'}$ and $\overline{B'C}$.

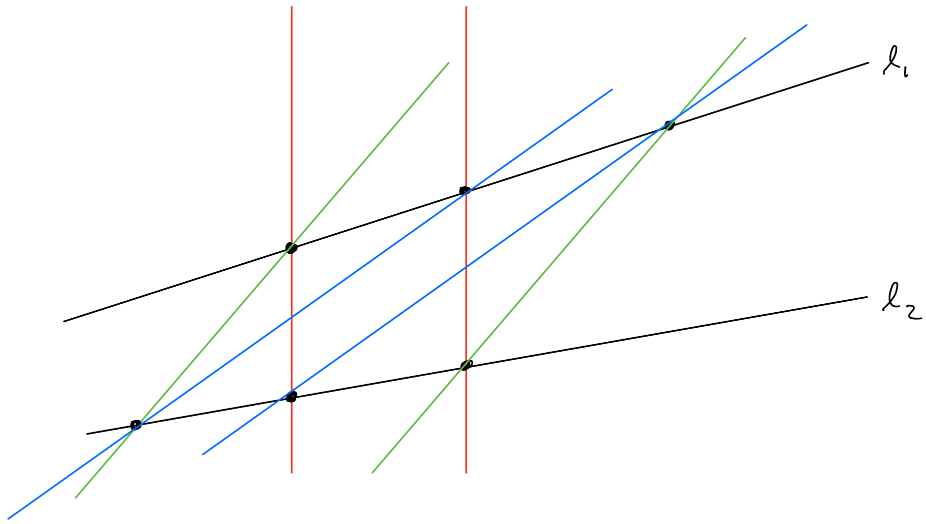


Then the conclusion of Pappus's theorem is that P, Q, R must be collinear. This is just background and you don't have to prove this.

Now suppose the picture is a little bit different. We again start with lines ℓ_1 and ℓ_2, (black in the picture below) which are not necessarily parallel. We have three points, A, B, C on one line and A', B', C' on the other. (You'll have to label the points as part of this problem.) Assume that

- the red lines are parallel,
- the green lines are parallel.

(See the picture below.)



Prove that the blue lines are parallel (viewed in $\mathbb{R}^2$ of course). To do this, assume that Pappus's theorem is a theorem about points in $\mathbb{P}^2_{\mathbb{R}}$ rather than $\mathbb{R}^2$ (you don't have to justify this part), and make a careful study of what happens at infinity.

Note:

- you could answer this with one sentence in a way that would be technically correct, but I want your answer to really reflect the fact that you understand the geometry going on and the difference between the affine geometry and the projective geometry. So please put some thought and detail into your answer!
- Please answer this using Pappus's theorem. **No argument with similar triangles or such!!**
- Make sure your answer includes a labelled picture! The black dots need to be labelled A, A', B, B', C, C' in some suitable order, and you should refer to this labelled picture in your proof.

3. Let $f(x,y) \in k[x,y]$ be a homogeneous polynomial. Show that there are finitely many points P of $\mathbb{P}^1_k$ (possibly none) where $f(P) = 0$ (in the sense discussed in class, meaning that f vanishes at P no matter which choice of coordinates we choose for P, which is ok since f is homogeneous).

4. We saw in class that if $f \in k[x_0, \ldots, x_n]$ then we can't view f as a function on $\mathbb{P}^n$ because for the same point $P = [a_0, \ldots, a_n]$, different representations of this point give different values when plugged into f. Let's explore what happens if we use rational functions instead of polynomials.

   Throughout this problem, assume that k is an infinite field. You can freely use the fact that over an infinite field, a non-constant polynomial (even in several variables) is never identically 0 at all points of $\mathbb{P}^n_k$. You don't have to prove this fact.

   a) Let $f, g \in k[x_0, \ldots, x_n]$. Prove that if f and g are homogeneous of the same degree then f/g gives a well-defined function on $\mathbb{P}^n \backslash \mathbb{V}(g)$. Be sure to mention why we have to restrict to $\mathbb{P}^n \backslash \mathbb{V}(g)$ instead of all of $\mathbb{P}^n$.

   b) For simplicity, now assume that we have only two variables, x, y. Give an example to show that if f and g are homogeneous of different degrees then f/g is not well-defined as a function on $\mathbb{P}^1 \backslash \mathbb{V}(g)$. (Make sure you explain why it's not well-defined – it's enough to exhibit one point where you show it's not well-defined.)

   c) Again assuming that $f, g \in k[x,y]$, give an example of polynomials f, g that are not homogeneous but have the same degree, such that f/g is again not well-defined as a function on $\mathbb{P}^1 \backslash \mathbb{V}(g)$. (Make sure you explain why it's not well-defined – it's enough to exhibit one point where you show it's not well-defined.)

d) Let $f \in k[x_0, \ldots, x_n]$ be a non-constant polynomial (which may or may not be homogeneous – this is not an issue in this part of the problem). In this part of the problem we'll explore in a bit more detail what we discussed in class.

