Math 40510, Algebraic Geometry

Problem Set 2, due March 24, 2023

<u>**Note**</u>: Answers that are sloppy, either from a mathematical point of view or because they are hard to read, will result in points being deducted even if they are technically correct.

Solutions

1. (7 points) In class we showed that if $R = k[x_1, \ldots, x_n]$, where k is a field, and if I and J are ideals for which $I + J = \langle 1 \rangle$ then $I \cap J = IJ$. Assume that I and J have this property, namely $I + J = \langle 1 \rangle$. Let s and t be arbitrary positive integers. Prove that it's also true that $I^s + J^t = \langle 1 \rangle$.

   Solution:

   Suppose $f \in I$ and $g \in J$ are such that $f + g = 1$. (f and g exist by the assumption $I + J = \langle 1 \rangle$.) Then

$$\begin{aligned} 1 &= 1^{s+t} \\ &= (f+g)^{s+t} \\ &= \left[f^{s+t} + \binom{s+t}{1} f^{s+t-1} g + \binom{s+t}{2} f^{s+t-2} g^2 + \cdots + \binom{s+t}{s} f^s g^t \right] + \\ &\quad \left[\binom{s+t}{s+1} f^{s-1} g^{t+1} + \cdots + \binom{s+t}{s+t-1} f g^{s+t-1} + g^{s+t} \right]. \end{aligned}$$

   Since $f \in I$ we have $f^s \in I^s$, so since every term in the part in the first set of brackets is divisible by f^s we have that the part in the first set of brackets is in I^s. Similarly, the part in the second set of brackets is in J^t. This proves $I^s + J^t = \langle 1 \rangle$.

2. Let $R = k[x_1, \ldots, x_n]$ and let I and J be ideals.

   (i) (7 points) Prove that if I is radical then so is $I : J$.

   Solution:

   Let $f \in R$ satisfy $f^t \in I : J$ for some $t \geq 1$. Let g be an arbitrary element of J. Notice that then $g^t \in J$ as well. Since $f^t \in I : J$ we have $f^t g^t \in I$. But $f^t g^t = (fg)^t \in I$, and I is radical ideal, so $fg \in I$. But g was an arbitrary element of J, so $f \in I : J$ so $I : J$ is a radical ideal.

   (ii) (7 points) Give an example to show that if I is not radical then $I : J$ need not be radical. (So you have to find a suitable I and a suitable J.)

   Solution:

   Let $R = k[x]$, $I = \langle x^3 \rangle$, $J = \langle x \rangle$. Then $x^2 \in I : J$ but $x \notin I : J$, so $I : J$ is not a radical ideal.

3. Let $I, J, K \subset k[x_1, \ldots, x_n]$ be ideals.

   (i) (7 points) Prove that $(I : J) : K = I : JK$.

   Solution:

   We'll prove both inclusions.

   $\subseteq$:

Let $f \in (I : J) : K$. We want to show that $f \in I : JK$, so let g be an arbitrary element of JK. We want to show that $fg \in I$. By definition, we can write

$$g = h_1k_1 + h_2k_2 + \cdots + h_sk_s$$

where $h_i \in J$ and $k_i \in K$ for all $1 \leq i \leq s$. Then we have

$$fg = fh_1k_1 + fh_2k_2 + \cdots + fh_sk_s$$

Focus on any one of these terms, say fh_ik_i. Since $f \in (I : J) : K$ and $k_i \in K$ we know $fk_i \in I : J$. But then since $h_i \in J$, we get $fh_ik_i = (fk_i)h_i \in I$. Finally, fg is a linear combination of terms all in I, so $fg \in I$.

$\supseteq$:
Let $f \in I : JK$. We want to show that $f \in (I : J) : K$. Let $k \in K$. We want to show that $fk \in I : J$. That is, we want to show that $(fk)h \in I$ for all $h \in J$. By associativity and commutativity we want to show that $f(kh) = f(hk) \in I$ for all $k \in K$ and all $h \in J$. But since $f \in I : JK$, this latter statement is automatically true and we are done.

(ii) (7 points) Prove that $(I \cap J) : K = (I : K) \cap (J : K)$.

Solution:

Again we'll prove the two inclusions.

$\subseteq$:
Let $f \in (I \cap J) : K$, so $fk \in (I \cap J)$ for all $k \in K$. We want to show that $f \in I : K$ and $f \in J : K$. For any $k \in K$ we know that $fk \in I \cap J$, so in particular $fk \in I$ and hence $f \in I : K$. Similarly $f \in J : K$ and we are done.

$\supseteq$:
Assume $f \in (I : K) \cap (I : J)$. Let $k \in K$ be an arbitrary element. Since $f \in I : K$ we have $fk \in I$. Since $f \in J : K$ we have $fk \in J$. Thus $fk \in I \cap J$, and $f \in (I \cap J) : K$.

(iii) (7 points) Prove that $I : (J + K) = (I : J) \cap (I : K)$.

Solution:

Again we'll prove the two inclusions.

$\subseteq$:
Let $f \in I : (J + K)$. We want to show that $f \in (I : J) \cap (I : K)$.

