Math 40510, Algebraic Geometry

Problem Set 3, due April 28, 2023

Solutions

In class we gave the axiomatic approach for a projective plane $\mathbb{P}^2$, and we now recall the axioms we used. Remember that $\mathbb{P}^2$ consists of a set $\mathfrak{P}$ of points and a collection $\mathfrak{L}$ of subsets called *lines*, satisfying these axioms (from Moorhouse):

(M1) Given any two points P, Q there is a unique line $\overline{PQ}$ containing both P and Q.
(M2) Given any two lines ℓ, m there is a unique point P lying both on ℓ and on m (i.e. any two lines meet in a point).
(M3) There exist four points such that no three are collinear.

We derived a lot from these axioms, and then we constructed the classical projective planes $\mathbb{P}^2_k$ (where k is a field) as examples of projective planes.

Later in the book Moorhouse also gives axioms for projective n-space. Let's just look at the case $n = 3$. Temporarily forget about the construction of the classical $\mathbb{P}^3$ that we gave (lines through the origin in k^4) and let's focus on the axioms.

Here are the axioms for $\mathbb{P}^3$ obtained by specializing the Moorhouse axioms and tweaking a little bit for convenience. We will say that:

$\mathbb{P}^3$ consists of a set $\mathfrak{P}$ of *points*, a collection $\mathfrak{L}$ of *lines* and a collection $\mathfrak{H}$ of *planes*,

satisfying the following axioms.

(S1) Any two distinct points lie on exactly one line.
(S2) Any two distinct planes meet in exactly one line.
(S3) If a plane contains a line, it contains all the elements of that line.
(S4) Two distinct lines meet in a point if and only if they lie in a common plane.
(S5) There exists a set of five points, of which no four lie in a common plane.
(S6) Every line contains at least three points.
(S7) if X is a plane and P_1, P_2 are points of X then X contains the entire line spanned by P_1 and P_2 (whose existence is guaranteed by (S1)).

For each of the following problems you can refer to any earlier problem in addition to using the axioms. You can do this regardless of whether you were able to prove the earlier problem or not.

Problem 1. Prove that three noncollinear points lie on a unique plane. (Be sure to prove uniqueness as part of your answer.)

Solution:

Let our three points be P_1, P_2, P_3. By (S1), P_1 and P_2 span the line $\overline{P_1P_2}$, and P_2 and P_3 span the line $\overline{P_2P_3}$. These lines are distinct since the three points are noncollinear. These two lines meet at the point P_2, so by (S4) they both lie in a plane H. In particular, by (S7) H contains the three points P_1, P_2, P_3.

Suppose that there is a different plane H' also containing the three points P_1, P_2, P_3. By (S7), H' contains the line $\overline{P_1P_2}$ and also the line $\overline{P_2P_3}$, which (again) are different lines. But these two lines are also contained in H. This contradicts (S2). Thus H is unique.

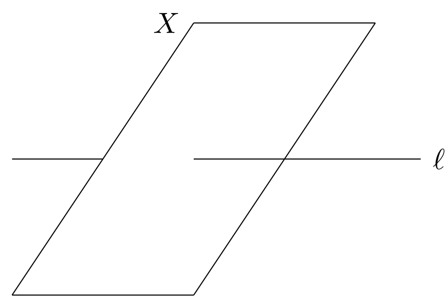
Problem 2. Given any line ℓ and any point P not on ℓ, prove that there exists a unique plane containing both P and ℓ.

Solution:

Let P_1, P_2 be points of ℓ (OK by (S6)). Then $\overline{P_1P_2} = \ell$ since two points determine a unique line (S1). By Problem 1 there is a unique plane H containing P_1, P_2, P.

We next claim that H has to contain ℓ as well. (So far all we know is that H contains P_1, P_2 and P.) The line $\overline{PP_1}$ meets the line ℓ at the point P_1, so by (S4) these two lines lie in a common plane, and by the argument in Problem 1 this plane is the unique plane containing P, P_1, P_2. So H contains ℓ and P and is the unique plane doing that.

Problem 3. Let X be any plane and let ℓ be any line not contained in X. Prove that X must meet ℓ in exactly one point.



