Math 40510, Algebraic Geometry

Problem Set 2, due March 24, 2023

<u>**Note**</u>: Answers that are sloppy, either from a mathematical point of view or because they are hard to read, will result in points being deducted even if they are technically correct.

1. In class we showed that if $R = k[x_1, \ldots, x_n]$, where k is a field, and if I and J are ideals for which $I + J = \langle 1 \rangle$ then $I \cap J = IJ$. Assume that I and J have this property, namely $I + J = \langle 1 \rangle$. Let s and t be arbitrary positive integers. Prove that it's also true that $I^s + J^t = \langle 1 \rangle$.

2. Let $R = k[x_1, \ldots, x_n]$ and let I and J be ideals.

   (i) Prove that if I is radical then so is $I : J$.

   (ii) Give an example to show that if I is not radical then $I : J$ need not be radical. (So you have to find a suitable I and a suitable J.)

3. Let $I, J, K \subset k[x_1, \ldots, x_n]$ be ideals.

   (i) Prove that $(I : J) : K = I : JK$.

   (ii) Prove that $(I \cap J) : K = (I : K) \cap (J : K)$.

   (iii) Prove that $I : (J + K) = (I : J) \cap (I : K)$.

4. If I is a prime ideal, show that $\sqrt{I} = I$.

5. For both of the following sets, find the Zariski closure and prove your answer.

   (i) $S = \{(t, t^2, t^3) \in \mathbb{R}^3 \mid t \in \mathbb{Z}\}$.

   (ii) $T = \{(p, q) \in \mathbb{R}^2 \mid p, q \in \mathbb{Z} \text{ are prime numbers } \}$.

6. In this problem we work over the real numbers, $\mathbb{R}$. Let
$$D = \{(t^a, t^b, t^c, t^d) \mid t \in \mathbb{R}^4\},$$
   where a, b, c, d are given positive integers. For any part of this problem you can use without proof the fact that D is a variety in $\mathbb{R}^4$ for any choice of a, b, c, d, but you can't choose your own values for a, b, c, d.

   (i) If $a = b = c = d = 3$, describe the variety geometrically and find the defining polynomials as a variety in $\mathbb{R}^4$.

   (ii) From now on assume a, b, c, d are arbitrary. Prove that $D \subset \mathbb{R}^4$ is irreducible.

   (iii) If P is the point $(1, 2, 3, 4)$ in $\mathbb{R}^4$, prove that the ideal $\mathbb{I}(D \cup \{P\})$ is not a prime ideal.

   (iv) If P is the point $(1, 1, 1, 1)$ in $\mathbb{R}^4$, prove that the ideal $\mathbb{I}(D \cup \{P\})$ *is* a prime ideal.

7. Let $R = \mathbb{C}[x, y, z, w]$ and let
$$D = \{(t^a, t^b, t^c, t^d) \mid t \in \mathbb{C}^4\},$$
   where a, b, c, d are given positive integers. You can use without proof the following two facts:

   - D is a variety in $\mathbb{C}^4$;
   - D is irreducible (since the argument in any case would be the same as what you just did in Problem #6 for $\mathbb{R}$).

However, as in Problem #6 you can't choose your own values for a, b, c, d.

Let

$$J = \mathbb{I}(D)^3 = \left\{ \sum_{i=1}^{s} a_i f_i g_i h_i \mid a_i \in R \text{ and } f_i, g_i, h_i \in \mathbb{I}(D) \right\}.$$

(I'm just reminding you what the cube of an ideal means.) Prove that $\sqrt{J}$ is a prime ideal.