Let $h_1 \in J$ and $h_2 \in K$. Notice that not only is $h_1 + h_2 \in J + K$, but in fact also $h_1 = h_1 + 0 \in J + K$ and $h_2 = 0 + h_2 \in J + K$. Our assumption is that $fg \in I$ whenever $g \in J + K$, so we get for free that $fh_1 \in I$ and $fh_2 \in I$, i.e. $f \in (I : J) \cap (I : K)$.

$\supseteq$:
Assume that $f \in (I : J) \cap (I : K)$. Let $h_1 \in J$ and $h_2 \in K$. We want to show that $f(h_1 + h_2) \in I$. But $fh_1 \in I$ since $f \in I : J$, and $fh_2 \in I$ since $f \in I : K$, so $f(h_1 + h_2) = fh_1 + fh_2 \in I$ and we are done.

4. (7 points) If I is a prime ideal, show that $\sqrt{I} = I$.

Solution:

We know that $I \subset \sqrt{I}$ for any ideal, so we only have to prove the reverse inclusion. We'll prove it by induction. If $f^1 \in I$ then obviously $f \in I$. Assume whenever $f \in f^t \in I$ then it follows that $f \in I$. Now consider f^{t+1}. But if $f^{t+1} = f^t f \in$, the fact that I is a prime ideal means that either $f^t \in I$ or $f \in I$. In the latter case we are done. In the former case we know by the inductive hypothesis that $f \in I$, so we are again done.

5. For both of the following sets, find the Zariski closure and prove your answer.

   (i) (7 points) $S = \{(t, t^2, t^3) \in \mathbb{R}^3 \mid t \in \mathbb{Z}\}$.

   Solution:

   The Zariski closure of S is $\overline{S} = \mathbb{V}(\mathbb{I}(S))$ so the main task is to find $\mathbb{I}(S)$. Notice that S lies on the twisted cubic $C = \{(t, t^2, t^3) \mid t \in \mathbb{R}\}$. We claim that $\mathbb{I}(Z) = \mathbb{I}(C)$. Since $S \subset C$, we have that $\mathbb{I}(S) \supseteq \mathbb{I}(C)$. Thus we just have to show that $\mathbb{I}(S) \subseteq \mathbb{I}(C)$. That is, if f vanishes at all the points of S, we have to show that f vanishes along all of C.

   So let $f(x, y, z) \in \mathbb{I}(S)$. That means that $f(t, t^2, t^3) = 0$ for all $t \in \mathbb{Z}$. But this is a polynomial in the single variable t, and it can't have infinitely many zeros unless it is the zero polynomial in t. This means that $f(t, t^2, t^3) = 0$ for all $t \in \mathbb{R}$. But thinking again of $t, t^2, t^3)$ as a point of the twisted cubic, this means that $f(x, y, z)$ vanishes along the whole twisted cubic, i.e. $f \in \mathbb{I}(C)$. So we are done: we get $\overline{S} = C$.

   Notice that we did not need to write explicit equations for f. However, just as extra information, recall that $\mathbb{I}(C) = \langle y - x^2, z - x^3 \rangle$.

   (ii) (8 points) $T = \{(p, q) \in \mathbb{R}^2 \mid p, q \in \mathbb{Z}$ are prime numbers $\}$.

   Solution:

   The Zariski closure of T is $\overline{T} = \mathbb{V}(\mathbb{I}(T))$, so again we have to find $\mathbb{I}(T)$. We claim that $\mathbb{I}(T) = \langle 0 \rangle$. This means that $\overline{T} = \mathbb{R}^2$. Let $f(x, y)$ be a polynomial vanishing at every point of T. We have to be a bit careful because T lives only in the first quadrant, and has no points (x, y) with $x = 0$ or 1 and no points with $y = 0, 1$.

   But let $f(x, y)$ be a polynomial vanishing at every point of T. Let's decompose $f(x, y)$ in terms of powers of y:

$$f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots + f_d(x)y^d, \tag{1}$$

   where d is the highest power of y occurring anywhere in $f(x, y)$.

   Consider the polynomial $f(2, y)$, which is a polynomial in one variable vanishing for every prime number that we plug in for y. There are infinitely many such primes that we could plug in, so $f(2, y)$ is the zero polynomial in y. This means that

$$f_0(2) = 0, \; f_1(2) = 0, \; f_2(2) = 0, \ldots, f_d(2) = 0.$$

   But we can do the same with $x = 3, 5, 7, 11, \ldots$. Thus for $0 \le i \le d$, $f_i(x)$ vanishes for infinitely many x (namely all the prime numbers), so $f_i(x)$ is the zero polynomial. But then using the decomposition (1), $f(x, y)$ is the zero polynomial, and we're done.

6. In this problem we work over the real numbers, $\mathbb{R}$. Let

$$D = \{(t^a, t^b, t^c, t^d) \mid t \in \mathbb{R}^4\},$$

where a, b, c, d are given positive integers. For any part of this problem you can use without proof the fact that D is a variety in $\mathbb{R}^4$ for any choice of a, b, c, d, but you can't choose your own values for a, b, c, d.