Solution:

Let P be any point of $\mathbb{P}^3$ not on ℓ. This is OK by (S5). It doesn't matter if P is on X or not. By Problem 2 there is a unique plane, say H, containing both P and ℓ. By (S2), X and H meet in a line, say m. Then m lies on X and m lies on H. But ℓ also lies on H, so by Problem 4, ℓ and m meet in a point, say Q. Clearly Q is on ℓ. But Q is also on m, which is entirely contained in X. So by (S3), Q also lies on X. Thus X meets ℓ at the point Q.

Problem 4. If X is a plane, show that it is a $\mathbb{P}^2$; i.e. show that axioms (M1), (M2), (M3) hold.

Solution:

(M1): this is the same as (S1).

(M2): If ℓ and m are lines on X then they lie on a common plane, so they meet in a point thanks to (S4).

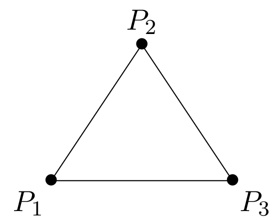
(M3): By (S5) there is a set of five points, call it $Z = \{P_1, P_2, P_3, P_4, P_5\}$, no four of which are in a common plane. In particular, at most three points of Z are on the plane X. Choose one of these five points that is not on X; say without loss of generality that it is P_1. Define a function $\pi_{P_1} : Z \backslash \{P_1\} \to X$ as follows (basically it is a projection). For any $Q \in Z \backslash \{P_1\}$, there is a unique line $\overline{P_1Q}$ containing both P_1 and Q by (S1). This line meets X in one point, by Problem 3, since $P_1 \notin X$. We define $\pi_{P_1}(Q)$ to be this point $X \cap \overline{P_1Q}$. (Note that if Q is on X then $Q = \pi_{P_1}(Q)$.)

In this way we get four points on X, namely $\pi_{P_1}(P_2), \pi_{P_1}(P_3), \pi_{P_1}(P_4), \pi_{P_1}(P_5)$. We claim that no three of these are on a line. Suppose, for example, that $\pi_{P_1}(P_2), \pi_{P_1}(P_3)$, $\pi_{P_1}(P_4)$ lie on a line ℓ. By Problem 2, ℓ and P_1 span a plane, H. By the way this has been constructed, and using (S3), this forces P_1, P_2, P_3, P_4 to all lie on H; for instance,

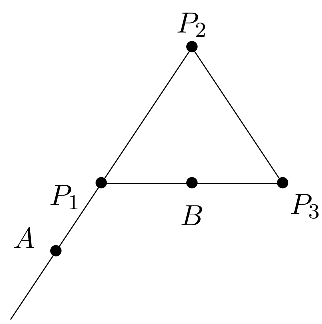
H contains $\pi_{P_1}(P_2)$ and it contains P_1, so it contains the line spanned by these points, which by construction contains P_2, so $P_2 \in H$. This contradicts the choice of the five points.

Problem 5. Prove that every line meeting two sides of a triangle, but none of its vertices, must also meet the third side. More precisely:

Consider the triangle P_1, P_2, P_3. We'll interpret this as the data consisting of the three points P_1, P_2, P_3 together with the corresponding three lines that they span pairwise, which we'll denote $\overline{P_1P_2}$, $\overline{P_1P_3}$, $\overline{P_2P_3}$ (this is OK by (S1)).



Choose a third point, A, on $\overline{P_1P_2}$ and a third point, B, on $\overline{P_1P_3}$ (this is ok by (S6)):



Then prove that the line $\overline{AB}$ has to meet the line $\overline{P_2P_3}$ in a point.

Solution:

By Problem 1, P_1, P_2, P_3 lie on a unique plane, H. By (S7), the lines $\overline{P_1P_2}$, $\overline{P_1P_3}$ and $\overline{P_2P_3}$ all lie on H, and by (S3) all the points on these lines also lie on H. In particular, A and B lie on H so again by (S7) the line $\overline{AB}$ lies on H. By Problem 4, the axioms for $\mathbb{P}^2$ hold for H. But then by (M2) the lines $\overline{AB}$ and the line $\overline{P_2P_3}$ meet at a point (on H).