   (i) Prove that there exists a point $P = (a_0, \ldots, a_n) \in k^{n+1}$ such that $f(ta_0, \ldots, ta_n)$ is not constant as a function of t. [Warning: it's not enough to just say that this is immediate since we assumed that f is not constant at the start of the problem. For example, suppose $f(x, y) = x^2 - y$. This is not constant as a polynomial in x, y, but if $P = (2, 4)$ then $f(2t, 4t) = 0$ no matter what t is, and this a constant.]

   (ii) Explain in a few words why this means that f does not give a well-defined function on $\mathbb{P}^n$ even if f is homogeneous.

5. For this problem recall that the twisted cubic, C, in $\mathbb{R}^3$ was determined, as an affine variety, by

$$C = \{(t, t^2, t^3) \mid t \in \mathbb{R}\} \quad \text{and} \quad \mathbb{I}(C) = \langle x_2 - x_1^2 \ , \ x_3 - x_1^3 \rangle.$$

We also saw that to extend this to $\mathbb{P}^3_{\mathbb{R}}$ we need to add just one point at infinity. Let's elaborate on this a bit.

Let

$$I = \langle x_0x_2 - x_1^2,\ x_0^2x_3 - x_1^3,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2, \rangle \subseteq \mathbb{R}[x_0, x_1, x_2, x_3] = R.$$

This is a homogeneous ideal, so $\mathbb{V}(I)$ defines a subvariety of $\mathbb{P}^3_{\mathbb{R}}$. For convenience, let

$$\begin{array}{rcl} f_1 & = & x_0x_2 - x_1^2 \\ f_2 & = & x_0^2x_3 - x_1^3 \\ f_3 & = & x_0x_3 - x_1x_2 \\ f_4 & = & x_1x_3 - x_2^2. \end{array}$$

a) One of the generators f_1, f_2, f_3, f_4 of I is already a linear combination of the others, so it is redundant (i.e. it can be removed without changing the ideal). Find which one is redundant, and show how it is a linear combination of the other three (with coefficients in R).

b) Once you remove the redundant generator in part 5a), you're left with three generators. Give the dehomogenizations of these three with respect to x_0, and show how one of the three dehomogenizations is again a linear combination of the other two (with coefficients in R).

c) Let $V = \mathbb{V}(I)$ (where again, I is the ideal given above). Using the ideas from class, show that $V \cap U_0 = C$ and use the equations to find the single point at infinity (remembering that the plane at infinity is given by $x_0 = 0$). Show your work to find the point at infinity, but for the whole problem feel free to quote anything from class.

6. In this problem you can use (without proof) the following facts to help you.

- if I is a homogeneous ideal, and if we define

$$[I]_d = \{\text{homogeneous polynomials of degree } d \text{ in } I\},$$

  then $[I]_d$ is a vector space over the field k.

- the number of monomials of degree d in R is $\binom{d+2}{2}$. (So for example if $d = 1$ there are $\binom{1+2}{2} = 3$, namely x, y, z while if $d = 2$ there are $\binom{2+2}{2} = 6$, namely $x^2, xy, xz, y^2, yz, z^2$.)

- If I is an ideal generated by monomials, then I is a homogeneous ideal, and you can always find a basis for $[I]_d$ consisting only of monomials.

For this problem let

$$I = \langle\ x^2,\ xy,\ y^2\ \rangle \subset R = k[x, y, z].$$

(Note that the generators involve only x and y, but the ring has 3 variables.)

a) A basis for $[I]_2$ is clearly given by x^2, xy and y^2, so $[I]_2$ is a 3-dimensional vector space. (You don't have to prove this.) Find a basis for $[I]_3$ and for $[I]_4$. [Hint: don't forget the third variable, and remember that you have to remove repeated terms to find a basis!] As a result, what is the dimension of $[I]_3$ and of $[I]_4$?

b) Based on patterns you see in 6a), find a formula for $\dim[I]_d$ for any $d \geq 2$. Explain your answer. As long as you see the right pattern, I won't be too fussy about proving it. [Hint: you'll need some binomial coefficients. Focus on a pattern for how much you have to subtract because of over-counting.]

c) We maintain the notation that $I = \langle\ x^2,\ xy,\ y^2\ \rangle$. For any degree d, define a function

$$h(d) = \binom{d+2}{2} - \dim[I]_d.$$

(Fun fact: this function is named after our old buddy Hilbert.) Prove that $h(d) = 3$ for all $d \geq 2$. You'll need your answer to part 6b).