(i) (7 points) If $a = b = c = d = 3$, describe the variety geometrically and find the defining polynomials as a variety in $\mathbb{R}^4$.

Solution:

If $a = b = c = d = 3$ then the four coordinates of any point of D are equal, and since the exponent is odd we can produce any point (s, s, s, s) in this way (since any real number has a cube root).

So D is a line in $\mathbb{R}^4$ defined as $\mathbb{V}(x - y, x - z, x - w)$.

(ii) (7 points) From now on assume a, b, c, d are arbitrary. Prove that $D \subset \mathbb{R}^4$ is irreducible.

Solution:

It's enough to prove that $\mathbb{I}(D)$ is a prime ideal. Suppose $fg \in \mathbb{I}(D)$. We want to show that either $f \in \mathbb{I}(D)$ or $g \in \mathbb{I}(D)$. We have $(fg)(P) = f(P)g(P) = 0$ for all $P \in D$. But every point of D is of the form (t^a, t^b, t^c, t^d) for some $t \in \mathbb{R}$. That is,

$$f(t^a, t^b, t^c, t^d)g(t^a, t^b, t^c, t^d) = 0$$

for all $t \in \mathbb{R}$. So now thinking of t as a variable, the polynomial $H(t) := f(t^2, t^3, t^5)g(t^2, t^3, t^5)$ has infinitely many roots, hence is the zero polynomial (since $\mathbb{R}$ is infinite). But $\mathbb{R}[t]$ is an integral domain, so either $f(t^a, t^b, t^c, t^d)$ or $g(t^a, t^b, t^c, t^d)$ is the zero polynomial in $\mathbb{R}[t]$. This means that either $f(x, y, z, w)$ or $g(x, y, z, w)$ vanishes at every point of D, so either $f \in \mathbb{I}(D)$ or $g \in \mathbb{I}(D)$. Hence $\mathbb{I}(D)$ is a prime ideal and D is irreducible.

(iii) (7 points) If P is the point $(1, 2, 3, 4)$ in $\mathbb{R}^4$, prove that the ideal $\mathbb{I}(D \cup \{P\})$ is not a prime ideal.

Solution:

It is enough to prove that $X := D \cup \{P\}$ is not irreducible. The key observation is that P does not lie on D, since if $(1, 2, 3, 4) = (t^a, t^b, t^c, t^d)$ then t would have to be either 1 or -1 to get the 1 in the first coordinate, and then we could never get the second, third or fourth coordinates of P. But we know that $W_1 = D$ is a variety and $W_2 = \{P\}$ is a variety, that $X = W_1 \cup W_2$, and that neither of W_1 or W_2 is equal to X. So X is not irreducible.

(iv) (7 points) If P is the point $(1, 1, 1, 1)$ in $\mathbb{R}^4$, prove that the ideal $\mathbb{I}(D \cup \{P\})$ *is* a prime ideal.

Solution:

This is the same situation as part (iii), but now $P \in D$ (take $t = 1$). Then $D \cup \{P\} = D$ so the union is irreducible and hence the ideal is prime.

7. (8 points) Let $R = \mathbb{C}[x, y, z, w]$ and let

$$D = \{(t^a, t^b, t^c, t^d) \mid t \in \mathbb{C}^4\},$$

where a, b, c, d are given positive integers. You can use without proof the following two facts:

- D is a variety in $\mathbb{C}^4$;
- D is irreducible (since the argument in any case would be the same as what you just did in Problem #6 for $\mathbb{R}$).

However, as in Problem #6 you can't choose your own values for a, b, c, d.

Let

$$J = \mathbb{I}(D)^3 = \left\{ \sum_{i=1}^{s} a_i f_i g_i h_i \mid a_i \in R \text{ and } f_i, g_i, h_i \in \mathbb{I}(D) \right\}.$$

(I'm just reminding you what the cube of an ideal means.) Prove that $\sqrt{J}$ is a prime ideal.

Solution:

Since we're working over an algebraically closed field now, we can use the Nullstellensatz. We thus have that

$$\mathbb{I}(\mathbb{V}(J)) = \sqrt{J}$$

so it's enough to prove that $\mathbb{V}(J)$ is irreducible. We claim that

$$\mathbb{V}(J) = \mathbb{V}(\mathbb{I}(D)) = D.$$

The second equality is immediate because D is a variety, so we want to show the first equality. We'll prove the two inclusions.

$\supseteq$:
We know that $J = \mathbb{I}(D)^3 \subset \mathbb{I}(D)$ (this is true for any ideal) so

$$\mathbb{V}(J) = \mathbb{V}(\mathbb{I}(D^3)) \supseteq \mathbb{V}(\mathbb{I}(D))$$

$\subseteq$:
Let $P \in \mathbb{V}(J)$. We want to show that $P \in \mathbb{V}(\mathbb{I}(D))$. That is, for any $f \in \mathbb{I}(D)$ we want to show that $f(P) = 0$. But $f^3 \in \mathbb{I}(D)^3 = J$, and $P \in \mathbb{V}(J)$, so $f^3(P) = (f(P))^3 = 0$. Then $f(P) = 0$ as desired.