For Problems 6, 7 and 8, feel free to use any facts we proved in class about $\mathbb{P}^2$.

Problem 6. If the $\mathbb{P}^3$ is finite, show that any two lines of $\mathbb{P}^3$ have the same number of elements, which we'll still call $d+1$. (Note that the two lines don't necessarily meet in a point, so they're not necessarily in the same plane.)

Solution:

Let ℓ_1 and ℓ_2 be distinct lines in $\mathbb{P}^3$. They can meet in either zero points or one point: indeed, by (S1) if they meet in two or more points then they are the same line.

<u>Case 1</u>: *ℓ_1 and ℓ_2 meet in a point.* Then by (S4) the two lines lie in a common plane. By Problem 4, this plane satisfies the axioms of a projective plane, so by what we did in class they have the same number of elements.

Case 2: *ℓ_1 and ℓ_2 are disjoint.* Let $P_1 \in \ell_1$ and $P_2 \in \ell_2$ be points. By (S1) there is a unique line, ℓ, spanned by P_1 and P_2. ℓ meets ℓ_1 and ℓ_2 in one point each, since two distinct lines meet in at most one point. By (S4), then, ℓ_1 and ℓ lie in a common plane and by Case 1 it follows that ℓ_1 and ℓ have the same number of points. Similarly, ℓ and ℓ_2 have the same number of points. Then ℓ_1 and ℓ_2 have the same number of points and we are done.

Problem 7. Show that any two planes contain the same number of elements. What is that number (in terms of the integer d in Problem 6)? Explain your answer. Again, feel free to use earlier problems or results from class.

Solution:

We have seen in Problem 4 that any plane satisfies the axioms for $\mathbb{P}^2$, and we showed in class that $\mathbb{P}^2$ has $d^2 + d + 1$ points. Thus any two planes have that number of points.

Problem 8. In terms of d (as in Problem 6), how many points are in $\mathbb{P}^3$? Explain your answer using the axioms and previous problems.

Solution:

Fix a line ℓ_1 and a line ℓ_2 disjoint from ℓ_1 in $\mathbb{P}^3$. Let $\mathcal{L}_1$ be the collection of planes containing ℓ_1. No plane H in $\mathcal{L}_1$ also contains ℓ_2, since otherwise we'd have two disjoint lines in H, and we saw in Problem 4 that H is a copy of $\mathbb{P}^2$, where no two lines can be disjoint.

There is a bijection between the elements of $\mathcal{L}_1$ and the points of ℓ_2 as follows:

$$\phi : \mathcal{L}_1 \to \ell_2$$

given by $\phi(H) = H \cap \ell_2$. This is well-defined because we just saw that H does not contain ℓ_2, so by Problem 3, H meets ℓ_2 in a point (of ℓ_2). It is a bijection because each H gives a different point of ℓ_2 (so ϕ is one-to-one), and given any $P \in \ell_2$ there is an H containing it by Problem 2 (so ϕ is onto).

Now we count. There are $d^2 + d + 1$ points on each H by Problem 4 and what we did in class, and there are $d + 1$ planes in $\mathcal{L}_1$ since that is the number on ℓ_2 and we have the bijection ϕ. This gives $(d^2 + d + 1)(d + 1)$ points. But we overcounted: we counted the points on ℓ_1 $(d + 1)$ times so the number of points in $\mathbb{P}^3$ is

$$(d^2 + d + 1)(d + 1) - d(d + 1) = (d^3 + 2d^2 + 2d + 1) - (d^2 + d) = d^3 + d^2 + d + 1.$$

Now we'll focus on the classical projective spaces $\mathbb{P}^n_k$ and their varieties, using homogeneous coordinates.

Problem 9. In this problem we work over $\mathbb{R}$. Let's look at the projective twisted cubic over the real numbers. Consider the mapping (function)

$$\phi : \mathbb{P}^1 \to \mathbb{P}^3$$

defined by $\phi([s, t]) = [s^3, s^2t, st^2, t^3]$.

(i) Show that this mapping is well-defined.

