Math 40510, Algebraic Geometry

Problem Set 3, due April 28, 2023

<u>**Note**</u>: Answers that are sloppy, either from a mathematical point of view or because they are hard to read, will result in points being deducted even if they are technically correct.

In class we gave the axiomatic approach for a projective plane $\mathbb{P}^2$, and we now recall the axioms we used. Remember that $\mathbb{P}^2$ consists of a set $\mathfrak{P}$ of points and a collection $\mathfrak{L}$ of subsets called *lines*, satisfying these axioms (from Moorhouse):

(M1) Given any two points P, Q there is a unique line $\overline{PQ}$ containing both P and Q.
(M2) Given any two lines ℓ, m there is a unique point P lying both on ℓ and on m (i.e. any two lines meet in a point).
(M3) There exist four points such that no three are collinear.

We derived a lot from these axioms, and then we constructed the classical projective planes $\mathbb{P}^2_k$ (where k is a field) as examples of projective planes.

Later in the book Moorhouse also gives axioms for projective n-space. Let's just look at the case $n = 3$. Temporarily forget about the construction of the classical $\mathbb{P}^3$ that we gave (lines through the origin in k^4) and let's focus on the axioms.

Here are the axioms for $\mathbb{P}^3$ obtained by specializing the Moorhouse axioms and tweaking a little bit for convenience. We will say that:

$\mathbb{P}^3$ consists of a set $\mathfrak{P}$ of *points*, a collection $\mathfrak{L}$ of *lines* and a collection $\mathfrak{H}$ of *planes*,

satisfying the following axioms.

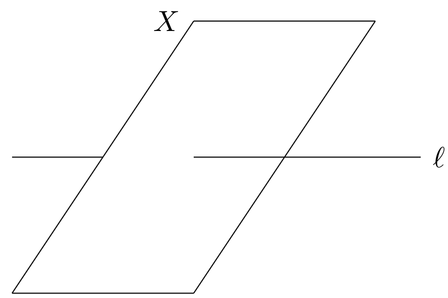
(S1) Any two distinct points lie on exactly one line.
(S2) Any two distinct planes meet in exactly one line.
(S3) If a plane contains a line, it contains all the elements of that line.
(S4) Two distinct lines meet in a point if and only if they lie in a common plane.
(S5) There exists a set of five points, of which no four lie in a common plane.
(S6) Every line contains at least three points.
(S7) if X is a plane and P_1, P_2 are points of X then X contains the entire line spanned by P_1 and P_2 (whose existence is guaranteed by (S1)).

For each of the following problems you can refer to any earlier problem in addition to using the axioms. You can do this regardless of whether you were able to prove the earlier problem or not.

Problem 1. Prove that three noncollinear points lie on a unique plane. (Be sure to prove uniqueness as part of your answer.)

Problem 2. Given any line ℓ and any point P not on ℓ, prove that there exists a unique plane containing both P and ℓ.

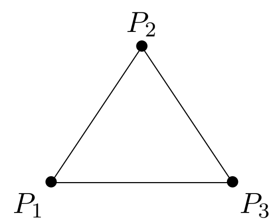
Problem 3. Let X be any plane and let ℓ be any line not contained in X. Prove that X must meet ℓ in exactly one point.



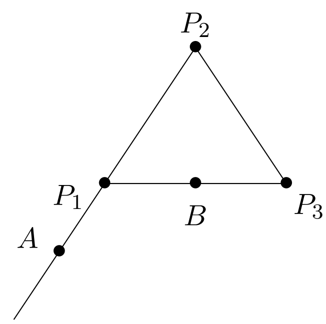
Problem 4. If X is a plane, show that it is a $\mathbb{P}^2$; i.e. show that axioms (M1), (M2), (M3) hold.

Problem 5. Prove that every line meeting two sides of a triangle, but none of its vertices, must also meet the third side. More precisely:

Consider the triangle P_1, P_2, P_3. We'll interpret this as the data consisting of the three points P_1, P_2, P_3 together with the corresponding three lines that they span pairwise, which we'll denote $\overline{P_1P_2}$, $\overline{P_1P_3}$, $\overline{P_2P_3}$ (this is OK by (S1)).



Choose a third point, A, on $\overline{P_1P_2}$ and a third point, B, on $\overline{P_1P_3}$ (this is ok by (S6)):



Then prove that the line $\overline{AB}$ has to meet the line $\overline{P_2P_3}$ in a point.

For Problems 6, 7 and 8, feel free to use any facts we proved in class about $\mathbb{P}^2$.

Problem 6. If the $\mathbb{P}^3$ is finite, show that any two lines of $\mathbb{P}^3$ have the same number of elements, which we'll still call $d+1$. (Note that the two lines don't necessarily meet in a point, so they're not necessarily in the same plane.)

Problem 7. Show that any two planes contain the same number of elements. What is that number (in terms of the integer d in Problem 6)? Explain your answer. Again, feel free to use earlier problems or results from class.

Problem 8. In terms of d (as in Problem 6), how many points are in $\mathbb{P}^3$? Explain your answer using the axioms and previous problems.

Now we'll focus on the classical projective spaces $\mathbb{P}^n_k$ and their varieties, using homogeneous coordinates. Specifically, for the final problem assume $k = \mathbb{R}$.

Problem 9. Again, in this problem we work over $\mathbb{R}$. Let's look at the projective twisted cubic over the real numbers. Consider the mapping (function)

$$\phi : \mathbb{P}^1 \to \mathbb{P}^3$$

defined by $\phi([s,t]) = [s^3, s^2t, st^2, t^3]$.

(i) Show that this mapping is well-defined.

(ii) Explain why (for example) the mapping $\phi'([s,t]) = [s^3, s^2t, st^2, t^4]$ would not be well-defined. Make sure you give enough detail in your answer.

(iii) Let $V = \phi(\mathbb{P}^1)$ be the image of $\mathbb{P}^1$ in $\mathbb{P}^3$ under the mapping ϕ. Prove that $V \cap U_0$ (where U_0 is the affine part of $\mathbb{P}^3$ as defined in class) is the affine twisted cubic mentioned in class.

(iv) Consider the matrix

$$A = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$$

Find the maximal minors of A.

(v) Let I be the ideal generated by the maximal minors of A that you found in (9iv). Show that $\mathbb{V}(I) = V$. (Make sure you show both inclusions.)