Math 40510, Algebraic Geometry

Problem Set 1, due February 17, 2023

Note: Answers that are sloppy, either from a mathematical point of view or because they are hard to read, will result in points being deducted even if they are technically correct.

1. Let k be a field and let $R = k[x_1, \dots, x_n]$, the polynomial ring in n variables over k.

   a) Prove that for any positive integer $n \geq 1$, $R = k[x_1, \dots, x_n]$ has the property that if $f, g \in R$ and $f \neq 0, g \neq 0$, then $fg \neq 0$. (Recall that this is the main step in showing that R is an integral domain.) [Hint: In class we showed this for $n = 1$, and you can use this fact without proving it again. Now use induction on n. Notice that we can lump together the terms according to the power of one variable, for example

$$y^2+z^3+xy^5z^4+xy^4+x^2y^3z^4+x^2yz^5+x^2z^7+x^3y^2z = x^0(y^2+z^3)+x(y^5z+y^4)+x^2(y^3z^4+yz^5+z^7)+x^3(y^2z).]$$

   b) If $f, g \in R$, prove that $\deg(fg) = \deg(f) + \deg(g)$. (We know it for monomials, but you have to show that things don't get messed up when you use polynomials even though cancelation of terms can occur in a product.) [Hint: collect terms of the same degree together. For example,

$$y^2 + z^2 + xy^5z^4 + xy^4 + x^2y^3z^5 + x^2yz^2 + x^3z^7 + x^3yz$$

$$= (y^2 + z^2) + (xy^4 + x^2yz^2 + x^3yz) + (xy^5z^4 + x^2y^3z^5 + x^3z^7).]$$

2. In class we proved that if k is an infinite field and $f \in k[x_1, \dots, x_n]$ then the following are equivalent:

   - f is the zero polynomial.
   - The evaluation function $f : k^n \to k$, defined by $f(P) = f(b_1, \dots, b_n)$ for $P = (b_1, \dots, b_n) \in k^n$, is the zero function. (I.e. f, evaluated at any point of k^n, vanishes.)

   Now instead we consider a finite field. Let p be a prime and consider the field $\mathbb{Z}_p$. Give an example of a polynomial $f \in \mathbb{Z}_p[x, y]$ for which $f : \mathbb{Z}_p^2 \to \mathbb{Z}_p$ vanishes at **all but one** point of $\mathbb{Z}_p^2$. (Specifically, it has to fail to vanish at one and only one point of $\mathbb{Z}_p^2$.) Be sure to prove why your example works – it's not enough to just give the polynomial. [Hint: Fermat's Little Theorem.]

3. If k is an infinite field, prove that the phenomenon in Problem 2 can't happen. That is, prove that if $f \in k[x, y]$ and $f(x, y) = 0$ when evaluated at every point of k^2 except one specific point (a, b) then we must also have $f(a, b) = 0$. (Make sure to indicate the relevance of the assumption that k is infinite.)

4. Let $D \subset \mathbb{R}^3$ be the set of points

$$D = \{(t^2, t^3, t^5) \mid t \in \mathbb{R}\}.$$

   For instance, the point $(2^2, 2^3, 2^5) = (4, 8, 32) \in \mathbb{R}^3$ is a point of D.

   a) Prove that D is an affine variety. Specifically, find polynomials $f_1, \dots, f_s$ (you get to decide what s is) so that $D = \mathbb{V}(f_1, \dots, f_s)$. Make sure you prove both inclusions, $\subseteq$ and $\supseteq$.

   b) If $f \in \mathbb{R}[x, y, z]$ and $D \not\subset \mathbb{V}(f)$ (i.e. f does not vanish on all of D), show that $\mathbb{V}(f) \cap D$ consists of at most $5 \cdot \deg(f)$ points.

   c) Let $\phi : \mathbb{R} \to \mathbb{R}^3$ be the function defined by $\phi(t) = (t^2, t^3, t^5)$. Prove that ϕ is one-to-one.

   d) We first make the following definition:

> For any field k, if W is a set in k^n and V is a variety in k^n, we say that W is a *subvariety* of V if $W \subseteq V$ and W is itself a variety in k^n. We say that W is a *proper* subvariety of V if, in addition, $W \subsetneq V$.

In the context of the current problem, prove that a set W is a proper subvariety of D (the variety defined in a)) if and only if W consists of a finite set of points on D. You can use anything we talk about in class in your answer. (The implication $\Leftarrow$ should only take a line or two but $\Rightarrow$ will take a little more work.)

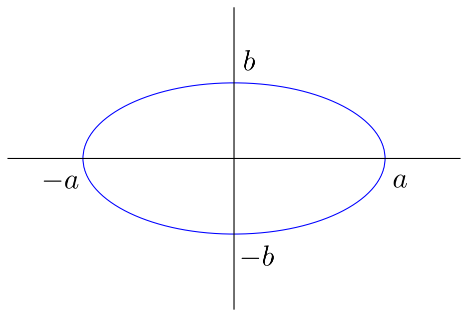
5. Let a and b be positive real numbers and let

$$V = \mathbb{V}(b^2x^2 + a^2y^2 - a^2b^2).$$

Notice that V is the solution set of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

so V is an ellipse. (I'm not asking you to prove this; I'm just pointing out the fact.)



a) Mimicking what we did in class, find a rational parametrization of V. [Hint: try setting t to be the slope of a line through the point $(-a, 0)$ as we did in class.]

b) Check your answer by plugging in two specific choices of t to see if you get a point of V both times.