Solution:

We know that $[s,t] = [\lambda s, \lambda t]$ for any $\lambda \in \mathbb{R}$. Then

$$\phi([\lambda s, \lambda t]) = [(\lambda s)^3, (\lambda s)^2(\lambda t), (\lambda s)(\lambda t)^2, (\lambda t)^3] = [\lambda^3 s^3, \lambda^3 s^2 t, \lambda^3 st^2, \lambda^3 t^3] = [s^3, s^2t, st^2, t^3]$$

since we pull out the same λ^3 from each component.

(ii) Explain why (for example) the mapping $\phi'([s,t]) = [s^3, s^2t, st^2, t^4]$ would not be well-defined. Make sure you give enough detail in your answer.

Solution:

Making the same calculation as in the last part, we'd have

$$\phi([\lambda s, \lambda t]) = [\lambda^3 s^3, \lambda^3 s^2 t, \lambda^3 st^2, \lambda^4 t^4].$$

But now we can't pull the same scalar from each component, since most of the time $\lambda^3 \neq \lambda^4$. So this is not equal to $[s^3, s^2t, st^2, t^3]$.

(iii) Let $V = \phi(\mathbb{P}^1)$ be the image of $\mathbb{P}^1$ in $\mathbb{P}^3$ under the mapping ϕ. Prove that $V \cap U_0$ (where U_0 is the affine part of $\mathbb{P}^3$ as defined in class) is the affine twisted cubic mentioned in class.

Solution:

We know $V = \{[s^3, s^2t, st^2, t^3] \mid [s,t] \in \mathbb{P}^1\}$. We also know $U_0 = \{[a,b,c,d] \in \mathbb{P}^3 \mid a \neq 0\}$. So every point of $V \cap U_0$ comes from a point $[s,t] \in \mathbb{P}^1$ such that $s \neq 0$ (since we need $s^3 \neq 0$). That is,

$$V \cap U_0 = \{[s^3, s^2t, st^2, t^3 \mid s \neq 0\} = \left\{ \left[1, \left(\frac{t}{s}\right), \left(\frac{t}{s}\right)^2, \left(\frac{t}{s}\right)^3 \right] \mid s \neq 0 \right\}$$

Since $\frac{t}{s}$ can take any real value, this is the affine twisted cubic under the identification of U_0 with $\mathbb{R}^3$.

(iv) Consider the matrix

$$A = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$$

Find the maximal minors of A.

Solution:

$$x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2.$$

(v) Let I be the ideal generated by the maximal minors of A that you found in (9iv). Show that $\mathbb{V}(I) = V$. (Make sure you show both inclusions.)

Solution:

We want to show that $V = \mathbb{V}(x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2)$.

$\subseteq$:
Let $P \in V$, so $P = [s^3, s^2t, st^2, t^3]$ for some s, t. Plugging in

$$x_0 = s^3,\ x_1 = s^2t,\ x_2 = st^2,\ x_3 = t^3$$

it's clear that all three given polynomials vanish at P.

$\supseteq$:
Let $P = [a,b,c,d] \in \mathbb{V}(x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2)$.

Case 1: $a = 0$, so $P = [0,b,c,d]$. We're assuming that each of the three polynomials vanishes at P. This gives

$$\begin{aligned}
&(0)(c) - b^2 = 0 \ \Rightarrow b = 0\\
&(0)(d) - (0)(c) = 0 \ \Rightarrow \quad \text{no new information}\\
&(0)(d) - c^2 = 0 \ \Rightarrow c = 0
\end{aligned}$$

where on each line we use information we got earlier. We conclude $P = [0,0,0,1]$. This is indeed a point of V (take $s = 0,\ t = 1$).

Case 2: $a \neq 0$, so without loss of generality we can take $P = [1,b,c,d]$. Since $P \in \mathbb{V}(x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_1x_3 - x_2^2)$, this means

$$c - b^2 = 0, \quad d - bc = 0, \quad bd - c^2 = 0.$$

Thus

$$c = b^2, \quad d = bc = b^3, \quad b^4 - b^4 = 0.$$

In particular, $P = [1,b,b^2,b^3]$ is of the desired form, taking $s = 1, t = b$.